Commentationes Mathematicae Universitatis Carolinae

Antonín Sochor

Addition of initial segments. II.

Commentationes Mathematicae Universitatis Carolinae, Vol. 29 (1988), No. 3, 519--528

Persistent URL: `http://dml.cz/dmlcz/106666`

COMMENTATIONES MATHEMATICAE UNIVERSTATIS CAROLINAE
29,3 (1988)

ADDITION OF INITIAL SEGMENTS II

Antonín SOCHOR

Abstract: In the previous paper [S] we extended addition and subtraction to the system of all initial segments of N. In this article we continue in the investigation of properties of these operations and we describe some examples.

Key words: Alternative set theory, natural number, finite natural number, initial segment, π-semiset, σ-semiset.

Classification: Primary 03E70
Secondary 03H15

This paper is a direct continuation of the article [S].

We use the notation usual in the alternative set theory (see [V]); complete subclasses of N are called initial segments and cuts are initial segments closed under the successor operation; the letters R, S and T are reserved for variables running through initial segments. The results of the paper [S] are cited using the number of the result in question only; we numer the results of the present article in accordance with [S].

For every two initial segments R, S we define (see [S])

$R' = R \cup \{R\}$

$R+S = \{\vartheta ; (\exists \alpha \in R')(\exists \beta \in S')\vartheta < \alpha + \beta\}$

$R \dot{-} S = \{\vartheta ; (\forall \beta \in S')\vartheta + \beta \in R\}$

$R^+ = \{\vartheta ; (\forall \alpha \in R')\vartheta + \alpha \in R\} = R \dot{-} R.$

The magnitude of the class R^+ determines considerably the behaviour of addition and subtraction on the class R (see e.g. (5)). The following result helps us to determine the magnitude of the cut R^+.

(20) a) $(R')^+ = R^+ = (R^+)^+$,

to prove this assertion it is sufficient to use the definition of R^+ and (14).□

b) $(R+S)^+ = R^+ \cup S^+$.

For every $\tau \in (R+S)'$ there are $\alpha \in R'$ and $\beta \in S'$ such that $\tau \leq \alpha + \beta$ (cf. (3b)); furthermore, for every $\vartheta \in R^+ \cup S^+$ either $\vartheta + \alpha \in R$ or $\vartheta + \beta \in S$

and therefore (cf. (3c))

$\vartheta+\tau\leq\vartheta+\alpha+\beta\in R+S.$

If $\vartheta\notin R^+\cup S^+$ then there is $\bar{\vartheta}\notin R^+\cup S^+$ with $2\bar{\vartheta}\leq\vartheta$ (see (14) and (16)) and using the definitions of R^+ and S^+ we can find $\alpha\in R'$ and $\beta\in S'$ with

$\alpha+\bar{\vartheta}\notin R\ \&\ \beta+\bar{\vartheta}\notin S$

and hence by (15) we get

$\alpha+\beta+\vartheta\geq(\alpha+\bar{\vartheta})+(\beta+\bar{\vartheta})\notin R+S\ \&\ \alpha+\beta\in(R+S)'.$ □

c) $S\subseteq R\rightarrow(R\div S)^+=R^+\cup S^+.$

Let $\vartheta\in R^+\cup S^+$ and let $\tau\in(R\div S)'$ (i.e. $(\forall\bar{\beta}\in S')\tau+\bar{\beta}\in R'$). We want to show $\vartheta+\tau\in R\div S$, i.e. that for every $\beta\in S'$ it is

$(\vartheta+\tau)+\beta\in R.$

However, this is trivial in both cases: if $\vartheta\in S^+$, then $\vartheta+\beta\in S$ and thence $\vartheta+\beta+1\in S'$ from which

$\vartheta+\beta+1+\tau\in R'$

follows. If $\vartheta\in R^+$ then it is sufficient to use the definition of R^+ because $\tau+\beta\in R'$.

If $\vartheta\notin R^+\cup S^+$, then we can again choose $\bar{\vartheta}\notin R^+\cup S^+$ with $2\bar{\vartheta}\leq\vartheta$ and further we are able to fix $\beta\in S'$ with $\beta+\bar{\vartheta}\notin S$. Without loss of generality we can assume $\beta+\bar{\vartheta}\in R$ because

$R=S\rightarrow(R\div S)^+=(R^+)^+=R^+$.

Therefore we are able to fix moreover $\alpha\in R'$ with

$\beta+\vartheta<\alpha\ \&\ \alpha+\bar{\vartheta}\notin R.$

The natural number $\alpha-(\beta+\bar{\vartheta})$ is an element of $(R\div S)'$ because for every $\bar{\beta}\in S'$ we have $\bar{\beta}\leq\beta+\bar{\vartheta}$ and therefore

$\alpha-(\beta+\bar{\vartheta})+\bar{\beta}\leq\alpha\in R'.$

However,

$(\alpha-(\beta+\bar{\vartheta})+\vartheta)+\beta\leq\alpha+\bar{\vartheta}\notin R,$

which implies

$\alpha-(\beta+\bar{\vartheta})+\vartheta\notin R\div S$

and consequently $\vartheta\notin(R\div S)^+$. □

The last result and (5) imply the equality

$(R+S)\div S^+=R+S$

for all initial segments and if $S\subseteq R$, then we get moreover

$(R\div S)+S^+=R\div S.$

Now we are going to investigate the behaviour of the operations + and $\div$ for monotonous systems of initial segments.

(21) a) If for every $n \in FN$ we have

$R_n \subseteq R_{n+1} \& S_n \subseteq S_{n+1}$,

then

$\bigcup\{R_n; n \in FN\} + \bigcup\{S_n; n \in FN\} = \bigcup\{R_n + S_n; n \in FN\}$. $\square$

b) If for each $n \in FN$ we have

$R_{n+1} \subseteq R_n \& S_n \subseteq S_{n+1}$,

then

$\bigcap\{R_n; n \in FN\} \dot{-} \bigcup\{S_n; n \in FN\} = \bigcap\{R_n \dot{-} S_n; n \in FN\}$,

because the following formulae are equivalent:

$\vartheta \in (\bigcap\{R_n; n \in FN\} \dot{-} \bigcup\{S_n; n \in FN\})$

$(\forall n \in FN)(\forall \beta \in S_n')\vartheta + \beta \in \bigcap\{R_m; m \in FN\}$

$(\forall n \in FN)(\forall \beta \in S_n')(\forall m \in FN)\vartheta + \beta \in R_m$

and the above stated formulae are equivalent furthermore to the formulae

$(\forall n \in FN)(\forall \beta \in S_n')\vartheta + \beta \in R_n$

$\vartheta \in \bigcap\{R_n \dot{-} S_n; n \in FN\}$

because the systems of R_n's and S_n's are supposed to be monotonous. (in detail: Let $n, m \in FN$ be given and let

$(\forall k \in FN)(\forall \beta \in S_k')\vartheta + \beta \in R_k$

hold. If $m \leq n$, then for every $\beta \in S_n'$ we have

$\vartheta + \beta \in R_n \subseteq R_m$;

if $n \leq m$, then $S_n \subseteq S_m$ and thus

$(\forall \beta \in S_n')\vartheta + \beta \in R_m$

follows from

$(\forall \beta \in S_m')\vartheta + \beta \in R_m$)). $\square$

If for every $n \in FN$ we have

$R_n \subseteq R_{n+1} \& S_{n+1} \subseteq S_n \& T_{n+1} \subseteq T_n$,

then the formulae

$\bigcup\{R_n \dot{-} S_n; n \in FN\} \subseteq \bigcup\{R_n; n \in FN\} \dot{-} \bigcap\{S_n; n \in FN\}$

and

$\bigcap\{T_n; n \in FN\} + \bigcap\{S_n; n \in FN\} \subseteq \bigcap\{T_n + S_n; n \in FN\}$

are trivial consequences of (7), however, the following examples show that these assertions cannot be strengthened to equalities.

Choosing the sequences $\{\vartheta_n; n \in FN\}$ and $\{\tau_n; n \in FN\}$ and $\xi \in N$ with

$\vartheta_n \dot{-} \vartheta_{n+1} \notin FN \& \tau_{n+1} \dot{-} \tau_n \notin FN \& \vartheta_0 < \xi \& (\forall n \in FN)\tau_n < \xi$

and putting

$R = \xi + \bigcap\{\vartheta_n; n \in FN\}$

$S_n = \vartheta_n \dot{-} FN$

$$R_n = \xi + \tau_n + FN$$
$$S = \bigcup\{\tau_n; n \in FN\}$$
$$T_n = (\xi - \tau_n) \dot{-} FN$$
we get
$$\bigcup\{R \dot{-} S_n; n \in FN\} \subset R \dot{-} \bigcap\{S_n; n \in FN\}$$
$$\bigcup\{R_n \dot{-} S; n \in FN\} \subset \bigcup\{R_n; n \in FN\} \dot{-} S$$
and
$$\bigcap\{T_n; n \in FN\} + S \subset \bigcap\{T_n + S; n \in FN\}$$
(and the initial segments in question are nonempty cuts with $S_n \subseteq R$ & $S \subseteq R_n$).

In fact, we have
$$\xi \in R \dot{-} \bigcap\{S_n; n \in FN\} = (\xi + \bigcap\{\vartheta_n; n \in FN) \dot{-} \bigcap\{\vartheta_n; n \in FN\}$$
and
$$\xi \notin R \dot{-} (\vartheta_n \dot{-} FN) \subseteq R \dot{-} \vartheta_{n+1};$$
furthermore
$$\xi \in \bigcup\{R_n; n \in FN\} \dot{-} S$$
and on the other hand,
$$\xi \notin R_n \dot{-} S$$
because
$$\xi + \tau_{n+1} \notin R_n.$$
To prove the last claim let us realize that for every $n \in FN$ it is
$$\xi = (\xi - \tau_{n+1}) + \tau_{n+1} \in T_n + S$$
and on the other hand, assuming
$$\xi \in \bigcap\{T_n; n \in FN\} + S$$
one could find α, β with
$$\alpha \in (\bigcap\{T_n; n \in FN\})' = \bigcap\{T_n; n \in FN\} \mathbin{\&} \beta \in S' = S \mathbin{\&} \xi < \alpha + \beta .$$
Thus there would be $n \in FN$ with
$$\xi < \alpha + \tau_n,$$
which contradicts the choice of the class T_n.

If R and S are real classes, then the classes R+S and $R \dot{-} S$ are real, too. In other words, if R and S belong to the system of all π-semisets and σ-semisets, then R+S and $R \dot{-} S$ belong to the same system. Furthermore, if we know whether R is a π-semiset or a σ-semiset and whether S is a π-semiset or a σ-semiset, then we can in some cases decide whether R+S ($R \dot{-} S$ respectively) is a π-semiset or a σ-semiset. Let us note that all possibilities which are not excluded by the following statement can be realized.

(22) a) If R and S are π-semiset (σ-semiset respectively), then R+S is also a π-semiset (σ-semiset respectively). Let

$$R=\bigcap\{\gamma_n;n\in FN\}\ \&\ S=\bigcap\{\delta_n;n\in FN\}\ \&\ (\forall n\in FN)(\gamma_{n+1}\leq\gamma_n\ \&\ \delta_{n+1}\leq\delta_n).$$

Evidently

$$R+S\subseteq\bigcap\{\gamma_n+\delta_n;n\in FN\}.$$

Using the prolongation axiom we can fix monotonous functions f, g with

$$(\forall n\in FN)(f(n)=\gamma_n\ \&\ g(n)=\delta_n).$$

If

$$\vartheta\in\bigcap\{\gamma_n+\delta_n;n\in FN\},$$

then there is $\mu\notin FN$ such that

$$(\forall\nu\leq\mu)\,\vartheta<f(\nu)+g(\nu)$$

by overspill. Evidently

$$f(\mu)\in(\bigcap\{\gamma_n;n\in FN\})'\ \&\ g(\mu)\in(\bigcap\{\delta_n;n\in FN\})'$$

and therefore

$$\vartheta<f(\mu)+g(\mu)$$

is an element of R+S. The second assertion is a trivial consequence of (21a). □

b) If R is a π-semiset and S is a σ-semiset, then $R\dot{-}S$ is a π-semiset and $S\dot{-}R$ is a σ-semiset.

The statement (21b) implies the first assertion. Let

$$R=\bigcap\{\gamma_n;n\in FN\}\ \&\ S=\bigcup\{\beta_n;n\in FN\}\ \&\ (\forall n\in FN)(\beta_n\leq\beta_{n+1}\ \&\ \gamma_{n+1}\leq\gamma_n).$$

The inclusion

$$S\dot{-}R\subseteq\bigcup\{\beta_n\dot{-}\gamma_n;n\in FN\}$$

follows from (7). According to the prolongation axiom we are able to choose monotonous functions f, g so that

$$(\forall n\in FN)(f(n)=\gamma_n\ \&\ g(n)=\beta_n).$$

Let

$$\vartheta\notin\bigcup\{\beta_n\dot{-}\gamma_n;n\in FN\}$$

be given. Using overspill we can find $\mu\notin FN$ with

$$(\forall\nu\leq\mu)\,\vartheta\geq g(\nu)\dot{-}f(\nu).$$

Obviously

$$g(\mu)\notin S\ \&\ f(\mu)\in R'$$

and thus

$$\vartheta+f(\mu)\geq g(\mu)$$

implies $\vartheta\notin S\dot{-}R$. □

The following six examples confirm our claim that all possibilities which are not excluded by the last statement can be realized.

At first let us realize that if R is a π-semiset (σ-semiset respectively), then for every ξ, the initial segment $\xi+R$ is a π-semiset (σ-semiset respectively) and $\xi\dot{-}R$ is a σ-semiset (π-semiset respectively).

If R and S are cuts closed under the operation +, then

$$R+S = \begin{cases} R, & \text{if } S \subseteq R \\ S, & \text{if } R \subseteq S \end{cases}$$

and thus assuming that R is a π-semiset and S is a σ-semiset we see that the sum of a π-semiset and σ-semiset can be sometimes a π-semiset and at some other time a σ-semiset.

Furthermore we have

$R \dot{-} R = R^+ = R$ and $S \dot{-} S^+ = S^+ = S$

and moreover for every ξ with

$R \cup S \subsetneq \xi$

we get

$(\xi \dot{-} R) \dot{-} (\xi \dot{-} R) = (\xi \dot{-} R)^+ = R^+ = R$

and

$(\xi \dot{-} S) \dot{-} (\xi \dot{-} S) = (\xi \dot{-} S)^+ = S^+ = S$

according to (21) (and (14)) and hence neither the system of all π-semisets nor the system of all σ-semisets is closed under the operation T^+.

In (19a) - (19c) we have described the class R+S under the assumption that both R and S are real. The following four results can be considered as a description of R+S, too: however, now we replace the assumption of reality of S by the assumption that S is closed under the operation + (we have $S=S^+$ in this case).

(23) If

$R = \bigcap \{\gamma_n ; n \in FN\}$

and if there is $\tau \in S^+$ such that

$(\forall n \in FN)\, \gamma_0 \dot{-} \gamma_n \leq \tau$,

then

$R+S = \gamma_0 + S$.

The inclusion

$R+S \subseteq \gamma_0 + S$

is a trivial consequence of (7). Let f be a monotonous function such that

$(\forall n \in FN) f(n) = \min(\gamma_0, \ldots, \gamma_n)$

(the existence of such a function is a consequence of the prolongation axiom). We can choose $\mu \notin FN$ so that

$(\forall \nu \leq \mu)(f(\nu) \geq f(\nu+1) \,\&\, f(0) - f(\nu) \leq \tau)$.

Evidently

$(\forall n \in FN) f(\mu) \leq \gamma_n$

and thus

$f(\mu) \in R'$.

Therefore for every $\beta \in S'$ we have

$$\gamma_0+\beta=f(\mu)+(f(0)-f(\mu))+\beta\leq f(\mu)+(\tau+\beta)\in R+S$$

because $\tau+\beta\in S$ is a consequence of $\tau\in S^+$. The converse inclusion is proved. $\square$

(24) If

$$R=\bigcap\{\gamma_n;n\in FN\}$$

and if

$$(\forall n\in FN)(\gamma_{n+1}\leq\gamma_n)\&(\forall n\in FN)(\forall\beta\in S)(\exists m\in FN)(\gamma_n-\gamma_m>\beta),$$

then

$$R+S=R.$$

If $\alpha\in R'$, $\beta\in S'$ and $n\in FN$ are given, then we can choose $m\in FN$ with $\gamma_n-\gamma_m\geq\beta$ and thus (since $\alpha\leq\gamma_m$) we get

$$\alpha+\beta\leq\gamma_m+((\gamma_n-\gamma_m)=\gamma_n.$$

We have proved

$$(\forall n\in FN)\alpha+\beta\leq\gamma_n$$

i.e. $\alpha+\beta\in R'$ and consequently we get $R+S\subseteq R$; the converse inclusion is trivial. $\square$

(25) If

$$R=\bigcup\{\alpha_n;n\in Fn\}$$

and if

$$(\forall n\in FN)\ \alpha_{n+1}\dot{-}\alpha_n\in S^+,$$

then

$$R+S=\alpha_0+S.$$

Really, if $\alpha\in R'$ and $\beta\in S'$, then there is $n\in FN$ such that $\alpha\leq\alpha_{n+1}$ and we have

$$\alpha+\beta\leq\alpha_{n+1}+\beta\leq\alpha_0+\sum_{k=0}^{n}(\alpha_{k+1}\dot{-}\alpha_k)+\beta\in\alpha_0+S$$

because

$$\sum_{k=0}^{n}(\alpha_{k+1}\dot{-}\alpha_k)\in S^+$$

according to (14) and because this fact implies

$$\sum_{k=0}^{n}(\alpha_{k+1}\dot{-}\alpha_k)+\beta\in S.$$

The converse inclusion is a trivial consequence of (7). $\square$

(26) If

$$R=\bigcup\{\alpha_n;n\in FN\}$$

and if

$(\forall n \in FN)\alpha_{n+1} \dot- \alpha_n \notin S)$,

then

$R+S=R$.

For each $\alpha \in R'$ and $\beta \in S'$ there is $n \in FN$ so that $\alpha \leq \alpha_n$ and therefore

$\alpha + \beta \leq \alpha_n + (\alpha_{n+1} \dot- \alpha_n) \leq \max(\alpha_n, \alpha_{n+1}) \in R'$

is a consequence of the implication

$\alpha_{n+1} \dot- \alpha_n \notin S \rightarrow \alpha_{n+1} \dot- \alpha_n \geq \beta$. □

At the end of the paper we are going to describe some particular examples.

(27) a) For every ξ, the initial segment

$\{\vartheta; (\exists n \in FN)\ n\xi > \vartheta\}$

is the smallest initial segment containing ξ and closed under the operation $+$; if $\xi > 1$, then the class

$\{\vartheta; (\exists n \in FN)\ n\xi > \vartheta\}$

is a cut. □

b) For every ξ, the initial segment

$\{\vartheta; (\forall n \in FN)\ n\vartheta < \xi\}$

is the maximal initial segment closed under the operation $+$ and not containing ξ ; if $\xi \notin FN$, then the class

$\{\vartheta; (\forall n \in FN)\ n\vartheta < \xi\}$ □

is a cut.

(28) For every $\xi \notin FN$ we have

$\xi \dot- \{\vartheta; (\forall n \in FN)\ n\vartheta < \xi\} = \{\vartheta; (\exists n \in FN)(n+1)\vartheta < n\xi\}$

and

$\xi + \{\vartheta; (\forall n \in FN)\ n\vartheta < \xi\} = \{\vartheta; (\forall n \in FN)\ n\vartheta < (n+1)\xi\}$.

Let us put

$R = \{\vartheta; (\forall n \in FN)\ n\vartheta < \xi\}$

and

$S = \{\vartheta; (\exists n \in FN)(n+1)\vartheta < n\xi\}$.

If $\alpha \in R'$ and $\beta \in S$, then there is $n \in FN$ so that

$(n+1)\beta < n\xi$ and $(n+1)\alpha < \xi$

because $R=R'$ by (27) and thus

$(n+1)(\alpha+\beta) = (n+1)\alpha + (n+1)\beta < \xi + n\xi = (n+1)\xi$

i.e. $\alpha + \beta < \xi$ which proves $S \subseteq \xi \dot- R$.

To prove the converse inclusion let us assume $\delta \notin S$ i.e.

$(\forall n \in FN)(n+1)\delta \geq n\xi$.

By the prolongation axiom we can choose $\mu \notin FN$ so that

$$(\mu+i)\sigma \geq \mu\xi$$

and furthermore we are able to fix α such that

$$(\mu+1)\alpha < \xi \leq (\mu+1)(\alpha+1).$$

By the definition of R we see that α is an element of R and consequently $\alpha+1 \in R$ and therefore the formula

$$(\mu+1)(\sigma+\alpha+1)=(\mu+1)\sigma+(\mu+1)(\alpha+1) \geq \mu\xi+\xi=(\mu+1)\xi$$

i.e. the formula

$$\sigma+\alpha+1 \geq \xi$$

implies $\sigma \notin \xi \div R$.

The second equality is evident - it is sufficient to use the following formulae

$$n\vartheta < \xi \longrightarrow n(\xi+\vartheta)= n\xi+n\vartheta < n\xi+\xi=(n+1)\xi$$

and

$$(\vartheta > \xi \ \& \ n\vartheta < (n+1)\xi) \longrightarrow n(\vartheta-\xi)=n\vartheta-n\xi < (n+1)\xi - n\xi = \xi .$$

Note. We investigated the operations on initial segments related to addition on natural numbers. Similarly we can deal with operations related to multiplication and more generally with any associative operation on natural numbers. Using similar methods as in [S], we are able to prove e.g.: If R is a real cut closed under the operation + , then there is a natural number ξ such that

$$R=\{\vartheta;(\exists\beta\in S)\,\vartheta<\beta\xi\}$$

or

$$R=\{\vartheta;(\forall\beta\in S)\,\vartheta\beta<\xi\}$$

where

$$S=\{\vartheta;(\forall\alpha\in R)\,\alpha\vartheta\in R\}.$$

We can also deal with two operations related to two operations on natural numbers - e.g. we are able to prove the distributive law for operations extending addition and multiplication, however, there are cuts R, S and T fulfilling the inequality

$$\{\vartheta;(\exists\alpha\in R\div S)(\exists\tau\in T)\,\vartheta<\alpha\tau\} \neq \{\vartheta;(\exists\alpha\in R)(\exists\tau\in T)\,\vartheta<\alpha\tau\}\div\{\vartheta;(\exists\beta\in S)(\exists\tau\in T)\,\vartheta<\beta\tau\}$$

(choose $\xi \notin FN$ and put $R=S=\xi+FN$ and $T=FN$).

References

[V] P. VOPĚNKA: Mathematics in the Alternative Set Theory, Teubner Texte, Leipzig 1979.

[S] A. SOCHOR: Addition of initial segments I, Comment. Math. Univ. Carolinae 29(1988). 501-517.

Math. Institute of Czechoslovak Acad. of Sci., Žitná 25, 11000 Praha 1, Czechoslovakia

(Oblatum 13.6. 1988)