Eliza Wajch
Compactifications and $L$-separation

Commentationes Mathematicae Universitatis Carolinae, Vol. 29 (1988), No. 3, 477--484

Persistent URL: http://dml.cz/dmlcz/106663

Terms of use:
© Charles University in Prague, Faculty of Mathematics and Physics, 1988

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project DML-CZ: The Czech Digital Mathematics Library http://project.dml.cz
COMPACTIFICATIONS AND L-SEPARATION

Eliza WAJCH

Abstract: In the paper, the notion of l-separation introduced by J.L. Blasco is applied to characterizing subsets of $C_0(X)$ which generate compactifications of a Tychonoff space $X$ (i.e. sets $F \subseteq C_0(X)$ such that the diagonal mapping $\triangle f$ is a homeomorphic embedding).

Key words: Compactifications, continuous functions, l-separation, homeomorphic embeddings, proximities, functional bases.

Classification: 43D35, 54D40, 54C20

Throughout this paper, $X$ denotes a Tychonoff space. The algebra of all bounded real-valued continuous functions on $X$ is denoted by $C_0(X)$.

Let $K(X)$ be the family of all compactifications of $X$. If $\alpha X$, $\gamma X \in K(X)$ and there is a continuous $\varphi: \alpha X \rightarrow \gamma X$ such that $\varphi \circ \alpha = \gamma$, then we write $\gamma X \leq \alpha X$. For $\alpha X \in K(X)$, let $C_{\alpha}$ denote the set of all functions $f \in C(X)$ continuously extendable to $\alpha X$. For $f \in C_{\alpha}$, let $f^{\alpha}$ be the continuous extension of $f$ to $\alpha X$ and, for $F \subseteq C_{\alpha}$, let $F^{\alpha} = \{ f^{\alpha} : f \in F \}$.

If $F \subseteq C_0(X)$ and the family $\bigwedge_{\alpha X \in K(X)} F \subseteq C_{\alpha} I$ has a minimal (with respect to the partial order $\leq$) element $\kappa_F X$, then $\kappa_F X$ is said to be determined by $F$. Denote by $\mathcal{D}(X)$ the family of subsets of $C_0(X)$ which determine compactifications of $X$.

Let $\mathcal{E}(X)$ be the family of all sets $F \subseteq C_0(X)$ such that the diagonal mapping $e_F = \triangle f$ is a homeomorphic embedding. If $F \subseteq \mathcal{E}(X)$, then the closure of $e_F(X)$ in $R^{|F|}$ is a compactification of $X$. This compactification is said to be generated by $F$ and is denoted by $e_F X$. If $\alpha X \in K(X)$, $F \subseteq \mathcal{E}(X)$ and $e_F X = \alpha X$, then we say that $F$ generates $\alpha X$.

Finally, let $\mathcal{G}(X)$ be the family of all sets $F \subseteq C_0(X)$ which separate points from closed sets. It is well known that $\mathcal{G}(X) \subseteq \mathcal{E}(X) \subseteq \mathcal{D}(X)$; however,
in general, both inclusions are proper.

The families $\mathcal{Y}(X)$ and $\mathcal{B}(X)$ were considered in [1] - [3] and [7]. J.L. Blasco introduced in [4] the notion of L-separation and used it to characterize those functions from $C^*(X)$ which are continuously extendable to $\varepsilon F X$ where $F \in \mathcal{Y}(X)$. In this paper we apply the notion of L-separation to investigate the family $\mathcal{Y}(X)$.

For notation and terminology not defined here, see [5] and [6].

Before proceeding to the body of the article, let us recall two more definitions and establish some useful facts.

**Definition 1** (cf. [4]). A set $G \subseteq C^*(X)$ L-separates a set $A \subseteq X$ from a set $B \subseteq X$ if there exist real numbers $a_j, b_j, c_j, d_j \in \mathbb{R}$ and functions $g_{j,k} \in G$ ($j=1,\ldots,m; k=1,\ldots,n$) such that $A \subseteq \bigcup_{j=1}^{m} \bigcap_{k=1}^{n} g_{j,k}^{-1}((b_j, c_j], c_j, k)$ and $B \subseteq \bigcap_{j=1}^{m} \bigcup_{k=1}^{n} g_{j,k}^{-1}((a_j, b_j], d_j, k)$.

**Proposition 1.** Suppose that $G \subseteq C^*(X)$ and let $A_i, B_i$ be subsets of $X$ for $i=1,2$.

1. If $G$ L-separates $A_1$ from $B_1$, then $G$ L-separates $B_1$ from $A_1$.
2. If $G$ L-separates $A_i$ from $B_i$ for $i=1,2$, then $G$ L-separates $A_1 \cup A_2$ from $B_1 \cap B_2$.
3. Subsets $A$ and $B$ of $X$ are completely separated if and only if $C^*(X)$ L-separates $A$ from $B$.

We omit simple proofs of (1) and (2). To show (3), it suffices to observe that if $C^*(X)$ L-separates $A$ from $B$, then $(\text{cl}_A X) \cap (\text{cl}_B X) = \emptyset$.

**Definition 2** (cf. [4]). A set $G \subseteq C^*(X)$ L-separates a function $f \in C^*(X)$ if, for any real numbers $a < b$, the sets $f^{-1}((-\infty; a])$ and $f^{-1}([b; +\infty))$ are L-separated by $G$. A set $F \subseteq C^*(X)$ is L-separated by $G$ if $G$ L-separates any function $f \in F$.

**Proposition 2.** A set $G \subseteq C^*(X)$ L-separates a function $f \in C^*(X)$ if and only if, for any real numbers $a < b \leq c < d$, the sets $f^{-1}([b; c])$ and $f^{-1}((-\infty; a) \cup [d; +\infty))$ are L-separated by $G$.

**Proof.** Let $a < b \leq c < d$. If $G$ L-separates $f^{-1}((-\infty; a])$ from $f^{-1}([b; +\infty))$ and $f^{-1}([d; +\infty))$ from $f^{-1}((-\infty; c])$, then, by Proposition 1 (2), the sets $f^{-1}((-\infty; a]) \cup [d; +\infty))$ and $f^{-1}([b; c])$ are L-separated by $G$. On the other hand, since $f$ is bounded, there is a real number $r > 0$ such
that \( f(X) \subset [\-r;r] \) and \( a, b \in (\-r;r) \). Then \( \mathcal{T}^{-1}((-\infty; a]) = \mathcal{T}^{-1}([-r; a]) \) and \( \mathcal{T}^{-1}([b;+\infty)) = \mathcal{T}^{-1}((-\infty; -2r) \cup [b;+\infty)) \), which completes the proof.

Now, we are in a position to prove the main theorems of this paper.

**Theorem 1.** If \( F \in \mathfrak{D}(X) \), \( G \subset C^*(X) \) and \( F \) is \( L \)-separated by \( G \), then \( G \in \mathfrak{D}(X) \) and \( \alpha F X \subset \alpha G X \).

Proof. Let us consider any \( \alpha X \in K(X) \) for which \( G \subset C^\infty \). Since \( C^\infty \) is \( L \)-separated \( F \), it follows from [4; Theorem 4] that \( F \subset C^\infty \). Hence the set \( C_F = \cap_i C_i : \alpha X \in K(X) \) and \( F \subset C^\infty \) is contained in \( C_G = \cap_i C_i : \alpha X \in K(X) \) and \( G \subset C^\infty \). This, together with [1; Theorem 3.1] or [5; Theorem 2.18], implies that \( C_\Phi \subset \mathfrak{D}(X) \) because \( C_\Phi \subset \mathfrak{D}(X) \). Using [1; Theorem 3.1] again, we conclude that \( G \in \mathfrak{D}(X) \) and \( \alpha F X \subset \alpha G X \).

The next theorem can be regarded as a generalization of Theorem 6 of [4].

**Theorem 2.** For sets \( F \in \mathfrak{D}(X) \) and \( G \subset C^*(X) \), the following conditions are equivalent:

1. \( G \in \mathfrak{D}(X) \) and \( \alpha F X \subset \alpha G X \);
2. \( F \) is \( L \)-separated by \( G \).

Proof. That (1) implies (2) follows from [4; proofs of Proposition 2 and Theorem 6].

Assume (2). Let \( A \) be a closed subset of \( X \) and let \( x \in X \setminus A \), by virtue of the theorem given in [6; Exercise 2.3.0], there exist \( f_1, \ldots, f_n \in F \) such that

\[
\bigcap_{i=1}^n f_i(x) \subset \text{cl} \bigcap_{i=1}^n f_i(A). \quad \text{We can find } \eta > 0 \text{ such that}
\]

\[
\bigcap_{i=1}^n \left( f_i(x) - \eta ; f_i(x) + \eta \right) \cap \bigcap_{i=1}^n f_i(A) = \emptyset. \quad \text{By Proposition 2, for each}
\]

\[
i \in \{1, \ldots, n\}, \quad \text{there exist functions } g_i, j, k \in G \text{ and real numbers } a_i, j, k < b_i, j, k \leq c_i, j, k < d_i, j, k \quad (j=1, \ldots, n_i; k=1, \ldots, m_i)
\]

such that

\[
\{y \in X : |f_i(y) - f_i(x)| \leq \eta \} \subset \bigcup_{j=1}^{n_i} \bigcap_{k=1}^{m_i} g_i, j, k([-b_i, j, k; c_i, j, k]) \quad \text{and}
\]

\[
\{y \in X : |f_i(y) - f_i(x)| > \eta \} \subset \bigcup_{j=1}^{n_i} \bigcap_{k=1}^{m_i} g_i, j, k((\-\infty; a_i, j, k] \cup [d_i, j, k; +\infty)).
\]

To each \( i \in \{1, \ldots, n\} \) assign some \( j_i \in \{1, \ldots, n_i\} \) such that

- 479 -
\[ \times_{k=1}^{m_i} g^{-1}_{i,j_i,k}(\Gamma_{i,j_i,k}^{c_i,j_i,k}) \]. Denote
\[ g = \Delta \{ g_{i,j_i,k} : i=1,...,n \text{ and } k=1,...,m_i \} \]
and
\[ V = \prod \{ \{ a_{i,j_i,k}, d_{i,j_i,k} \} : i=1,...,n \text{ and } k=1,...,m_i \} \].

Then \( V \) is an open subset of \( R^m \) where \( m = \sum_{i=1}^{n} m_i \), and \( g(x) \in V \). It is easily seen that \( g(A) \cap V = \emptyset \), so \( g(x) \notin \operatorname{cl}_{R^n} g(A) \). Using the theorem of [6; Exercise 2.3.0], we obtain that \( G \in \mathcal{C}(X) \). Theorem 1 yields that \( e_x X \in e_G X \).

For a nonempty set \( F \subset C^*(X) \), let \( M_F \) denote the family of all functions of the form \( g \circ \Delta_{f \in F} f \) where \( g \in C^*(R^{|F|}) \) (cf. [2],[3] and [7]). It follows from [7; Remark 1.5 and Corollary 1.12] that \( M_F \) is the smallest subalgebra of \( C^*(X) \) closed under uniform convergence, containing \( F \) and all constant functions.

**Corollary 1.** For sets \( F \subset \mathcal{C}(X) \) and \( G \subset C^*(X) \), the following conditions are equivalent:

1. \( G \subset \mathcal{C}(X) \) and \( e_F X \in e_G X \);
2. \( M_F \) is \( L \)-separated by \( G \);
3. \( F \) is \( L \)-separated by \( M_G \).

**Proof.** By virtue of [7; Corollary 2.6] (or [2; Theorem 2.3]), \( M_F \) generates \( e_F X \), so the implication \((1) \Rightarrow (2) \) follows from Theorem 2. The implication \((2) \Rightarrow (3) \) is obvious. If we assume \((3) \), then Theorem 2 yields that \( M_G \subset \mathcal{C}(X) \) and, moreover, the compactification generated by \( M_G \) is not less than \( e_F X \). From [7; Corollary 2.6] (or [2; Theorem 2.3]) we deduce that \((3) \Rightarrow (1) \).

**Corollary 2.** For sets \( F \subset \mathcal{C}(X) \) and \( G \subset C^*(X) \), the following conditions are equivalent:

1. \( G \subset \mathcal{C}(X) \) and \( e_F X \in e_G X \);
2. \( M_F \) is \( L \)-separated by \( G \) and \( M_G \) is \( L \)-separated by \( F \);
3. \( F \) is \( L \)-separated by \( M_G \) and \( G \) is \( L \)-separated by \( M_F \).

Since \( M_F = G_{\infty} \) for any \( F \subset \mathcal{C}(X) \) such that \( e_F X = \infty X \) (cf. [2; Theorem 2.3], [3; Theorem 3.1] or [7; Theorem 2.12]), our next corollary is an immediate consequence of Corollary 2.

**Corollary 3.** For any \( F \subset C^*(X) \) and \( \infty X \in K(X) \), the following conditions are equivalent:

- 480 -
(1) \( F \in \mathcal{C}(X) \) and \( e_F X = \infty X \);
(2) \( F \subseteq C_\infty \) and \( C_\infty \) is \( L \)-separated by \( F \);
(3) \( F \subseteq C_\infty \) and \( C_\infty \) is \( L \)-separated by \( M_F \).

Let \( \equiv \) be the equivalence relation on \( \mathcal{C}(X) \) defined by the condition: \( F \equiv G \) if and only if \( F \) \( L \)-separates \( G \) and \( G \) \( L \)-separates \( F \). The equivalence class of \( \equiv \) containing \( F \in \mathcal{C}(X) \) will be denoted by \([F]_L\). For \( F, G \in \mathcal{C}(X) \), putting \([F]_L \preceq [G]_L\) if and only if \( G \) \( L \)-separates \( F \), we define a partial order on the set \( \mathcal{C}(X)/L \) of all equivalence classes of \( \equiv \). The corollaries from Theorem 2 imply the following:

**Theorem 3.** By assigning to any \([F]_L \in \mathcal{C}(X)/L\) the compactification \( e_F X \) of \( X \), one establishes an isomorphism of the partially ordered set \( (\mathcal{C}(X)/L, \preceq) \) onto the partially ordered set \( (K(X), \subseteq) \).

Now, we are going to study interrelations between elements of \( \mathcal{C}(X) \) and proximities on \( X \).

For \( \infty X \in K(X) \), denote by \( \sigma(\infty) \) the proximity on \( X \) induced by \( \infty X \); i.e. \( \sigma(\infty) \) is defined by letting: \( A \sigma(\infty) B \) if and only if \( (\text{cl}_X A) \cap (\text{cl}_X B) \neq \emptyset \) (cf. [6, p. 561]).

Let \( F \subseteq C^*(X) \). We shall say that two sets \( A, B \subseteq X \) are close with respect to \( \sigma(F) \) if \( F \) does not \( L \)-separate \( A \) from \( B \).

**Theorem 4.** For any \( F \subseteq C^*(X) \), the following conditions are equivalent:

1. \( F \in \mathcal{C}(X) \), and \( \sigma(F) \) is a proximity on \( X \) such that \( \sigma(F) = \sigma(e_F) \);
2. \( F \in \mathcal{C}(X) \);
3. \( \sigma(F) \) is a proximity on \( X \).

**Proof.** According to the proof of Proposition 2 in [4], we deduce that \((2) \implies (3)\).

Assume \((3)\) and let \( \infty X \in K(X) \) be such that \( \sigma(F) = \sigma(\infty) \). By virtue of [4; Corollary 3], \( C_\infty \) is \( L \)-separated by \( F \). On the other hand, if \( F \in F \) and \( a < b \) \((a, b \in R)\), then the sets \( f^{-1}((-\infty; a]) \) and \( f^{-1}([b; +\infty)) \) are \( L \)-separated by \( F \), so their closures in \( \infty X \) are disjoint. Using [4; Corollary 3] again, we obtain that \( F \subseteq C_\infty \). By our Corollary 3, \( F \in \mathcal{C}(X) \) and \( e_F X = \infty X \); hence \((3) \implies (1)\).

**Theorem 5.** By assigning to any \([F]_L \in \mathcal{C}(X)/L\) the proximity \( \sigma(F) \) on \( X \), we establish a one-to-one correspondence between elements of \( \mathcal{C}(X)/L \) and all proximities on the space \( X \).
To give another necessary and sufficient condition for $F$ to be in $\mathscr{C}(X)$, we need some notation.

Suppose that $F \subseteq C^*(X)$. Denote by $\mathcal{Z}_F$ the family of all sets of the form

$$\bigcup_{j=1}^{m} \bigcap_{k=1}^{n} f_{j,k}^{-1}(\{a_{j,k}; b_{j,k}\})$$

where $f_{j,k} \in F$ and $a_{j,k} \leq b_{j,k}(a_{j,k}, b_{j,k}, k \in \mathbb{R})$ for $j=1,...,m$; $k=1,...,n$ ($m,n \in \mathbb{N}$). One can easily check that the family $\mathcal{Z}_F$ is closed under finite unions and intersections; moreover, $\mathcal{Z}_F$ consists of zero-sets of $X$.

**Theorem 6.** A set $F \subseteq C^*(X)$ is an element of $\mathscr{C}(X)$ if and only if the family $\mathcal{Z}_F$ is a closed base for $X$.

**Proof.** Let $A$ be a closed subset of $X$ and let $x \in X \setminus A$. If $F \subseteq \mathscr{C}(X)$, then from [4; proof of Proposition 2] we deduce that $F$ L-separates $A$ from $\{x\}$; hence there exists $Z \in \mathcal{Z}_F$ such that $A \subseteq Z$ and $x \notin Z$, which means that $\mathcal{Z}_F$ is a closed base for $X$.

Conversely, if $\mathcal{Z}_F$ is a closed base for $X$, then there exist functions $f_{j,k} \in F$ and real numbers $a_{j,k}, b_{j,k}$ ($j=1,...,m$; $k=1,...,n$) such that $A \subseteq \bigcup_{j=1}^{m} \bigcap_{k=1}^{n} f_{j,k}^{-1}(\{a_{j,k}; b_{j,k}\})$ and $x \notin \bigcup_{j=1}^{m} \bigcap_{k=1}^{n} f_{j,k}^{-1}((-\infty ; a_{j,k}; b_{j,k}; +\infty))$.

To each $j \in \{1,\ldots,m\}$ assign some $k_j \in \{1,\ldots,n\}$ such that $x \in f_{j,k_j}^{-1}(\{a_{j,k_j}; b_{j,k_j}\})$. Denote $f = \bigtriangleup_{j=1}^{m} f_{j,k_j}$ and $V = \bigcap_{j=1}^{m} \{(-\infty ; a_{j,k_j}; b_{j,k_j}; +\infty)\}$. Then $V \cap f(A) = \emptyset$, so $f(x) \notin \mathbb{R}^m f(A)$.

Applying the theorem given in [6; Exercise 2.3.3], we obtain that $F \subseteq \mathscr{C}(X)$.

Let $\alpha X \in K(X)$. In [1; p.9] B.J. Ball and Shoji Yokura introduced the cardinal number $e(\alpha X) = \min \{|F|: F \subseteq \mathscr{C}(X) \text{ and } e_F = \alpha X\}$. We shall call this number the functional weight of $\alpha X$. As shown in [1; Theorem 4.21], if the functional weight of $\alpha X$ is infinite, then it is equal to the weight of $\alpha X$.

It seems natural to call every set generating $\alpha X$ a functional base for $\alpha X$. It is worth mentioning that $F \subseteq C^*(X)$ is a functional base for $\alpha X$ if and only if $M_F = C_\alpha$ (cf. [2; Definition 1.2 and Theorem 2.3]). Our final theorem points out that functional bases have some property similar to that of open bases for topological spaces.

**Theorem 7.** If $\alpha X \in K(X)$ is of infinite functional weight, then every
functional base for \( \kappa X \) contains a functional base for \( \kappa X \) of cardinality \( e(\kappa X) \).

Proof. Consider any functional base \( F \) for \( \kappa X \). There exists a functional base \( H \) for \( \kappa X \) such that \( |H|=e(\kappa X) \). Denote by \( Q \) the set of rational numbers and let \( P = \{ < a, b > : Q^{2}: a < b \} \). By Corollary 2, \( H \) is \( L \)-separated by \( F \). Therefore, to each \( h \in H \) and \( < a, b > \in P \) we can assign a finite set \( F(h; < a, b >) \subseteq F \) which \( L \)-separates \( h^{-1}((-\infty; a]) \) from \( h^{-1}([b; +\infty)) \). Let \( G = \bigcup \{ F(h; < a, b >) : h \in H \land < a, b > \in P \} \). First of all, observe that \( |G| \leq |H| \) and \( H \) is \( L \)-separated by \( G \). Since \( G \subseteq F \) and, by Corollary 2, \( F \) is \( L \)-separated by \( H \), we have that \( G \) is \( L \)-separated by \( H \). Applying Corollary 2 again, we deduce that \( G \) is a functional base for \( \kappa X \) and, consequently, \( |G|=|H| \).

The assumption that \( e(\kappa X) \) is infinite cannot be omitted in the above theorem.

Example 1. Let \( \kappa x = [-1; 1] \) and \( F = \{ f_{1}, f_{2} \} \) where
\[
\begin{align*}
f_{1}(x) &= \begin{cases} 
0 & \text{for } -1 < x \leq 0, \\
0 & \text{for } 0 < x < 1
\end{cases} \\
f_{2}(x) &= \begin{cases} 
0 & \text{for } -1 < x \leq 0, \\
0 & \text{for } 0 < x < 1.
\end{cases}
\end{align*}
\]
Then \( F \) is a functional base for \( \kappa X \) (cf. [3; Theorem 2.3]), \( e(\kappa X) = 1 \), but none of the sets \( \{ f_{1} \}, \{ f_{2} \} \) generates \( \kappa X \).

Observe that Theorem 4.3 of [1], our Theorem 2 and the proof of Theorem 7 imply that if \( \kappa X \in \mathcal{K}(X) \) are of infinite functional weight, \( \kappa X \not\leq \gamma X \) and \( F \) is a functional base for \( \gamma X \), then there exists a set \( G \subseteq F \) such that \( G \leq e(\kappa X) \) and \( \kappa X \leq G X \leq \gamma X \); however, \( F \) need not contain any functional base for \( \kappa X \).

Example 2. Consider the space \( [0; \omega_{1}] \) of ordinal numbers \( < \omega_{1} \) with the order topology. Let \( X = [0; \omega_{1}] \times \{ 0, 1 \} \) and \( F = \{ f \in C^{*}(X) : f^{\beta} ( < \omega_{1}, 0 >) \neq f^{\beta} ( < \omega_{1}, 1 >) \} \). Since \( F \) separates points of \( \beta X \), it follows from [3; Theorem 2.3] that \( F \) is a functional base for \( \beta X \). No function from \( F \) is continuously extendable to the one-point compactification of \( X \); hence, no subset of \( F \) is a functional base for the one-point compactification of \( X \).

References


Institute of Mathematics, University of Łódź, Banacha 22, 90-238 Łódź, Poland.

(Oblatum 3.8. 1987, revisum 8.2. 1988)