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Stability of eigenvalues and eigenvectors of variational inequalities

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Abstract: There is studied the dependence of eigenvectors and eigenvalues of variational inequalities on continuous deformations of the cone. The lower semicontinuity of eigenvalues for general continuous deformations and continuity for continuous invertible deformations is proved.

Key words: Variational inequalities, eigenvalues, eigenvectors, sup min principle.

Classification: 49G05, 35J85, 35P30

0. Introduction. We shall study a variational inequality

\( u \in K \wedge S, \quad (Au - Au, v - u) \geq 0 \quad \forall v \in K, \)

where \( K \) is a closed convex cone with vertex in the origin of a real separable Hilbert space \( H \) and \( A \) is a linear, completely continuous, symmetric and positive operator on \( H \). We denote by \((\cdot , \cdot)\) and \( \| \cdot \| \) the scalar product and the norm in \( H \), respectively, and \( S = \{ u \in H; \| u \| = 1 \} \). A real number \( \lambda \) is called an eigenvalue (and \( u \) the corresponding eigenvector) of the inequality \((A,K)\) if there exists \( u \) satisfying \((A,K)\).

It is well known (see [7]) that the inequality \((A,K)\) has at least one but need not have more than one eigenvalue \( \lambda_1 \) which can be found as follows:

\( \lambda_1 = \max_{u \in K \wedge S} (Au, u) \)

Thus its dependence on the deformations of \( K \) can be studied directly.

The existence of the higher eigenvalues of \((A,K)\) can be proved (under additional assumptions on \( A \) and \( K \)) by different methods (see [1],[2],[3],[4],[5],[6],[7]). We have chosen the variational method due to E. Miersemann ([3], [4],[5]) and we shall restrict ourselves to the existence and continuous dependence of the eigenvalues found by this approach. Thus Section 1 is devoted to the brief description of the sup min principle used for the definition
of the $k$-th eigenvalue.

In Section 2, the concept of deformation of the cone $K$ is introduced and there are proved theorems on lower semicontinuity and continuity of eigenvalues.

In Section 3 we deal with the behaviour of sets of eigenvectors corresponding to eigenvalues on deformed cones. These results are valid under more general conditions than in Section 2.

I am indebted to Jana Stará for many valuable advices during my work on the subject.

1. The sufficient conditions for the existence of higher eigenvalues.

An abstract condition for the existence of higher eigenvalues is given in Theorem 1. Some assumptions on the cone and the operator $A$ which guarantee that this abstract condition is satisfied, are given in Theorems 2 and 3.

Denote $\{\lambda_j\}_{j=1}^{\infty}$ the eigenvalues of $A$ (numbered according to their magnitude $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$) and $\{u_j\}_{j=1}^{\infty}$ the corresponding eigenvectors. Let $L_k$ be the span of the first $k$ eigenvectors of the operator $A$ and $\mathcal{M}_k = \{F \subseteq K \cap S; F$ is compact, $F$ is not contractible within $H-L_k^\perp\}$. The class $\mathcal{M}_k$ may be empty; if it is not the case put

$$c_k = \sup_{F \in \mathcal{M}_k} \min_{u \in F} (Au,u).$$

**Theorem 1** (see [4]). Let $\mathcal{M}_k$ contain a set $F_0$ such that

$$\min_{u \in F_0} (Au,u) \geq \lambda_{k+1} \eta$$

for a positive constant $\eta$.

Then $c_k$ is an eigenvalue of the variational inequality $(A,K)$.

**Remark.** Clearly, $c_k > \lambda_{k+1}$. If, moreover, $\text{Ker}(\lambda_{k+1} I - A) \notin K$, then $c_k < \lambda_{k+1}$, and the eigenvalue $\lambda_{k+1}$ of $(A,K)$ is not an eigenvalue of the operator $A$.

The next two theorems give conditions guaranteeing that the assumption of Theorem 1 holds.

**Theorem 2** (see [5]). Let there exist a closed linear subspace $\tilde{H}$ of the space $H$ so that $\tilde{H} \subseteq K \cap H$. Let $\tilde{T}:H \to \tilde{H}$ be the orthogonal projection onto $\tilde{H}$. 

- 542 -
Denote $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ the eigenvalues of the operator $A = PA^*H$. Let $\lambda_k > \lambda_{k+1}$.

Then $\lambda_k$ is an eigenvalue of $(A,K)$.

**Theorem 3** (see [6]). Denote $B_k = \{ u \in L^2_k; \| u \| \leq 1 \}$

$$V_k = \{ v \in L^2_k; \| v \| \leq 1 \}.$$ 

Let there exist $v_0 \in V_k$ so that

$$\lambda_k - \lambda_{k+1} > \lambda_{k+1} \| v_0 \|^2 - (Av_0, v_0).$$

Then $\lambda_k$ is an eigenvalue of $(A,K)$.

2. **The stability of the eigenvalues.** The concepts of the deformation or of the invertible $f$-deformation are introduced in Definitions 1, 2 and some examples are given. The lower semicontinuity or continuity dependence of the eigenvalue on the deformation or on the invertible $f$-deformation of the cone $K$ are proved in Theorems 4, 5, respectively.

**Definition 1.** Let $\sigma_0 > 0$ and $T: <0, \sigma_0> \times KnS \rightarrow S$ be such that

1) $T_0(u) = u \quad \forall u \in KnS$

2) $T$ is a continuous mapping

3) $K_\sigma = \{ tT_\sigma(u); tu \in KnS \}$ is a convex closed cone for every $\sigma \in <0, \sigma_0>$. 

Then $T$ is called the (admissible) deformation of the cone $K$.

**Example 1.** Let $<a, b, c> > 0$ be such that $0 < a - c < a < b, b + <c < 1$. Denote by $W^{1,2}(0,1)$ the Sobolev space of absolutely continuous functions with square integrable derivatives and by $K$ a cone in $W^{1,2}(0,1)$ which does not depend on the set $M = \{ a-c, b+c \}$ (i.e. if two functions $u, v \in W^{1,2}(0,1)$ coincide on $<0,1>-M$, then $u \equiv v$ iff $v \equiv K$). ($K$ can be given as a set of functions satisfying some unilateral or bilateral boundary conditions.) Define:

$$K_\sigma = \{ u \in K; u(x) \equiv 0 \quad \forall x \in <a, b, c> \}$$

for $\sigma \in <0, \sigma_0>$. 

Then there exists $\sigma_0 \in <0, \sigma_0>$ and a deformation $T: <0, \sigma_0> \times KnS \rightarrow S$ transforming the cone $K_0$ onto the cone $K_\sigma$ for every $\sigma \in <0, \sigma_0>$.

**Remark.** The mapping $T$ from Example 1 can be defined as follows:

$$T_\sigma(u)(x) = \frac{u(x) - \sigma \Phi(x)}{\|u(x) - \sigma \Phi(x)\| W^{1,2}(0,1)} ,$$

- 543-
where \( \varphi \) is a continuous differentiable function on \(<0,1>\) such that 
\[
\varphi \wedge <a-\frac{\alpha}{2},b+\frac{\beta}{2}> = 1, \quad \varphi \wedge <0,1> = -(a-\infty ,b+\infty ) = 0, \quad \varphi (x) \in <0,1> \text{ for every } x \in <0,1>.
\]

In an analogous way the deformation can be defined for the cones:
\[
K_\varphi = \{ u \in \mathcal{K}; u(x) \in <0,1> \text{ a.e. on } <a+b>,b+\varphi > \}
\]
or (for a measurable set \( M \subset <a,b> \))
\[
K_\varphi = \{ u \in \mathcal{K}; \oint_M u(x+\varphi)dx \geq 0 \},
\]
etc.

Denote \( M_{k,\varphi} = f \in K_\varphi \cap S; \mathcal{F} \text{ compact, } \mathcal{F} \text{ is not contractible within } H-l_k \).

If \( M_{k,\varphi} \) is not empty, we define:
\[
c_{k,\varphi} = \sup_{\mathcal{F}_{k,\varphi}} \min (Au,u).
\]

Denote \( P_k: H \rightarrow l_k \) the orthogonal projection of the space \( H \) onto \( l_k \).

**Lemma 1** (see [4]). Let \( \lambda_1 > \lambda_{k+1} \). Then the inequality
\[
\|P_ku\|^2 \geq \frac{(Au,u) - \lambda_{k+1}}{\lambda_1 - \lambda_{k+1}}
\]
holds for every \( u \in S \).

**Theorem 4.** Let \( H, A, K, k \) satisfy the assumptions of Theorem 1. Suppose \( \sigma_0 > 0 \) and \( T: <0,\sigma_0> \times K \cap S \rightarrow S \) be a deformation.

Then there exists \( \sigma_1 \in (0,\sigma_0) \) such that for every \( \sigma \in (0,\sigma_1) \) \( c_{k,\sigma} \) is an eigenvalue of the inequality \((A,K_\sigma)\) and, moreover,
\[
\forall \sigma > 0 \exists \sigma_1 \in (0,\sigma_0) \quad \forall \sigma \in (0,\sigma_1): c_{k,\sigma} = c_k - \varepsilon
\]
holds.

**Proof.** Let \( \varepsilon > 0 \). Let \( F_1 \in M_k \) be such that
\[
\min_{u \in F_1} (Au,u) \geq c_k - \varphi,
\]
where \( \varphi = \min \{ \frac{\beta}{2}, \frac{\alpha}{4} \} \). According to Lemma 1 we have
\[
\|P_ku\|^2 \geq \frac{\lambda_1 - \lambda_{k+1}}{\lambda_1 - \lambda_{k+1}}.
\]
As \( F_1 \) is compact, we can find \( \sigma > 0 \) such that:
\[
\forall \sigma \in (0,\sigma) \quad \forall u \in F_1: \|T_{k,\sigma}(u)-u\| < \varepsilon,
\]

- 544 -
where \( q_{l} = \min \{ \frac{1}{2} \sqrt{4(A_{1} - A_{k+1})}, 2\} \). Since \( P_{k} \) is nonexpansive, we get
\[
\forall x \in F_{1} \quad \forall \sigma \in (0, \sigma_{0}): \| P_{k}^{T_{\sigma}}(u) \| \geq \| P_{k}^{T_{\sigma}}(u) - T_{\sigma}(u) - u \| > 0.
\]
Thus \( T_{\sigma}F_{1} \) is a homotopy of the set \( F_{1} \) onto the set \( T_{\sigma}(F_{1}) \) in the space \( H-L_{k} \). Hence, \( T_{\sigma}(F_{1}) \in \mathcal{M}_{k, \sigma} \). Further it holds
\[
\min_{u \in I_{\sigma}(F_{1})} (A_{1}, u) \leq \min_{u \in F_{1}} (A_{1}, u) - \max_{u \in F_{1}} \left( A_{1}(T_{\sigma}(u) - u), A_{1}(T_{\sigma}(u) - u) \right) -\]
\[
(7) \quad c_{k} + 2\| \tau \| c_{k} - 2\phi > 0,
\]
and according to the choice of \( \sigma \)
\[
\min_{u \in I_{\sigma}(F_{1})} (A_{1}, u) \leq c_{k} - \gamma < 0 \quad \forall \sigma \in \mathcal{M}_{k, \sigma}.
\]
Theorem 4 implies that \( c_{k, \sigma} \) is an eigenvalue of the inequality \( (A, K_{\sigma}) \).

From the same estimate (7) we get
\[
c_{k, \sigma} \geq \min_{u \in I_{\sigma}(F_{1})} (A_{1}, u) \geq c_{k} - 2\| \tau \| c_{k} - \varepsilon.
\]

**Remark.** Particularly, if \( A, K, k \) satisfy the assumptions of Theorem 2 or 3 and if \( T \) is a deformation of the cone \( K \), then \( c_{k, \sigma} \) is an eigenvalue of \( (A, K_{\sigma}) \) for all sufficiently small positive \( \sigma \), although the assumptions of Theorem 2 or 3 need not be satisfied for \( A, K_{\sigma}, k \).

**Definition 2.** Let \( \sigma_{0} > 0, T: \langle 0, \sigma_{0} \rangle \times K \times S \rightarrow S \) be a deformation of a cone \( K \). Moreover, let there exist a nondecreasing function \( f: \langle 0, \sigma_{0} \rangle \rightarrow \langle 0, a \rangle \), continuous in \( \sigma \), \( f(0) = 0 \) such that
\[
\forall \sigma_{1}, \sigma_{2} \in \langle 0, \sigma_{0} \rangle \quad \forall u \in K \times S
\]
\[
\| T_{\sigma_{1}}^{T_{\sigma_{2}}}(u) - T_{\sigma_{1}}^{T_{\sigma_{2}}}(u) \| \leq f(\| \sigma_{1} - \sigma_{2} \|).
\]

Then we call \( T \) the \( f \)-deformation of the cone \( K \).

If, moreover, \( T_{\sigma} \) is continuously invertible for every \( \sigma \in \langle 0, \sigma_{0} \rangle \), then we call \( T \) the invertible \( f \)-deformation of the cone \( K \).

**Example 2.** Let \( a \in (0, 1) \), \( k \in N, \sigma_{0} \in (0, \min \{ a, 1-a \}) \). Let \( K \subset W^{1,2}(\langle 0, 1 \rangle) \) be a cone independent of the set \( \langle a-\sigma_{0}, a+\sigma_{0} \rangle \) (see Example 1). Define:
\[
K = \{ u \in K; u(\alpha) \leq 0 \}
\]
\[
K_{\sigma} = \{ u \in K; u(\alpha+\sigma) \leq 0 \}
\]
\[
(\text{or } K_{\sigma} = \{ u \in K; u(\alpha) \leq 0 \} \quad \forall x \in \langle a, a+\sigma \rangle \}), \quad \sigma \in (0, \sigma_{0}).
\]
Then for any $\delta$ sufficiently small there is an invertible $f$-deformation of the cone $K$ onto $K_f$ with $f(\delta) = L\sqrt{\delta}$ (L being a positive constant).

**Remark.** The mapping $T$ from the example can be defined as follows:

$$T^f(u)(x) = \frac{u(x) + \varphi(x)(u(a) - u(a+\delta))}{\max_{x \in [a, a+\delta]} u(x)} \|w,2(0,1)\|^2,$$

or

$$T^f(u)(x) = \frac{\max_{x \in [a, a+\delta]} u(x)}{\max_{x \in [a, a+\delta]} u(x)} \|w,2(0,1)\|^2,$$

where $\varphi \in W^{1,2}(0,1)$ is any function such that

$$\varphi(x) \in (0,1) \quad \forall x \in (0,1).$$

The next example deals with deformations of halfspace in the Hilbert space $H$.

**Example 3.** Let $L$ be a positive constant, $g \in S$, $\delta_0 \in (0, \frac{1}{3L})$. Let $h \in H$ and $\|h\| \leq L$. Define

$$K = \{u \in S; (u,g) \geq 0 \},$$

$$K_f = \{u \in S; (u,g + \delta h) \geq 0 \} \quad \text{for} \quad \delta \in (0, \delta_0).$$

Then there exists $f$-deformation $T$ transforming the cone $K$ onto the cone $K_f$ for $f(\delta) = 9L\delta$ ($\delta \in (0, \delta_0)$).

**Remark.** One of the possibilities of defining $T_f$ is the following one:

$$T_f(u) = \frac{u - \delta(u,h)}{1 + \delta(g,h)} g.$$

For example for $H=W^{1,2}_0((0,1))$, $g_\delta(x) = \sinh|x-(a+\delta)| + k_1^\delta e^x + k_2^\delta e^{-x}$ (where $k_1^\delta, k_2^\delta$ are uniquely determined so that $g_\delta \in W^{1,2}_0((0,1)))$, $h_\delta = \frac{g_\delta - g_0}{\delta}$ we get the mapping of the cone $K = \{u \in W^{1,2}_0((0,1)); u(a) \geq 0 \}$ onto the cone $K_f = \{u \in W^{1,2}_0((0,1)); u(a+\delta) \geq 0 \}$.

**Theorem 5.** Let the assumption of Theorem 1 be satisfied and let $f$ be a function from Definition 2.

Then there exists $\delta_1 \in (0, \delta_0)$ such that for every $f$-deformation of the
cone K and for every \( \phi \in (0, \phi_1) \), \( c_k \phi \) is an eigenvalue of the inequality 
\((A, K^k)\) and

\[ c_k \phi \geq c_k - 2 \lambda_1 f(\phi). \]  

If \( T \) is, moreover, the invertible \( f \)-deformation, then

\[ |c_k \phi - c_k| \geq 2 \lambda_1 f(\phi). \]  

**Sketch of the proof.** Choose \( \phi_1 \in (0, \phi_0) \) such that \( f(\phi_1) \leq \frac{1}{4} \sqrt{\frac{\eta}{\lambda_1 - \lambda_{k+1}}} \). Let \( F_0 \) be the set given in the assumption of Theorem 1. According to Lemma 1 we have

\[
\forall u \in F_0, \forall \phi \in (0, \phi_1), \| P_k T_\phi (u) \| \geq \| P_k u \| - \| T_\phi (u) - u \| \geq \sqrt{\frac{\eta}{\lambda_1 - \lambda_{k+1}}} - f(\phi) \geq \frac{3}{4} \sqrt{\frac{\eta}{\lambda_1 - \lambda_{k+1}}}. 
\]

Thus \( T \) is a continuous homotopy of the set \( F_0 \) onto the set \( T_\phi (F_0) \) in the space \( H^1_{-L_k} \) and, hence, the set \( T_\phi (F_0) \) is compact and it is not contractible within \( H^1_{-L_k} \).

We have

\[
\min (A T_\phi (u), T_\phi (u)) \geq \min (A u, u) - \max \| A T_\phi (u), T_\phi (u) - u \| + \max (\| A T_\phi (u), T_\phi (u) - u \|, u \in K^k S)
\]

\[ + (A T_\phi (u) - u), u \| \geq \frac{\eta}{\lambda_{k+1}} - 2 \lambda_1 \max \| T_\phi (u) - u \| \geq \lambda_{k+1}^2 + \eta. \]

Hence, the assumptions of Theorem 1 are satisfied and \( c_k \phi \) is an eigenvalue of the inequality \((A, K^k)\) for \( \phi \in (0, \phi_1) \).

Let \( \epsilon \in (0, \frac{1}{16} \eta) \) be arbitrary, \( F_1 \in M_k \) such that \( \min (A u, u) \leq c_k - \epsilon \). Lemma 1 implies that

\[
\forall u \in F_1, \| P_k u \| \geq \frac{3}{4} \sqrt{\frac{\eta}{\lambda_1 - \lambda_{k+1}}}. 
\]

Similarly as before we can prove

\[ \| P_k T_\phi (u) \| \geq \frac{1}{2} \sqrt{\frac{\eta}{\lambda_1 - \lambda_{k+1}}}, \]

\[ T_\phi (u) \in M_k \phi \]

and

\[ c_k \phi \geq c_k - 2 \lambda_1 f(\phi). \]

Since \( \epsilon \in (0, \frac{1}{16} \eta) \) is arbitrary, (8) holds.

Let \( T \) be, moreover, invertible. Define the homotopy
\[ \Gamma: \langle 0, \sigma, \sigma \rangle \times K_{\sigma} \rightarrow S \rightarrow S \] for every \( \sigma \in (0, \sigma_1] \)

by

\[ \widetilde{\sigma}^\sigma (u) = \Gamma_\sigma^{-1}(\sigma, u). \]

We prove easily that \( \Gamma \) is an \( f \)-deformation of the cone \( K_\sigma \) onto the cone \( K_{\sigma}^\sigma \).

Let \( \varepsilon \in (0, \frac{\sigma}{4}] \), \( f_1 \in M_{K_\sigma} \) be such that

\[ \min_{\lambda \in \mathbb{R}} (\lambda f_1) \in C_{K_\sigma} \] and \( \min_{\lambda \in \mathbb{R}} (\lambda f_1) \leq -2 \lambda_1 f(\sigma) \).

Since \( \varepsilon \in (0, \frac{\sigma}{4}] \) is arbitrary, the inequality

\[ (10) \quad c_{k, \sigma} \leq -2 \lambda_1 f(\sigma) \]

holds.

From (8) and (10) we get (9).

3. Stability of eigenvectors. In this section we study the behaviour of the set of eigenvectors corresponding to eigenvalues on deformed cones. The results do not depend on the way how the eigenvalues were found and thus more general deformations of cones are considered.

Definition 3. Let \( K \) and \( \tilde{K} \) be two convex closed cones. The number

\[ \Phi(K, \tilde{K}) = \max \{ \sup_{u \in K, \sigma} \inf_{u \in \tilde{K}, \sigma} \| u - \tilde{u} \| ; \sup_{u \in K, \sigma} \inf_{u \in \tilde{K}, \sigma} \| u - \tilde{u} \| \} \]

will be called the distance between the cones \( K \) and \( \tilde{K} \).

Remark.

a) \( \Phi \) is a metric in the space of closed convex cones (it is the Hausdorff metric of their intersections with unit sphere).

b) If \( \Gamma \) is an \( f \)-deformation of the cone \( K_0 \), then for every \( \sigma \in (0, \sigma_1] \),

\[ \Phi(K_0, K_\sigma) \leq f(\sigma) \]

holds.

Theorem 6. Let \( K_{\sigma}, \sigma \in (0, \sigma_1] \) be a system of closed convex cones such that \( \Phi(K_0, K_{\sigma}) \rightarrow 0 \) for \( \sigma \rightarrow 0+ \). Let for every \( \sigma \in (0, \sigma_1] \), \( c_{\sigma} \) be an eigenvalue of the inequality \( (A, K_{\sigma}) \). Denote by \( V_{\sigma} \) the set of all eigenvectors of the inequality \( (A, K_{\sigma}) \) corresponding to the eigenvalue \( c_{\sigma} \). Let \( c_{\sigma} \) tend to a positive \( c_0 \) for \( \sigma \rightarrow 0+ \).

Then \( c_0 \) is an eigenvalue of the inequality \( (A, K_0) \), and

\[ \lim_{\sigma \rightarrow 0+} \sup_{u \in V_{\sigma}, V_0} \{ \text{dist}(u_{\sigma}, V_0) : u_{\sigma} \in V_{\sigma} \} = 0. \]

Lemma 2. Let the assumptions of Theorem 6 be satisfied and let

\[ \{ u_{k, \sigma} \}_{k=1}^{\infty} \]

be an arbitrary sequence such that \( u_{k, \sigma} \in V_{k, \sigma} \), \( u_{k, \sigma} \rightarrow u_0 \in B, \sigma \rightarrow 0+ \).
for $k \to \infty$. Then $u_0 \in V_0$ and $u_{\sigma_k} \to u_0$.

**Proof of Lemma 2.** (We write in this proof $c_k,u_k,v_k,K_k$ instead of $c_{\sigma_k}$, $u_{\sigma_k},v_{\sigma_k},K_{\sigma_k}$.) Let $v_0$ be an arbitrary element of $K_0$. According to the assumptions of Theorem 6 we find $\{v_k\}_{k=1}^{\infty}$ such that $v_k \in K_k$, $\|v_k\| = \|v_0\|$, $\|v_k - v_0\| \leq 2 \|v_0\|\phi(K_0,K_k)$. Let $M$ be a positive constant such that $|c_k| < M$ for every $k \in \mathbb{N}$. Since $c_k \to c_0$, $u_k \in S$, $u_0 \in B$,

$$
\lim_{k \to \infty} c_k(u_k,u_k) \phi(c_0(u_0,u_0)).
$$

It holds

$$
|c_k(u_k,Au_k,v_k-v_0)| \leq 2(M + \|A\|) \|v_0\|\phi(K_0,K_k) \to 0 \text{ for } k \to \infty,
$$

$$(c_k(u_k,Au_k,v_k-v_0) \to (c_0(u_0,Au_0,v_0),
$$

$$(Au_k,u_k) \to (Au_0,u_0).$$

Since $c_k,u_k$ are an eigenvalue and a corresponding eigenvector of the inequality $(A,K_k)$, we get

$$
0 \leq \lim_{k \to \infty} (c_k(u_k,Au_k,v_k-v_0)) = \lim_{k \to \infty} [(c_k(u_k,Au_k,v_k-v_0)) + (c_k(u_k,Au_k,v_k-v_0)) - (c_k(u_k,Au_k,u_k))] \leq (c_0(u_0,Au_0,v_0) + (Au_0,u_0) - c_0(u_0,u_0) =
$$

$$
= (c_0(u_0,Au_0,v_0-u_0).$$

For every $u_k$ we find $\tilde{u}_k \in K_k$ such that $\|u_k - \tilde{u}_k\| \leq 2 \phi(K_0,K_k)$. Then $\tilde{u}_k \to u_0$ and since the cone $K_0$ is weakly closed we get $u_0 \in K_0$.

If we put $v_0 = 2u_0$ in (13), we prove easily that $\|u_0\| \geq 1$. Hence, we get $u_0 \in V_0$ and Lemma 2 is proved.

**Proof of Theorem 6.** It follows easily from Lemma 2 that $V_0 \downarrow \emptyset$. Since any sequence $\{u_{\sigma_n}\}_{n=1}^{\infty}$ contains a subsequence $\{u_{\sigma_{k_n}}\}_{k=1}^{\infty}$ so that $u_{\sigma_{k_n}} \to u_0$ for $k \to \infty$, $u_0 \in B$, it follows from Lemma 2 for every $\{u_{\sigma_n}\}_{n=1}^{\infty}$

$$
\lim_{n \to \infty} \text{dist } (u_{\sigma_n},V_0) = 0
$$

and also for arbitrary $u_{\sigma} \in V_{\sigma}$, $\sigma > 0$

$$
\lim_{\sigma \to 0^+} \text{dist } (u_{\sigma},V_0) = 0.
$$

It means

$$
\lim_{\sigma \to 0^+} \sup \{\text{dist } (u_{\sigma},V_0); u_{\sigma} \in V_{\sigma} \} = 0.
$$
Remark. It follows from Lemma 2 that the set $V$ is weakly closed.

Remark. It can be shown that the following assertions do not hold:

a) $V u_0 \in V \exists \delta'_n \to 0 \exists \delta_n \in V \delta'_n \to u_0$

b) $\exists u_0 \in V \exists \delta'_0 > 0 \forall \delta \in (0, \delta'_0) \exists u_\delta \in V \delta : u \to u_0$ for $\delta \to 0$.

The counterexamples can be found in $R^3$.

References


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