Commentationes Mathematicae Universitatis Carolinae

Jaroslav Resler

Stability of eigenvalues and eigenvectors of variational inequalities

Commentationes Mathematicae Universitatis Carolinae, Vol. 29 (1988), No. 3, 541--550

Persistent URL: http://dml.cz/dmlcz/106668

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
29,3 (1988)

STABILITY OF EIGENVALUES AND EIGENVECTORS OF VARIATIONAL INEQUALITIES

Jaroslav RESLER

Abstract: There is studied the dependence of eigenvectors and eigenvalues of variational inequalities on continuous deformations of the cone. The lower semicontinuity of eigenvalues for general continuous deformations and continuity for continuous invertible deformations is proved.

Key words: Variational inequalities, eigenvalues, eigenvectors, sup min principle.

Classification: 49G05, 35J85, 35P30

---

0. Introduction. We shall study a variational inequality

$$(A,K) \qquad \begin{array}{l} u\in K\cap S \\ (\lambda u-Au,v-u)\geq 0 \qquad \forall\, v\in K, \end{array}$$

where K is a closed convex cone with vertex in the origin of a real separable Hilbert space H and A is a linear, completely continuous, symmetric and positive operator on H. We denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the scalar product and the norm in H, respectively, and $S=\{u\in H;\ \|u\|=1\}$. A real number λ is called an eigenvalue (and u the corresponding eigenvector) of the inequality (A,K) if there exists u satisfying (A,K).

It is well known (see [7]) that the inequality (A,K) has at least one but need not have more than one eigenvalue μ_1 which can be found as follows:

$$(2) \qquad \mu_1=\max_{u\in K\cap S}(Au,u)$$

Thus its dependence on the deformations of K can be studied directly.

The existence of the higher eigenvalues of (A,K) can be proved (under additional assumptions on A and K) by different methods (see [1],[2],[3],[4], [5],[6],[7]). We have chosen the variational method due to E. Miersemann ([3], [4],[5]) and we shall restrict ourselves to the existence and continuous dependence of the eigenvalues found by this approach. Thus Section 1 is devoted to the brief description of the sup min principle used for the definition of the k-th eigenvalue.

In Section 2, the concept of deformation of the cone K is introduced and there are proved theorems on lower semicontinuity and continuity of eigenvalues.

In Section 3 we deal with the behaviour of sets of eigenvectors corresponding to eigenvalues on deformed cones. These results are valid under more gneral conditions than in Section 2.

I am indebted to Jana Stará for many valuable advices during my work on the subject.

1. The sufficient conditions for the existence of higher eigenvalues. An abstract condition for the existence of higher eigenvalues is given in Theorem 1. Some assumptions on the cone and the operator A which guarantee that this abstract condition is satisfied, are given in Theorems 2 and 3.

Denote $\{\lambda_j\}_{j=1}^{\infty}$ the eigenvalues of A (numbered according to their magnitude $\lambda_1 \geq \lambda_2 \geq \ldots \geq 0$) and $\{u_j\}_{j=1}^{\infty}$ the corresponding eigenvectors. Let L_k be the span of the first k eigenvectors of the operator A and $\mathcal{M}_k = \{F \subset K \cap S;$ F is compact, F is not contractible within $H - L_k^{\perp}\}$. The class $\mathcal{M}_k$ may be empty; if it is not the case put

$$c_k = \sup_{F \in \mathcal{M}_k} \min_{u \in F} (Au,u). \tag{3}$$

Theorem 1 (see [4]). Let $\mathcal{M}_k$ contain a set F_0 such that

$$\min_{u \in F_0} (Au,u) \geq \lambda_{k+1} + \eta \tag{4}$$

for a positive constant η .

Then c_k is an eigenvalue of the variational inequality (A,K).

Remark. Clearly, $c_k > \lambda_{k+1}$. If, moreover,

$$\mathrm{Ker}(\lambda_k I - A) \not\subset K,$$

then $c_k < \lambda_k$. Thus the eigenvalue c_k of (A,K) is not an eigenvalue of the operator A.

The next two theorems give conditions guaranteeing that the assumption of Theorem 1 holds.

Theorem 2 (see [5]). Let there exist a closed linear subspace $\tilde{H}$ of the space H so that $\tilde{H} \subset K \subset H$. Let $\tilde{P}: H \to \tilde{H}$ be the orthogonal projection onto $\tilde{H}$.

Denote $\tilde{\lambda}_1 \geq \ldots \geq \tilde{\lambda}_n \geq \ldots \geq 0$ the eigenvalues of the operator $\tilde{A}=\tilde{P}A \wedge \tilde{H}$. Let $\tilde{\lambda}_k > \lambda_{k+1}$.

Then c_k is an eigenvalue of (A,K).

Theorem 3 (see [6]). Denote $B_k = \{u \in L_k; \|u\| \leq 1\}$

$$V_k = \{v \in L_k^{\perp}; u+v \in K \quad \forall u \in B_k\}.$$

Let there exist $v_0 \in V_k$ so that

$$\lambda_k - \lambda_{k+1} > \lambda_{k+1} \|v_0\|^2 - (Av_0, v_0).$$

Then c_k is an eigenvalue of (A,K).

2. The stability of the eigenvalues. The concepts of the deformation or of the invertible f-deformation are introduced in Definitions 1, 2 and some examples are given. The lower semicontinuity or continuity dependence of the eigenvalue on the deformation or on the invertible f-deformation of the cone K are proved in Theorems 4, 5, respectively.

Definition 1. Let $\delta_0 > 0$ and $T: \langle 0, \delta_0 \rangle \times K \cap S \longrightarrow S$ be such that

1) $T_0(u)=u \quad \forall u \in K \cap S$
2) T is a continuous mapping
3) $K_\delta = \{tT_\delta(u); t \geq 0, \ u \in K \cap S\}$ is a convex closed cone for every $\delta \in \langle 0, \delta_0 \rangle$.

Then T is called the (admissible) deformation of the cone K.

Example 1. Let $\alpha, a, b > 0$ be such that $0 < a - \alpha$, $a < b$, $b + \alpha < 1$. Denote by $W^{1,2}(\langle 0,1 \rangle)$ the Sobolev space of absolutely continuous functions with square integrable derivatives and by $\tilde{K}$ a cone in $W^{1,2}(\langle 0,1 \rangle)$ which does not depend on the set $M = \langle a-\alpha, b+\alpha \rangle$ (i.e. if two functions $u,v \in W^{1,2}(\langle 0,1 \rangle)$ coincide on $\langle 0,1 \rangle - M$, then $u \in \tilde{K}$ iff $v \in \tilde{K}$). ($\tilde{K}$ can be given as a set of functions satisfying some unilateral or bilateral boundary conditions.) Define:

$$K_\delta = \{u \in \tilde{K}; u(x) \geq 0 \quad \forall x \in \langle a+\delta, b+\delta \rangle$$

for $\delta \in \langle 0, \frac{\alpha}{2} \rangle$.

Then there exists $\delta_0 \in (0, \frac{\alpha}{2} \rangle$ and a deformation $T: \langle 0, \delta_0 \rangle \times K \cap S \longrightarrow S$ transforming the cone K_0 onto the cone K_δ for every $\delta \in \langle 0, \delta_0 \rangle$.

Remark. The mapping T from Example 1 can be defined as follows:

$$T_\delta(u)(x) = \frac{u(x - \delta\varphi(x))}{\|u(x - \delta\varphi(x))\|_{W^{1,2}(\langle 0,1 \rangle)}},$$

where φ is a continuous differentiable function on $\langle 0,1\rangle$ such that $\varphi \wedge \langle a-\frac{\alpha}{2}, b+\frac{\alpha}{2}\rangle \equiv 1$, $\varphi \wedge \langle 0,1\rangle - (a-\alpha, b+\alpha) \equiv 0$, $\varphi(x) \in \langle 0,1\rangle$ for every $x \in \langle 0,1\rangle$.

In an analogous way the deformation can be defined for the cones:

$$K_\delta = \{u \in \tilde{K}; u'(x) \geq 0 \text{ a.e. on } \langle a+\delta, b+\delta\rangle\}$$

or (for a measurable set $M \subset \langle a,b\rangle$)

$$K_\delta = \{u \in \tilde{K};\ \int_M u(x+\delta)dx \geq 0\},$$

etc.

Denote $\mathcal{M}_{k\delta} = \{F \subset K_\delta \cap S; F \text{ compact, } F \text{ is not contractible within } H-L_k^{\perp}\}$.

If $\mathcal{M}_{k\delta}$ is not empty, we define:

$$c_{k\delta} = \sup_{F \in \mathcal{M}_{k\delta}} \min_{u \in F} (Au,u).$$

Denote $P_k: H \rightarrow L_k$ the orthogonal projection of the space H onto L_k.

Lemma 1 (see [4]). Let $\lambda_1 > \lambda_{k+1}$. Then the inequality

$$\|P_k u\|^2 \geq \frac{(Au,u) - \lambda_{k+1}}{\lambda_1 - \lambda_{k+1}} \tag{6}$$

holds for every $u \in S$.

Theorem 4. Let H, A, K, k satisfy the assumptions of Theorem 1. Suppose $\delta_0 > 0$ and $T: \langle 0, \delta_0\rangle \times K \cap S \rightarrow S$ be a deformation.

Then there exists $\delta_1 \in (0, \delta_0\rangle$ such that for every $\delta \in (0, \delta_1\rangle$ $c_{k\delta}$ is an eigenvalue of the inequality (A, K_δ) and, moreover,

$$\forall \varepsilon > 0\ \exists \tilde{\delta} \in (0, \delta_1\rangle \quad \forall \delta \in (0, \tilde{\delta}\rangle : c_{k\delta} \geq c_k - \varepsilon$$

holds.

Proof. Let $\varepsilon > 0$. Let $F_1 \in \mathcal{M}_k$ be such that

$$\min_{u \in F_1} (Au,u) \geq c_k - \rho,$$

where $\rho = \min\{\frac{\varepsilon}{2}, \frac{\eta}{4}\}$. According to Lemma 1 we have

$$\|P_k u\| \geq \sqrt{\frac{3\eta}{4(\lambda_1 - \lambda_{k+1})}}.$$

As F_1 is compact, we can find $\tilde{\delta} > 0$ such that:

$$\forall \delta \in (0, \tilde{\delta}\rangle \quad \forall u \in F_1: \|T_\delta(u) - u\| < \tilde{\rho},$$

where $\tilde{\varphi} = \min\left\{\frac{1}{2}\sqrt{\frac{3\eta}{4(\lambda_1-\lambda_{k+1})}}, \frac{\varphi}{2\lambda_1}\right\}$. Since P_k is nonexpansive, we get

$$\forall x \in F_1 \quad \forall \delta \in (0,\tilde{\delta}\rangle : \|P_k T_\delta(u)\| \geq \|P_k u\| - \|T_\delta(u)-u\| > \tilde{\varphi} .$$

Thus $T/\!\!\!\wedge F_1$ is a homotopy of the set F_1 onto the set $T_\delta(F_1)$ in the space $H-L_k^{\perp}$. Hence, $T_\delta(F_1) \in \mathfrak{M}_{k\delta}$. Further it holds

$$\min_{\tilde{u}\in T_\delta(F_1)} (A\tilde{u},\tilde{u}) \geq \min_{u\in F_1}(Au,u) - \max_{u\in F_1}|(A(T_\delta(u)-u),u)-(AT_\delta(u),T_\delta(u)-u)| >$$

$$> c_k - \varphi - 2\lambda_1\tilde{\varphi} \geq c_k - 2\varphi \;, \tag{7}$$

and according to the choice of φ

$$\min_{\tilde{u}\in T_\delta(F_1)} (A\tilde{u},\tilde{u}) > c_k - \eta \geq \lambda_{k+1}.$$

Theorem 4 implies that $c_{k\delta}$ is an eigenvalue of the inequality (A,K_δ).

From the same estimate (7) we get

$$c_{k\delta} \geq \min_{u\in T_\delta(F_1)}(Au,u) \geq c_k - 2\varphi \geq c_k - \varepsilon .$$

Remark. Particularly, if A, K, k satisfy the assumptions of Theorem 2 or 3 and if T is a deformation of the cone K, then $c_{k\delta}$ is an eigenvalue of (A,K_δ) for all sufficiently small positive δ, although the assumptions of Theorem 2 or 3 need not be satisfied for A, K_δ, k.

Definition 2. Let $\delta_0 > 0$, $T:\langle 0,\delta_0\rangle \times K\cap S \to S$ be a deformation of a cone K. Moreover, let there exist a nondecreasing function $f:\langle 0,\delta_0\rangle \to \langle 0,\infty)$, continuous in zero, $F(0)=0$ such that

$$\forall \delta_1,\delta_2 \in \langle 0,\delta_0\rangle \quad \forall u \in K\cap S :$$
$$\|T_{\delta_1}(u) - T_{\delta_2}(u)\| \leq f(|\delta_1-\delta_2|).$$

Then we call T the f-deformation of the cone K.

If, moreover, T_δ is continuously invertible for every $\delta \in \langle 0,\delta_0\rangle$, then we call T the invertible f-deformation of the cone K.

Example 2. Let $a \in (0,1)$, $k \in \mathbf{N}$, $\delta_0 \in (0,\min\{a,1-a\}\rangle$. Let $\tilde{K} \subset W^{k,2}(\langle 0,1\rangle)$ be a cone indpendent of the set $\langle a-\delta_0, a+\delta_0\rangle$ (see Example 1). Define:

$$K = \{u \in \tilde{K}; u(a) \geq 0\}$$
$$K_\delta = \{u \in \tilde{K}; u(a+\delta) \geq 0\}$$
$$(\text{or } K_\delta = \{u \in \tilde{K}; u(x) \geq 0 \quad \forall x \in \langle a,a+\delta\rangle\}),\; \delta \in (0,\delta_0\rangle .$$

Then for any δ sufficiently small there is an invertible f-deformation of the cone K onto K_δ with $f(\delta)=L\sqrt{\delta}$ (L being a positive constant).

Remark. The mapping T from the example can be defined as follows:

$$T_\delta(u)x)=\frac{u(x)+\varphi(x)(u(a)-u(a+\delta))}{\|u(x)+\varphi(x)(u(a)-u(a+\delta))\|_{W^{k,2}(\langle 0,1\rangle)}}$$

$$\left(\text{or}\quad T_\delta(u)(x)=\frac{u(x)+\varphi(x)(u(a)-\min\limits_{t\in\langle a,a+\delta\rangle} u(t))}{\|u(x)+\varphi(x)(u(a)-\min\limits_{t\in\langle a,a+\delta\rangle} u(t))\|_{W^{k,2}(\langle 0,1\rangle)}}\right),$$

where $\varphi\in W^{k,2}(\langle 0,1\rangle)$ is any function such that

$$\varphi/\langle a-\frac{\delta_0}{2}\ \ a+\frac{\delta_0}{2}\rangle\equiv 1,\quad \varphi/\langle 0,1\rangle-(a-\delta_0,a+\delta_0)\equiv 0$$

$$\varphi(x)\in\langle 0,1\rangle\quad\forall x\in\langle 0,1\rangle .$$

The next example deals with deformations of halfspace in the Hilbert space H.

Example 3. Let L be a positive constant, $g\in S$, $\delta_0\in(0,\frac{1}{3L}\rangle$. Let $h\in H$ and $\|h\|\leqq L$. Define

$$K=\{u\in S;(u,g)\geqq 0\}$$
$$K_\delta=\{u\in S;(u,g+\delta h)\geqq 0\}\ \text{for}\ \delta\in\langle 0,\delta_0\rangle .$$

Then there exists f-deformation T transforming the cone K onto the cone K_δ for $f(\delta)=9L\delta$ $(\delta\in\langle 0,\delta_0\rangle)$.

Remark. One of the possibilities of defining T_δ is the following one:

$$T_\delta(u)=\frac{u-\frac{\delta\,(u,h)}{1+\delta(g,h)}\,g}{\|u-\frac{\delta\,(u,h)}{1+\delta(g,h)}\,g\|}.$$

For example for $H=W_0^{1,2}(\langle 0,1\rangle)$, $g_\delta(x)=\sinh|x-(a+\delta)|+k_{1\delta}e^x+k_{2\delta}e^{-x}$ (where $k_{1\delta}$, $k_{2\delta}$ are uniquely determined so that $g_\delta\in W_0^{1,2}(\langle 0,1\rangle)$), $h_\delta=\frac{g_\delta-g_0}{\delta}$ we get the mapping of the cone $K=\{u\in W_0^{1,2}(\langle 0,1\rangle);u(a)\geqq 0\}$ onto the cone $K_\delta=$ $=\{u\in W_0^{1,2}(\langle 0,1\rangle);\ u(a+\delta)\geqq 0\}$.

Theorem 5. Let the assumption of Theorem 1 be satisfied and let f be a function from Definition 2.

Then there exists $\delta_1\in(0,\delta_0\rangle$ such that for every f-deformation of the cone K and for every $\delta \in (0, \delta_1\rangle$, $c_{k\delta}$ is an eigenvalue of the inequality (A, K_δ) and

(8) $$c_{k\delta} \geq c_k - 2\lambda_1 f(\delta).$$

If T is, moreover, the invertible f-deformation, then

(9) $$|c_{k\delta} - c_k| \leq 2\lambda_1 f(\delta).$$

Sketch of the proof. Choose $\delta_1 \in (0, \delta_0)$ such that $f(\delta_1) \leq$ $\leq \min\{\frac{1}{4}\sqrt{\frac{\eta}{\lambda_1 - \lambda_{k+1}}}, \frac{\eta}{4\lambda_1}\}$. Let F_0 be the set given in the assumption of Theorem 1. According to Lemma 1 we have

$$\forall u \in F_0 \quad \forall \delta \in (0, \delta_1\rangle : \|P_k T_\delta(u)\| \geq \|P_k u\| - \|T_\delta(u) - u\| \geq \sqrt{\frac{\eta}{\lambda_1 - \lambda_{k+1}}} -$$
$$- f(\delta) \geq \frac{3}{4}\sqrt{\frac{\eta}{\lambda_1 - \lambda_{k+1}}}$$

. Thus T is a continuous homotopy of the set F_0 onto the set $T_\delta(F_0)$ in the space $H - L_k^\perp$ and, hence, the set $T_\delta(F_0)$ is compact and it is not contractible within $H - L_k^\perp$.

We have

$$\min_{u \in F_0} (AT_\delta(u), T_\delta(u)) \geq \min_{u \in F_0} (Au, u) - \max_{u \in K \cap S} |(AT_\delta(u), T_\delta(u) - u) +$$
$$+ (AT_\delta(u) - u), u)| \geq \lambda_{k+1} + \eta - 2\lambda_1 \max_{u \in K \cap S} \|T_\delta(u) - u\| \geq \lambda_{k+1} + \frac{\eta}{2}.$$

Hence, the assumptions of Theorem 1 are satisfied and $c_{k\delta}$ is an eigenvalue of the inequality (A, K_δ) for $\delta \in (0, \delta_1)$.

Let $\varepsilon \in (0, \frac{7}{16}\eta\rangle$ be arbitrary, $F_1 \in \mathcal{M}_k$ such that $\min_{u \in F_1} (Au, u) \geq c_k - \varepsilon$. Lemma 1 implies that

$$\forall u \in F_1 : \|P_k u\| \geq \frac{3}{4}\sqrt{\frac{\eta}{\lambda_1 - \lambda_{k+1}}}.$$

Similarly as before we can prove

$$\|P_k T_\delta(u)\| \geq \frac{1}{2}\sqrt{\frac{\eta}{\lambda_1 - \lambda_{k+1}}},$$

$$T_\delta(u) \in \mathcal{M}_{k\delta}$$

and

$$c_{k\delta} \geq c_k - \varepsilon - 2\lambda_1 f(\delta).$$

Since $\varepsilon \in (0, \frac{7}{16}\eta\rangle$ is arbitrary, (8) holds.

Let T be, moreover, invertible. Define the homotopy

$\tilde{T}: \langle 0,\tilde{\sigma}\rangle \times K_{\tilde{\sigma}} \cap S \to S$ for every $\tilde{\sigma} \in (0,\sigma_1\rangle$

by

$$\tilde{T}_{\sigma}(\tilde{u}) = T_{\tilde{\sigma}-\sigma} T_{\tilde{\sigma}}^{-1}(\tilde{u}).$$

We prove easily that $\tilde{T}$ is an f-deformation of the cone $K_{\tilde{\sigma}}$ onto the cone $K_{\tilde{\sigma}-\sigma}$.

Let $\varepsilon \in (0, \frac{\eta}{4}\rangle$, $F_1 \in \mathcal{M}_{k\tilde{\sigma}}$ be such that $\min_{\tilde{u}\in F_1} (A\tilde{u},\tilde{u}) \geqq c_{k\tilde{\sigma}} - \varepsilon$. In a similar way as before we prove that $\tilde{T}_{\tilde{\sigma}}(\tilde{F}_1) \in \mathcal{M}_k$ and $c_k \geqq c_{k\tilde{\sigma}} - \varepsilon - 2\lambda_1 f(\tilde{\sigma})$. Since $\varepsilon \in (0, \frac{\eta}{4}\rangle$ is arbitrary, the inequality

(10) $\quad c_k \geqq c_{k\tilde{\sigma}} - 2\lambda_1 f(\tilde{\sigma})$

holds.

From (8) and (10) we get (9).

3. Stability of eigenvectors. In this section we study the behaviour of the set of eigenvectors corresponding to eigenvalues on deformed cones. The results do not depend on the way how the eigenvalues were found and thus more general deformations of cones are considered.

Definition 3. Let K and $\tilde{K}$ be two convex closed cones. The number

$$\Phi(K,\tilde{K}) = \max \{ \sup_{u\in K\cap S} \inf_{\tilde{u}\in \tilde{K}\cap S} \|u-\tilde{u}\| ; \sup_{\tilde{u}\in \tilde{K}\cap S} \inf_{u\in K\cap S} \|u-\tilde{u}\| \}$$

will be called the distance between the cones K and $\tilde{K}$.

Remark.

a) Φ is a metric in the space of closed convex cones (it is the Hausdorff metric of their intersections with unit sphere).

b) If T is an f-deformation of the cone K_0, then for every $\sigma \in \langle 0,\sigma_0\rangle$, $\Phi(K_0,K_{\sigma}) \leqq f(\sigma)$ holds.

Theorem 6. Let K_{σ}, $\sigma \in \langle 0,\sigma_0\rangle$ be a system of closed convex cones such that $\Phi(K_0,K_{\sigma}) \to 0$ for $\sigma \to 0+$. Let for every $\sigma \in (0,\sigma_0\rangle$, c_{σ} be an eigenvalue of the inequality (A,K_{σ}). Denote by V_{σ} the set of all eigenvectors of the inequality (A,K_{σ}) corresponding to the eigenvalue c_{σ}. Let c_{σ} tend to a positive c_0 for $\sigma \to 0+$.

Then c_0 is an eigenvalue of the inequality (A,K_0), and

$$\lim_{\sigma\to 0+} \sup \{\operatorname{dist}(u_{\sigma},V_0); u_{\sigma} \in V_{\sigma}\} = 0.$$

Lemma 2. Let the assumptions of Theorem 6 be satisfied and let $\{u_{\sigma_k}\}_{k=1}^{\infty}$ be an arbitrary sequence such that $u_{\sigma_k} \in V_{\sigma_k}$, $u_{\sigma_k} \rightharpoonup u_0 \in B$, $\sigma_k \to 0+$.

for $k \to \infty$. Then $u_0 \in V_0$ and $u_{\delta_k} \to u_0$.

Proof of Lemma 2. (We write in this proof c_k, u_k, v_k, K_k instead of c_{δ_k}, $u_{\delta_k}, v_{\delta_k}, K_{\delta_k}$.) Let v_0 be an arbitrary element of K_0. According to the assumptions of Theorem 6 we find $\{v_k\}_{k=1}^{\infty}$ such that $v_k \in K_k$, $\|v_k\| = \|v_0\|$, $\|v_k - v_0\| \leq 2\|v_0\| \Phi(K_0, K_k)$. Let M be a positive constant such that $|c_k| < M$ for every $k \in \mathbf{N}$. Since $c_k \to c_0$, $u_k \in S$, $u_0 \in B$,

$$\lim_{k\to\infty} c_k(u_k,u_k) \geq c_0(u_0,u_0).$$

It holds

$$|(c_k u_k - Au_k, v_k - v_0)| \leq 2(M + \|A\|)\|v_0\| \Phi(K_0,K_k) \to 0 \text{ for } k \to \infty,$$

$$(c_k u_k - Au_k, v_0) \to (c_0 u_0 - Au_0, v_0),$$

$$(Au_k, u_k) \to (Au_0, u_0).$$

Since c_k, u_k are an eigenvalue and a corresponding eigenvector of the inequality (A, K_k), we get

$$\begin{aligned} 0 &\leq \lim_{k\to\infty} (c_k u_k - Au_k, v_k - u_k) = \lim_{k\to\infty} [(c_k u_k - Au_k, v_0) + (c_k u_k - Au_k, v_k - v_0) - \\ &- (c_k u_k - Au_k, u_k)] \leq (c_0 u_0 - Au_0 v_0) + (Au_0, u_0) - c_0(u_0, u_0) = \\ &= (c_0 u_0 - Au_0, v_0 - u_0). \end{aligned} \tag{13}$$

For every u_k we find $\tilde{u}_k \in K_0$ such that $\|\tilde{u}_k - u_k\| \leq 2\Phi(K_0,K_k)$. Then $\tilde{u}_k \rightharpoonup u_0$ and since the cone K_0 is weakly closed we get $u_0 \in K_0$.

If we put $v_0 = 2u_0$ in (13), we prove easily that $\|u_0\| \geq 1$. Hence, we get $u_0 \in V_0$ and Lemma 2 is proved.

Proof of Theorem 6. It follows easily from Lemma 2 that $V_0 \neq \emptyset$. Since any sequence $\{u_{\delta_n}\}_{n=1}^{\infty}$ contains a subsequence $\{u_{\delta_{n_k}}\}_{k=1}^{\infty}$ so that $u_{\delta_{n_k}} \to u_0$ for $k \to \infty$, $u_0 \in B$, it follows from Lemma 2 for every $\{u_{\delta_n}\}_{n=1}^{\infty}$

$$\lim_{n\to\infty} \operatorname{dist}(u_{\delta_n}, V_0) = 0$$

and also for arbitrary $u_\delta \in V_\delta$, $\delta > 0$

$$\lim_{\delta\to 0^+} \operatorname{dist}(u_\delta, V_0) = 0.$$

It means

$$\lim_{\delta\to 0^+} \sup\{\operatorname{dist}(u_\delta, V_0); u_\delta \in V_\delta\} = 0.$$

Remark. It follows from Lemma 2 that the set V_0 is weakly closed.

Remark. It can be shown that the following assertions do not hold:

a) $\forall u_0 \in V_0 \exists \delta_n \to 0+ \exists u_{\delta_n} \in V_{\delta_n} : u_{\delta_n} \to u_0$

b) $\exists u_0 \in V_0 \exists \delta_0 > 0 \ \forall \delta \in (0, \delta_0 \rangle \exists u_\delta \in V_\delta : u \to u_0$ for $\delta \to 0+$.

The counterexamples can be found in $\mathbf{R}^3$.

References

[1] M. KUČERA: A new method for obtaining eigenvalues of variational inequalities based on bifurcation theory, Čas. pěst. mat. 104(1979), 389-411.

[2] M. KUČERA: Bifurcation points of variational inequalities, Czechoslovak Math. J. 32(107)(1982), 208-226.

[3] M. KUČERA, J. NEČAS, and J. SOUČEK: The eigenvalue problem for variational inequalities and a new version of the Ljusternik-Schnirelmann theory, Nonlinear Analysis, A Collection of Papers in Honor of Erich H. Rothe, Academic Press, New York, San Francisco, London, 1978.

[4] E. MIERSEMANN: Über höhere Verzweigungspunkte nichtlinearer Variationsungleichungem Math. Nachr. 85(1978), 195-213.

[5] E. MIERSEMANN: Höhere Eigenwerte von Variationsungleichungen, Beiträge zur Analysis 17(1981), 65-68.

[6] E. MIERSEMANN: On higher eigenvalues of variational inequalities, Comment. Math. Univ. Carolinae 24(1983), 657-665.

[7] P. QUITNER: A note to E. Miesermann's papers on higher eigenvalues of variational inequalities, Comment. Math. Univ. Carolinae 26 (1985), 665-674.

[8] P. QUITNER: Spectral analysis of variational inequalities, To appear in Comment. Math. Univ. Carolinae.

[9] J. STARÁ, O. JOHN: Funkcionální analýza - nelineární úlohy , SPN,Praha, 1986.

SVÚSS, 190 11 Praha - Běchovice, Czechoslovakia

(Oblatum 17.3. 1988)