Aleksander Błaszczyk; Dok Yong Kim
A topological version of a combinatorial theorem of Katětov

Commentationes Mathematicae Universitatis Carolinae, Vol. 29 (1988), No. 4, 657--663

Persistent URL: http://dml.cz/dmlcz/106681

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
29,4 (1988)

A TOPOLOGICAL VERSION OF A COMBINATORIAL THEOREM OF KATĚTOV

Aleksander BŁASZCZYK, KIM DOK YONG

Dedicated to Professor M. Katětov on his seventieth birthday

Abstract: We prove that for every fixed-point-free homeomorphism f of a 0-dimensional paracompact space X onto a closed subset of X there exists a partition $\{U_1,U_2,U_3\}$ of X consisting of closed-open sets such that $f(U_i)\cap U_i=\emptyset$ for every $i\leq 3$.

Key words: Fixed-point-free homeomorphism, 0-dimensional space, paracompact space, clopen partition.

Classification: 54C10, 54D18

The theorem mentioned in the title says that if f is a mapping of a set X into itself such that $f(x)\neq x$ for all $x\in X$, then X is the union of disjoint sets A_1,A_2,A_3 such that $f(A_i)\cap A_i=\emptyset$ for all $i\leq 3$; see M. Katětov [4]. There is a natural question if the sets A_i can be open whenever X is a topological space. In this paper we present a partial answer to this question as well as some consequences of our result. Namely, we prove that if f is a homeomorphism of a 0-dimensional paracompact space X onto a closed subspace of X and $f(x)\neq x$ for all $x\in X$, then X is the union of disjoint clopen (= closed and open) sets U_1, U_2, U_3 such that $f(U_i)\cap U_i=\emptyset$ for all $i\leq 3$. In particular, if X is 0-dimensional and metrizable, then for every homeomorphism f of X onto a closed subset of X, there exists a partition $\{F,U_1,U_2,U_3\}$ of X such that F is just the set of fixed points of f and $f(U_i)\cap U_i=\emptyset$ for $i\leq 3$ and all sets U_i are open in X. By the Stone Representation Theorem and the fact that compact sets are paracompact, we obtain the following corollary: if B is a Boolean algebra and h is a homomorphism of B onto B with the property that for every ultrafilter $x\subset B$ there exists $u\in x$ such that $h(u)\notin x$, then there exist disjoint elements $u_1,u_2,u_3\in B$ such that $u_1\vee u_2\vee u_3=1$ and $h(u_i)\wedge u_i=0$ for all $i\leq 3$. For complete

Boolean algebras we obtain a short proof of the well known theorem due to Z. Frolík [3]: if h is a homomorphism of a complete Boolean algebra B onto itself, then there exist disjoint elements $u_1,u_2,u_3.u_4 \in B$ such that $u_1 \vee u_2 \vee u_3 \vee u_4=1$ and h is the identity on the partial algebra $B \upharpoonright u_1$ and $h(u_i) \wedge u_i=0$ holds for all i with $2 \leq i \leq 4$.

The authors are deeply obliged to Professors Sabine Koppelberg and Władysław Kulpa and Marian Turzański.

All spaces in the paper are assumed to be Tychonoff. A space X is 0-dimensional if dim X=0, i.e. for every two disjoint functionally closed sets A, $B \subset X$ there exists a clopen set $U \subset X$ such that $A \subset U \subset X-B$. In the case of compact spaces 0-dimensionality simply means that a space has a base consisting of clopen sets.

Lemma 1. For every continuous mapping f of a 0-dimensional space X into itself such that $f(x) \neq x$ for every $x \in X$, there exists a covering P of X consisting of clopen sets such that $f(U) \cap U=\emptyset$ for every $U \in P$.

Proof of this lemma is clear since if H and G are disjoint clopen neighbourhoods of x and f(x) respectively, then there exists a clopen set $U \subset H$ such that $f(U) \subset G$.

Lemma 2. Let f be a homeomorphism of a 0-dimensional normal space A onto a closed subset of X and let $\{U_1,\dots,U_4\}$ be a family of disjoint clopen sets in X such that $f(U_i) \cap U_i=\emptyset$ for every $i \leq 4$. Then there exist disjoint clopen sets $H_1,H_2,H_3 \subset X$ such that:

(1) $H_1 \cup H_2 \cup H_3=U_4$,

(2) $f(U_i \cup H_i) \cap (U_i \cup H_i)=\emptyset$ for all $i \leq 3$.

Proof. Since f is a homeomorphism and f(X) is closed in X, the family $\{U_4 \cap f(U_1),\ U_4 \cap f(U_2),\ U_4 \cap f(U_3)\}$ consists of disjoint closed subsets of X. Hence, by normality and 0-dimensionality of X, there exist disjoint clopen sets $F_1,F_2,F_3 \subset X$ such that the union of these sets equals U_4 and

(3) $U_4 \cap f(U_i) \subset F_i$ for all $i \leq 3$.

Since the sets U_i are disjoint and clopen, there exist disjoint clopen sets $G_1,G_2,G_3 \subset X$ the union of which equals U_4 and such that

(4) $U_4 \cap f^{-1}(U_i) \subset G_i$ for all $i \leq 3$.

We set

$W_i= \bigcup \{F_j \cap G_k: j,k \leq 3 \text{ and } j \neq i \text{ and } k \neq i\}$.

Clearly $W_1 \cup W_2 \cup W_3=U_4$ and $f(W_i) \cap W_i=\emptyset$ for all $i \leq 3$ since $f(U_4) \cap U_4=\emptyset$. By the condition (3), $f(U_i)\cap W_i=\emptyset$ and by the condition (4), $f(W_i)\cap U_i=\emptyset$. Therefore, for arbitrary $i\leq 3$, we get

$$f(U_i\cup W_i)\cap(U_i\cup W_i)=\emptyset.$$

Now it suffices to set $H_1=W_1$, $H_2=W_2-W_1$ and $H_3=W_3-(W_1\cup W_2)$.

Remark. One can easily observe that the assumption that X is normal and f(X) is closed can be replaced by the assumption that the sets U_1, U_2, U_3 are compact.

Recall that a family R of subsets of a space X is called locally finite if every point of X has a neighbourhood which intersects at most finitely many members of R. It is easy to see that if R is locally finite, then $\mathrm{cl}(\bigcup R)=\bigcup\{\mathrm{cl}\, A: A\in R\}$. A topological (Hausdorff) space X is said to be paracompact if every open covering of X has a locally finite refinement. All compact spaces and all metrizable spaces are paracompact; see e.g. R. Engelking [2].

Theorem 1. For every homeomorphism f of a 0-dimensional paracompact space X onto a closed subspace of X such that $f(x)\neq x$ for all $x\in X$, there exists a disjoint family $\{U_1, U_2, U_3\}$ of clopen sets covering X such that $f(U_i)\cap U_i=\emptyset$ for all $i\leq 3$.

Proof. By Lemma 1, there exists a family P of open subsets of X such that $\bigcup P=X$ and $f(U)\cap U=\emptyset$ for every $U\in P$. Since X is paracompact we can assume that P is locally finite. We set $P=\{V_\alpha:\alpha<\tau\}$, where $\tau=|P|$. By transfinite induction we can pick for every $\alpha<\tau$ a clopen set $W_\alpha\subset X$ such that

$$X-(\bigcup\{W_\xi:\xi<\alpha\}\cup\bigcup\{V_\eta:\alpha<\eta<\tau\})\subset W_\alpha\subset V_\alpha .$$

Such a choice is possible since X is 0-dimensional and normal. Then the resulting family $\{W_\alpha:\alpha<\tau\}$ is a locally finite covering of X and consists of clopen sets. Now we set

$H_0=W_0$,

$H_\alpha=W_\alpha-\bigcup\{W_\xi:\xi<\alpha\}$ for $0<\alpha<\tau$.

Clearly, the family $\{H_\alpha:\alpha<\tau\}$ is a covering of X consisting of disjoint clopen sets such that $f(H_\alpha)\cap H_\alpha=\emptyset$ for all $\alpha<\tau$. Now for every $\alpha\in\tau-\{0,1,2\}$ we construct, using Lemma 2, a disjoint family $\{G_0^\alpha, G_1^\alpha, G_2^\alpha\}$ of clopen sets such that

$H_\alpha=G_0^\alpha\cup G_1^\alpha\cup G_2^\alpha$ and $G_i^\alpha\cap G_j^\alpha=\emptyset$ for $i\neq j$ and

$f(H_i\cup\bigcup\{G_i^\xi:2<\xi\leq\alpha\})\cap(H_i\cup\bigcup\{G_i^\xi:2<\xi\leq\alpha\})=\emptyset$ for $i\leq 2$.

This is possible since for every $i\in\{0,1,2\}$, the family

$\{\bigcup\{G_i^{\xi}:2<\xi\leq\beta\}:\beta<\alpha\}$ is increasing and consists of clopen sets, because the members of this family are unions of locally finite families of clopen sets. It is easy to check that the sets

$$U_i=H_i\cup\bigcup\{G_i^{\alpha}:2<\alpha<\tau\}\text{ for } i\in\{0,1,2\}$$

have the required properties.

Corollary 1. If f is a homeomorphism of a 0-dimensional compact space X into itself and $f(x)\neq x$ for every $x\in X$, then X is the union of a disjoint family $\{U_1,U_2,U_3\}$ of clopen sets such that $f(U_i)\cap U_i=\emptyset$ for $i\leq 3$.

Remark. The Boolean version of this corollary was formulated in the introduction. A proof can be also derived directly from Lemma 1 and Lemma 2. Indeed, in compact case the family P in Lemma 1 can assume to be a finite family of disjoint clopen sets. Then, using Lemma 2 in finitely many steps we obtain the conclusion of Corollary 1.

Corollary 2. For every homeomorphism f of a 0-dimensional metrizable space X onto a closed subspace of X there exists a disjoint family $\{F,U_0,U_1,U_2\}$ covering X and such that F is the set of all fixed points of f, the sets U_i are open and $f(U_i)\cap U_i=\emptyset$ for all $i\leq 2$.

To prove the corollary it suffices to apply Theorem 1 to the mapping f restricted to X-F.

Corollary 3. If a homeomorphism f of a 0-dimensional paracompact space X onto a closed subspace of X does not have fixed points, then the extension of f over βX does not have fixed points as well.

Proof. Let the family $\{U_1,U_2,U_3\}$ be like in Theorem 1. Then the family $\{\text{cl } U_1,\text{cl } U_2,\text{cl } U_3\}$, where cl stands for the closure in the topology of βX, is a covering of βX consisting of disjoint clopen sets. For every $i\notin\{j,k\}$ we have $f(U_i)\subset U_j\cup U_k$. Then, for the extension βf of f we get $\beta f(\text{cl } U_i)\subset\text{cl } U_j\cup\text{cl } U_k$. Thus $\beta f(x)\neq x$ for every $x\in\beta X$ since the sets cl U_i, for $i\leq 3$, are pairwise disjoint and cover βX.

Our Lemma 2 can also be used to obtain a simple proof of the Frolík's Theorem mentioned in the introduction. First we note the following consequence of the lemma:

Lemma 3. Let f be a homeomorphism of a space X into itself and let $\{V_n:n<\omega\}$ be a sequence of compact clopen sets such that $f(V_n)\cap V_n=\emptyset$ for every $n<\omega$. Then there exists a family $\{U_1,U_2,U_3\}$ of disjoint open sets such that

(5) $U_1 \cup U_2 \cup U_3 = \cup \{V_n : n < \omega\}$

(6) $f(U_i) \cap U_i = \emptyset$ for $i \leq 3$.

Proof. First we note that there exists a family $\{W_n : n < \omega\}$ of disjoint compact clopen sets such that $f(W_n) \cap W_n = \emptyset$ for all $n < \omega$ and $\cup \{W_n : n < \omega\} = \cup \{V_n : n < \omega\}$. Then we proceed like in the proof of Theorem 1. Using Lemma 2 (cf. the remark after the lemma), we construct by induction for every $n > 3$ a disjoint family of compact clopen sets $\{G_1^n, G_2^n, G_3^n\}$ such that:

$G_1^n \cup G_2^n \cup G_3^n = W_n$ for $n > 3$ and

$f(W_i \cup G_i^4 \cup \ldots \cup G_i^n) \cap (W_i \cup G_i^4 \cup \ldots \cup G_i^n) = \emptyset$ for $i \leq 3$.

Finally, for $i \leq 3$ we set $U_i = W_i \cup \cup \{G_i^n : 4 \leq n < \omega\}$.

Theorem 2 (Z. Frolík [3]). If f is a homeomorphism of a locally compact extremally disconnected space X into itself, then X is the union of a disjoint family $\{U_0, U_1, U_2, U_3\}$ of clopen sets such that $f(x) = x$ for every $x \in U_0$ and $f(U_i) \cap U_i = \emptyset$ whenever $0 < i \leq 3$.

Proof. Let R be the set of all disjoint families $\{V_1, V_2, V_3\}$ consisting of clopen sets such that:

$f(V_1 \cup V_2 \cup V_3) \subset V_1 \cup V_2 \cup V_3$ and

$f(V_i) \cap V_i = \emptyset$ for all $i \leq 3$.

We claim that $R \neq \emptyset$ whenever f is not the identity. Indeed, since X is locally compact, there exists a compact clopen set $V_1 \subset X$ such that $f(V_1) \cap V_1 = \emptyset$. Let us choose a compact clopen set $V_2 \subset X$ such that $V_2 \cap V_1 = \emptyset$ and $V_2 \cap f(X) = f(V_1)$. Since f is one-to-one, $f(V_1) \cap f(V_2) = \emptyset$. Hence $f(V_2) \cap V_2 = \emptyset$. Going by induction we construct a sequence $\{V_n : n < \omega\}$ of compact open sets such that for every $n < \omega$ we have

(7) $f(V_n) = f(X) \cap V_{n+1}$ and $f(V_n) \cap V_n = \emptyset$ and $V_n \cap V_{n+1} = \emptyset$.

Then by Lemma 3 we get a disjoint family $\{W_1, W_2, W_3\}$ of open sets such that

$f(W_i) \cap W_i = \emptyset$ for all $i \leq 3$ and

$f(W_1 \cup W_2 \cup W_3) \subset W_1 \cup W_2 \cup W_3$ (cf. the condition (5)).

Since X is extremally disconnected, the family $\{\mathrm{cl}\ W_1, \mathrm{cl}\ W_2, \mathrm{cl}\ W_3\}$ is disjoint and belongs to R. Using Kuratowski-Zorn Lemma it is quite easy to show that if R is ordered by the relation

$\{W_1, W_2, W_3\} < \{V_1, V_2, V_3\}$ iff $W_i \subset V_i$ for all $i \leq 3$, then there exists an element $\{U_1,U_2,U_3\}$ which is maximal in R. It remains to show that f is the identity on the set $X - (U_1 \cup U_2 \cup U_3)$. Assume the contrary. Then by the same argument as above we construct a sequence $\{V_n : n < \omega\}$ of compact open sets for which the condition (7) holds true and moreover $V_0 \cap (U_1 \cup U_2 \cup U_3) = \emptyset$. There are two possibilities:

Case 1. For every $n < \omega$ and every $i \leq 3$, $f(V_n) \cap U_i = \emptyset$. Then also $V_n \cap U_i = \emptyset$ for every $n < \omega$ and every $i \leq 3$, because $f(U_i) \subset U_j \cup U_k$ whenever $i \notin \{j,k\}$. Using Lemma 3 once again we get a disjoint family $\{G_1,G_2,G_3\}$ of open sets satisfying conditions analogous to (5) and (6) and such that $G_i \cap U_j = \emptyset$ for $i,j \leq 3$. Hence the family $\{\mathrm{cl}\, G_1 \cup U_1,\ \mathrm{cl}\, G_2 \cup U_2,\ \mathrm{cl}\, G_3 \cup U_3\}$ belongs to R; a contradiction.

Case 2. For some $n < \omega$ and some $i \leq 3$ we have $f(V_n) \cap U_i \neq \emptyset$. We can assume that i=1 and n is minimal with this property. By the condition (7), $f^{-1}(V_{n+1}) = V_n$. We can also assume (see the construction of the sets V_n) that $U_i \cap V_k = \emptyset$ for every $i \leq 3$ and every $k \leq n$. Now we consider the sets $H_0, \dots, H_n$ defined by the formula:

$$H_i = V_i \cap f^{i-n-1}(U_1).$$

These sets are non-empty and have the following properties:

$$H_i \cap H_{i+1} = \emptyset \text{ and } f(H_i) \subset H_{i+1} \text{ for all } i \leq n \text{ and } f(H_n) \subset U_1.$$

If n is even we set $G_1 = U_1 \cup H_1 \cup H_3 \cup \dots \cup H_{n-1}$, $G_2 = U_2 \cup H_2 \cup H_4 \cup \dots \cup H_n$. If n is odd we set $G_1 = U_1 \cup H_2 \cup H_4 \cup \dots \cup H_{n-1}$, $G_2 = U_2 \cup H_1 \cup H_3 \cup \dots \cup H_n$. In both cases $G_3 = U_3$. Now it is easy to check that $\{U_1,U_2,U_3\} < \{G_1,G_2,G_3\}$, which leads to a contradiction completing the proof.

We end the paper with an example which shows that there exists a fixed-point-free homeomorphism f of a 0-dimensional locally compact space X onto itself for which does not exist any finite covering P consisting of disjoint clopen sets such that $f(U) \cap U = \emptyset$ for all $U \in P$.

Example. Let $X = \{-1,0,1\}^{\omega_1} - \{\mathbf{0}\}$, where $\mathbf{0}$ is the point of the cube all coordinates of which equal zero. The mapping $f: X \to X$ is defined by the formula

$$f(x)_\alpha = -x_\alpha \text{ for all } \alpha < \omega_1,$$

where x_α is the α-th coordinate of the point x. One can easily show (see e.g. B. Efimov [1]) that every real-valued continuous function on X can be extended over the cube $\{-1,0,1\}^{\omega_1}$. Thus $\beta X = \{-1,0,1\}^{\omega_1}$ and the point $\mathbf{0}$ is the unique fixed point of the extension of f over βX. The same argument as in the proof of Corollary 3 shows that for every finite covering P of X consisting of disjoint clopen sets there exists $U \in P$ such that $f(U) \cap U \neq \emptyset$.

References

[1] B. EFIMOV: Dyadic bicompacta, Trans. Mosc. Math. Soc. 14(1965), 229-267.

[2] R. ENGELKING: General Topology, PWN, Warsaw, 1977.

[3] Z. FROLÍK: Fixed points of maps of extremally disconnected spaces and complete Boolean algebras, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 16(1968), 269-275.

[4] M. KATĚTOV: A theorem on mappings, Comment. Math. Univ. Carolinae 8 (1967), 431-433.

Instytut Matematyki, Uniwersytet Śląski, Katowice, ul. Bankowa 14, 40-007 Katowice, Polska

(Oblatum 18.4. 1988)