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AN ELEMENTARY PROOF OF NOBLE S THEOREM ON NORMALITY OF POWERS

Ryszard ENGELKING

Dedicated to Professor M. Katětov on his seventieth birthday

Abstract: We show in a simple way that if all powers of a space are normal, then the space itself is compact.

Key words: Cartesian product, normality, compactness.

Classification: 54B10, 54D15, 54D30

One of the important results in the theory of normality of Cartesian products, originated in 1948 by M. Katětov and A.H. Stone (see [2] and [6]), is the theorem due to N. Noble [4] which states that if all powers of a space are normal, then the space itself is compact. The theorem has been originally obtained in the frame of a general theory developed by N. Noble, and this prompted several authors to propose simpler and more direct proofs (see [1], [3] and [5]). In all these proofs A.H. Stone's theorem on the non-normality of \( N^1 \) is applied and, together with a conveniently chosen rather strong topological result, yields Noble's theorem.

It turns out that the Noble theorem can also be established in an elementary way by a variant of the argument A.H. Stone used to prove the non-normality of \( N^1 \).

We shall show that if for a topological space \( X \) the power \( X^m \) is normal for every \( m \), then \( X \) is compact.

Suppose that \( X \) is not compact and consider a family \( \{ F_s : s \in S \} \) of closed subsets of \( X \) which has the finite intersection property and an empty intersection; denote by \( m \) the cardinality of \( S \). The set \( F = \prod_{s \in S} F_s \subset X^m = \prod_{s \in S} X_s \), where \( X_s = X \) for \( s \in S \), is closed and disjoint from the diagonal \( \Delta \subset X^m \).

Consider an open set \( U \) containing \( F \).

Let \( x_1 \) be an arbitrary point in \( F \). There exists a finite set \( S_1 \subset S \) such
that \( p_{S_1}^{-1} p_{S_1} (x_1) \subseteq U \). Define a point \( x_2 \in F \) by letting \( p_S(x_2) = a_1 \) for \( s \in S_1 \), where \( a_1 \) is an arbitrary point in \( \bigcap_{s \in S_1} F_s \), and \( p_S(x_2) = p_S(x_1) \) for \( s \notin S_1 \), and enlarge \( S_1 \) to a finite set \( S_2 \subseteq S \) such that \( p_{S_2}^{-1} p_{S_2} (x_2) \subseteq U \). By induction we can define points \( x_1, x_2, x_3, \ldots \) in \( F \), finite sets \( S_1 \subseteq S_2 \subseteq S_3 \subseteq \cdots \subseteq S \) and points \( a_1, a_2, a_3, \ldots \) in \( X \) such that

\[
p_S(x_n) = a_{n-1} \text{ for } s \in S_{n-1} \text{ and } p_{S_n}^{-1} p_{S_n} (x_n) \subseteq U.
\]

Since, by A.H. Stone's theorem, \( X \) does not contain a closed copy of \( N \), there exists a point \( a_0 \in X \) every neighbourhood of which contains infinitely many \( a_n \)'s. The points \( y_1, y_2, y_3, \ldots \) of \( X^m \) defined by

\[
p_S(y_n) = p_S(x_n) \text{ for } s \in S_m \text{ and } p_S(y_n) = a_0 \text{ for } s \notin S_m
\]

belong to \( U \) and - as one easily sees - every neighbourhood of the point \( y \in \Delta \) all of whose coordinates are equal to \( a_0 \), contains a \( y_n \). Thus \( \Delta \cap \bar{U} \neq \emptyset \); since this is in contradiction with the normality of \( X^m \), it follows that \( X^m \) is compact.

References


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- 678 -