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SEQUENTIAL STRUCTURES INDUCED BY MEROTOPIES

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Dedicated to Professor M. Katětov on his seventieth birthday

Abstract: Every merotopy on a set X induces a sequential structure and a uniformly sequential structure on X. This note characterizes those (uniformly) sequential structures on X which arise in this way.

Key words: Merotopy, convergent sequence, adjacent sequences, Cauchy sequence, Galois correspondence.

Classification: 54A20, 54E15, 06A15, 18B30

Background. A merotopy on a set X specifies certain collections of subsets of X as micromeric, subject to the following axioms:

(Mer 1) any collection of subsets of X which contains a member with at most one element, is micromeric,

(Mer 2) if $\mathcal{A}$ and $\mathcal{B}$ are collections of subsets of X such that $\mathcal{A}$ is micromeric and $\mathcal{B}$ minorizes $\mathcal{A}$ (i.e., if for each $A \in \mathcal{A}$ there exists $B \in \mathcal{B}$ with $B \subset A$), then $\mathcal{B}$ is micromeric,

(Mer 3) if $\mathcal{A} \cup \mathcal{B}$ is micromeric, then $\mathcal{A}$ or $\mathcal{B}$ is micromeric.

For further details on merotopies see [3] and [5] and the references given there. For convergence structures, induced by merotopies, see [4].

A sequential structure on a set X specifies, which sequences in X converge to which points in X (notation: $(x_n) \longrightarrow x$), subject to the following axioms:

(Seq 1) $\dot{x} \longrightarrow x$ (for each $x \in X$, where x denotes the constant sequence with value x,

(Seq 2) if a sequence converges to x, so does each of its subsequences.

A uniformly sequential structure on X specifies, which sequence pairs in X are adjacent, subject to the following axioms:

(USeq 1) each sequence in X is adjacent to itself,

(USeq 2) if a sequence-pair is adjacent, then so is each of its subsequence-pairs.

Formally a sequence-pair in X is a map $f:\mathbf{N} \longrightarrow X^2$ and a subsequence-pair of f is a composite $f \circ \sigma$ of f with a strictly increasing map $\sigma:\mathbf{N} \longrightarrow \mathbf{N}$.

For further details on (uniformly) sequential structures see [1] and the references given there. For sequential structures, induced by topologies or by closure operators, see [6].

Sequential structures induced by merotopies

Definition:

(1) In a merotopic space (X, Γ) a sequence (x_n) is said to converge to x provided the following equivalent conditions are satisfied:

(a) for each infinite subset M of **N** the collection $\{\{x_m, x\} | m \in M\}$ is micromeric,

(b) for each infinite subset M of **N** the sets $\{x\}$ and $\{x_m | m \in M\}$ are near,

(c) for each uniform cover $\mathcal{U}$ the set star $(x, \mathcal{U})$ contains x_n for almost all n.

(2) A sequential C on X is said to be merotopy-induced provided there exists a merotopy Γ on X, such that $(x_n) \xrightarrow{C} x$ iff $(x_n) \xrightarrow{\Gamma} x$.

Remarks:

(1) If Γ is a merotopy on X and Γ' is its contigual reflection, then Γ and Γ' induce the same sequential structure on X. This follows immediately from (b) above.

(2) For nearness spaces the above conditions (a) - (c) are equivalent to

(d) each uniform cover has a member which contains x and almost all x_n, but for merotopic spaces (d) is properly stronger than (a) - (c).

Proposition: A sequential structure on X is merotopy-induced if and only if it satisfies the following conditions:

(Seq 3) [Urysohn condition] if each subsequence of (x_n) contains a subsequence, which converges to x, then (x_n) converges to x,

(Seq 4) [Koutník condition] if for each n the constant sequence $\dot{x}_n$ converges to x, then (x_n) converges to x,

(Seq 5) [Symmetry condition] if $\dot{x}$ converges to y, then $\dot{y}$ converges to x.

Proof: Obviously the above conditions are necessary. To show the converse, let the sequential structure C on X satisfy the above conditions (Seq i). Call a collection $\mathcal{A}$ of subsets of X micromeric whenever $\emptyset \in \mathcal{A}$ or there exists a convergent sequence $(x_n) \xrightarrow{C} x$ such that $\mathcal{A}$ minorizes $\{\{x_n,x\} | n \in \mathbf{N}\}$. By (Seq 1) and (Seq 2) this defines a merotopy Γ on X. Moreover, $(x_n) \xrightarrow{C} x$ implies $(x_n) \xrightarrow{\Gamma} x$. Hence, Γ induces C, if $(x_n) \xrightarrow{\Gamma} x$ implies $(x_n) \xrightarrow{C} x$. If this were not the case, there would exist (x_n) and x with $(x_n) \xrightarrow{\Gamma} x$ such that (x_n) does not C-converge to x. By (Seq 3) we may assume that no subsequence of (x_n) C-converges to x. Hence, by (Seq 4), we may assume that $\dot{x}_n$ C-converges to x for no $n \in \mathbf{N}$. In particular $x_n \neq x$ for each $n \in \mathbf{N}$. Since $(x_n) \xrightarrow{\Gamma} x$, the collection $\{\{x_n,x\} | n \in \mathbf{N}\}$ is micromeric. Thus there exists $(y_n) \xrightarrow{C} y$ such that $\{\{x_n,x\} | n \in \mathbf{N}\}$ minorizes $\{\{y_n,y\} | n \in \mathbf{N}\}$. Hence for each $n \in \mathbf{N}$ we can select $m(n) \in \mathbf{N}$ with

$$\{x_{m(n)},x\} \subset \{y_n,y\}.$$

Since $x_{m(n)} \neq x$, this implies $\{x_{m(n)},x\} = \{y_n,y\}$.

Case 1: $M = \{m(n) | n \in \mathbf{N}\}$ is infinite.

Then there exist strictly increasing maps $\sigma : \mathbf{N} \longrightarrow \mathbf{N}$ and $\tau : \mathbf{N} \longrightarrow \mathbf{N}$ with $\{x_{\sigma(n)},x\} = \{y_{\tau(n)},y\}$ for each $n \in \mathbf{N}$. If x=y, then $x_{\sigma(n)} = y_{\tau(n)}$ for each $n \in \mathbf{N}$, contradicting the fact that $(y_{\tau(n)})$ C-converges to y but $(x_{\sigma(n)})$ does not C-converge to x. If $x \neq y$, then $x_{\sigma(n)} = y$ and $y_{\tau(n)} = x$ for each $n \in \mathbf{N}$. Hence $\dot{x}$ C-converges to y, but $\dot{y}$ does not C-converge to x, contradicting (Seq 5). Thus Case 1 is impossible.

Case 2: $M = \{m(n) | n \in \mathbf{N}\}$ is finite.

Then there exists an element $m \in \mathbf{N}$ and a strictly increasing map $\sigma : \mathbf{N} \longrightarrow \mathbf{N}$ with $\{x_m,x\} = \{y_{\sigma(n)},y\}$ for each $n \in \mathbf{N}$. If x=y, then $x_m = y_{\sigma(n)}$ for each $n \in \mathbf{N}$, contradicting the fact that $(y_{\sigma(n)})$ C-converges to y, but $\dot{x}_m$ does not C-converge to x. If $x \neq y$, then $x_m = y$ and $y_{\sigma(n)} = x$ for each $n \in \mathbf{N}$. Hence $\dot{x}$ C-converges to y, but $\dot{y}$ does not C-converge to x, contradicting (Seq 5). Thus Case 2 is impossible as well. This proves that C is induced by Γ.

Remark: If Mer is the construct of merotopic spaces and continuous maps, Seq is the construct of sequential spaces (i.e., sets supplied with a sequential structure) and sequentially continuous maps, $G:\text{Mer} \longrightarrow \text{Seq}$ is the concrete functor associating with any merotopy on X its induced sequential structure on X, and $F:\text{Seq} \longrightarrow \text{Mer}$ is the concrete functor associating with any sequential structure C on X the merotopy whose micromeric collections are precisely those $\mathcal{A}$ which contain $\emptyset$ or minorize $\{\{x_n,x\}\mid n\in\mathbf{N}\}$ for some $(x_n)\longrightarrow x$, then $\underline{\mathrm{Mer}} \underset{F}{\overset{G}{\rightleftarrows}} \underline{\mathrm{Seq}}$ is a Galois connection of the third kind (over Set) in the sense of [2]. The above proposition characterizes the corresponding Galois closed objects in Seq. More obviously, a merotopic space is Galois-closed provided that every micromeric collection contains $\emptyset$ or minorizes $\{\{x_n,x\}\mid n\in\mathbf{N}\}$ for some convergent sequence $(x_n)\longrightarrow x$.

Uniformly sequential structures induced by merotopies

Definition:

(1) In a merotopic space (X,Γ) a sequence-pair (x_n,y_n) is said to be adjacent provided the following equivalent conditions are satisfied:

(a) for each infinite subset M of $\mathbf{N}$ the collection $\{\{x_m,y_m\}\mid m\in\mathbf{N}\}$ is micromeric,

(b) for each infinite subset M of $\mathbf{N}$ the collection $\{A\subset X\mid A\cap\{x_m,y_m\}\neq\emptyset$ for each $m\in\mathbf{N}\}$ is near,

(c) for each uniform cover $\mathcal{U}$ there exists $n\in\mathbf{N}$ such that for each $m\geq n$ there exists $U\in\mathcal{U}$ with $\{x_m,y_m\}\subset U$.

(2) A uniformly sequential structure C on X is said to be merotopy induced provided there exists a merotopy Γ on X such that a sequence-pair is C-adjacent if it is adjacent in (X,Γ).

Proposition: A uniformly sequential structure on X is merotopy-induced if and only if it satisfies the following conditions:

(USeq 3) if each subsequence-pair of (x_n,y_n) contains an adjacent subsequence-pair, then (x_n,y_n) is adjacent,

(USeq 4) if for each n the constant sequence-pair $(\dot{x}_n,\dot{y}_n)$ is adjacent, then so is (x_n,y_n),

(USeq 5) if (x_n,y_n) is adjacent, then so is (y_n,x_n).

Proof: The proof is completely parallel to the proof of the sequential version, if we observe that the conditions (USeq i) imply:

(USeq 6) if (x_n,y_n) and (x_n',y_n') are sequence-pairs such that (x_n,y_n) is adjacent and $\{x_n,y_n\}=\{x_n',y_n'\}$ for each n, then (x_n',y_n') is adjacent.

Remark: Consider the following conditions:

(a) if (x_n,y_n) and (y_n,z_n) are adjacent, then so is (x_n,z_n),

(b) if $(x_n,\dot{x})$ and $(y_n,\dot{x})$ are adjacent, then so is (x_n,y_n).

Then in a uniform space (a) and (b) hold, in a nearness space (b) but not necessarily (a) holds, in a merotopic space neither (a) nor (b) need be true.

Remark (Cauchy sequences). The concept of adjacent sequences can be considered as a generalization of the concept of convergent sequences ($(x_n) \to x$ iff $(x_n, \dot{x})$ is adjacent), being less unsymmetric and less point-bound. For uniform (and, more generally, for nearness) spaces there is a more familiar such concept namely that of Cauchy sequences, i.e., of "potentially convergent" sequences, i.e., of sequences converging in a suitable extension of the given space. For merotopic spaces, however, there seems to be no reasonable concept of Cauchy sequences. The natural candidates

(a) $\{\{x_m | m \geq n\} | n \in \mathbf{N}\}$ is micromeric,

(b) any pair of subsequences of (x_n) is adjacent,

are too restrictive, since not even satisfied by all convergent sequences. Even worse: every merotopic space (X, Γ) can be embedded into a merotopic space in which every sequence converges: just add a point ∞ to X and call a collection $\mathcal{A}$ of subsets of $X \cup \{\infty\}$ micromeric provided $\{A \cap X | A \in \mathcal{A}\}$ is micromeric in (X, Γ). Then every sequence converges to ∞. Hence, in a merotopic space (as in a topological space) every sequence is "potentially" convergent.

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