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BADE'S THEOREM ON THE UNIFORMLY CLOSED ALGEBRA GENERATED BY A BOOLEAN ALGEBRA *

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Abstract: We give a new proof of the classical result of W.Bade stating that the uniformly closed algebra generated by a complete Boolean algebra of projections in a Banach space coincides with the weak operator closed algebra that it generates. The method of proof does not rely on the use of Gelfand theory for commutative Banach algebras or the existence of Bade functionals.

Key words: Boolean algebra of projections, spectral measure.

Classification: 47D30

The uniformly closed algebra generated by a complete Boolean algebra of projection operators M in a Banach space X is a full algebra equivalent to the algebra of continuous functions on its maximal ideal space. Furthermore, for each $x \in X$ there is a continuous linear functional x' with the properties

(i) $\langle Bx,x'\rangle \geq 0$, for each $B \in M$, and

(ii) if $\langle Bx,x'\rangle = 0$ for some $B \in M$, then $Bx = 0$.

An element x' satisfying (i) and (ii) is often called a Bade functional for x with respect to M. These two facts are the essential ingredients in the proof of the reflexivity theorem of W.Bade [1; Theorem 4.3] which states that the uniformly closed operator algebra generated by M consists of all bounded linear operators in X which leave invariant each closed subspace of X which is invariant for each member of M. The classical result of W.Bade [1; Theorem 4.5] stating that the closed algebra generated by M with respect to the weak (or, equivalently, strong) operator topology coincides with the uniformly closed algebra generated by M follows immediately. So, the idea is to exhibit a criterion describing the elements of the uniformly closed algebra generated by M (a Banach algebra) and then check that elements of the weakly (or strongly) closed algebra generated by M (a locally convex algebra) satisfy this criterion.

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The aim of this note is to show that this strategy can be "reversed", thereby providing an alternative proof of Bade's result. Namely, we give a description of the elements of the strongly closed algebra $\langle M\rangle_s$ generated by M from which it follows that $\langle M\rangle_s$ is a part of the uniformly closed algebra. The techniques are based on the theory of integration with respect to spectral measures, recently developed and successfully employed in [2;3;4] for the case of non-normable spaces. In the Banach space setting such techniques make no use of Banach algebra theory and Bade functionals play no role.

Let X be a Banach space with continuous dual space X', and let L(X) denote the space of all continuous linear operators of X into itself. Then $L_s(X)$ denotes the space L(X) equipped with the strong operator topology (i.e. the topology of pointwise convergence on X) and $L_u(X)$ denotes the space L(X) equipped with the uniform operator topology. We remark that $L_s(X)$ is quasicomplete. The definition of a Boolean algebra of projections M is standard; see [1] or [5], for example. It is assumed that the unit element of M is the identity operator I in X. The closed algebras generated by M in $L_s(X)$ and $L_u(X)$ are denoted by $\langle M\rangle_s$ and $\langle M\rangle_u$, respectively. For the notions of σ-completeness and completeness of Boolean algebras of projections in Banach spaces we refer to [1]. Such Boolean algebras are always uniformly bounded in $L_u(X)$, [1; Theorem 2.2]. We remark that $\langle M\rangle_s$ coincides with the closed algebra generated by M with respect to the weak operator topology in L(X).

Theorem (W.Bade). *Let M be a complete Boolean algebra of projections in a Banach space X. Then $\langle M\rangle_u = \langle M\rangle_s$.*

The inclusion $\langle M\rangle_u \subseteq \langle M\rangle_s$ is clear. It is the reverse containment which is not so obvious. The basic idea of the proof goes as follows:

(i) Realize M as the range of a suitable spectral measure P.

(ii) Using the results of [2] characterize $\langle M\rangle_s$ as an L^1-space with respect to P.

(iii) Observe that the only P-integrable functions are the P-essentially bounded functions.

(iv) Approximate (uniformly) by P-simple functions.

We now formulate these steps more precisely.

A spectral measure is a σ-additive map $P : \Sigma \to L_s(X)$ satisfying $P(\Omega) = I$ and $P(E \cap F) = P(E)P(F)$, for every $E,F \in \Sigma$, where Σ is a σ-algebra of subsets of some set Ω. It is known that the range $R(P) = \{P(E);\ E \in \Sigma\}$ of P is a σ-complete Boolean algebra. Furthermore, $R(P)$ is a complete Boolean algebra if and only if it is a closed subset of $L_s(X)$, [2; p.148]. Conversely, any complete Boolean algebra is the range of a spectral measure defined on the Borel sets of its

Stone space; see [1; p.349], for example. So, formally (i) means that there exists a spectral measure $P : \Sigma \to L_s(X)$, with $R(P)$ a closed subset of $L_s(X)$, such that $M = R(P)$.

Let $P : \Sigma \to L_s(X)$ be any spectral measure. The σ-additivity of P is equivalent to the σ-additivity of each $\mathbb{C}$-valued set function

$$\langle Px,x'\rangle : E \to \langle P(E)x,x'\rangle , \qquad E \in \Sigma,$$

for each $x \in X$ and $x' \in X'$. A Σ-measurable function f on Ω is said to be P-integrable if it is $\langle Px,x'\rangle$-integrable, for every $x \in X$ and $x' \in X'$, and there exists an element $P(f) = \int_\Omega f dP$ in $L(X)$ satisfying

(1) $$\langle P(f)x,x'\rangle = \int_\Omega f d\langle Px,x'\rangle,$$

for every $x \in X$ and $x' \in X'$. This definition agrees with that for more general vector measures (in the sense of [6]); see [2; Proposition 1.2]. In this case

$$\int_E f dP = P(f)P(E) = P(E)P(f), \qquad E \in \Sigma.$$

A P-integrable function f is called P-null if $P(f) = 0$. A typical seminorm generating the topology of $L_s(X)$ is of the form

$$q_x : T \to \|Tx\| , \qquad T \in L(X),$$

for some $x \in X$. This seminorm then induces a seminorm $q_x(P)$ in the space $L(P)$ of all P-integrable functions via the formula

(2) $$q_x(P)(f) = \sup \{\| (\int_E f dP)x\| ; E \in \Sigma\}, \qquad f \in L(P).$$

The family of all such seminorms (2), as x varies through X, generates a locally convex topology in $L(P)$. The quotient space of $L(P)$ with respect to the space of P-null functions is denoted by $L^1(P)$. Then $L^1(P)$ is a (Hausdorff) commutative, locally convex algebra with unit (the constant function 1) and $L^1(P)$ is complete if and only if $R(P)$ is a closed subset of $L_s(X)$, [2; Proposition 1.4]. If $R(P)$ is a closed set in $L_s(X)$, then the integration mapping

(3) $$\Phi_P : f \to P(f) = \int_\Omega f dP, \qquad f \in L^1(P),$$

is a bicontinuous isomorphism of the (complete) locally convex algebra $L^1(P)$ onto the operator algebra $\langle R(P)\rangle_s$, [2; Proposition 1.5]. Accordingly, $\langle M\rangle_s$ is isomorphic to $L^1(P)$ for any spectral measure P such that $M = R(P)$.

The notion of P-essentially bounded functions proceeds as for (finite) numerical-valued measures; see [5; Chapter XVII, §2], for example. In particular, such functions are necessarily P-integrable. Indeed, if $f = \Sigma_{j=1}^n \alpha_j \chi_{E(j)}$ is a Σ-simple function, then it is clear that f is P-integrable and

$$P(f) = \int_\Omega f dP = \Sigma_{j=1}^n \alpha_j P(E(j)).$$

It is then possible to define, for each P-essentially bounded function f on Ω, an operator $P(f) \in L(X)$ satisfying (1) by continuous extension from the Σ-simple functions; for the details we refer to [5; Chapter XVII] or [7; §1]. In particular, there is a constant κ such that

(4) $$|f|_P \le \| \int_\Omega f dP\| \le \kappa |f|_P ,$$

for every P-essentially bounded function f on Ω, [5; XVII Theorem 2.10], where $|\cdot|_P$ denotes the P-essential supremum norm. Actually, the P-essentially bounded functions are the only P-integrable functions. This follows from [5; XVIII Theorem 2.11(c)] and Proposition 1.8 of [2]; see also the remarks following this proposition.

Having formulated the steps (i)-(iv) more precisely the proof of Bade's theorem follows immediately. Indeed, let $P : \Sigma \to L_s(X)$ be any spectral measure such that $R(P) = M$. If $T \in \langle M \rangle_s$, then it follows from the surjectivity of (3) that there exists $f \in L^1(P)$ such that

$$T = \int_\Omega f dP \ .$$

But, f is then P-essentially bounded and so there exist Σ-simple functions $\{f_n\}$ which converge to f with respect to the norm $|\cdot|_P$. It follows from (4) that $\{P(f_n)\}$ converges to $P(f)$ in $L_u(X)$. Since each operator $P(f_n)$, $n = 1,2,\ldots$, belongs to $\langle R(P) \rangle_u = \langle M \rangle_u$ it follows that also $T = P(f)$ belongs to $\langle M \rangle_u$. This completes the proof.

Remark. The representation of $\langle M \rangle_s$ as an L^1-space seems not to have been sufficiently exploited in the Banach space setting.

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