Commentationes Mathematicae Universitatis Carolinae

Mary Ellen Rudin

A few topological problems

Commentationes Mathematicae Universitatis Carolinae, Vol. 29 (1988), No. 4, 743--746

Persistent URL: http://dml.cz/dmlcz/106692

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
29,4 (1988)

A FEW TOPOLOGICAL PROBLEMS

Mary Ellen RUDIN

Dedicated to Professor M. Katětov on his seventieth birthday

Problems have always fascinated me. I will discuss here a few problems: they are not equally difficult nor equally fundamental but are problems I have worked on or would like to work on.

We begin with a purely set theoretic question asked by Dowker in 1955 (one recognizes the normal, not collectionwise normal topological pathology in it). Does there exist a set X and filter F on X having the following properties?

(1) If $f:X \longrightarrow F$, then there is $x \neq y$ in X with $x \in f(y)$ and $y \in f(x)$, but

(2) if $Y \subset X$, then there is $f:X \longrightarrow F$ such that, for all $x \in (X-Y)$ and $y \in Y$, either $x \notin f(y)$ or $y \notin f(x)$.

It was already known to Dowker [1] that a counterexample must have cardinality greater than ω_1. The answer to a related problem is found in [2], but there are no partial results known on this problem.

A trio of fundamental set theory questions which are also interesting topologically are the L-space, $\omega^* = \omega_1^*$, and box product problems: Is there a regular, hereditarily Lindelof, nonseparable space (called an L-space)? We would like a "real" L-space or a model for ZFC containing no L-space, for we know many models of ZFC contain L-spaces. (The related S-space problem has been solved [3] by showing it is undecidable in ZFC.) Similarly, [4], we know that in many models for ZFC, ω^* is not homeomorphic to ω_1^*. (By ω_1^* we mean the remainder in the Stone-Čech compactification of a discrete set of cardinality ω_1 when the discrete set is removed.) Is it just a theorem that ω^* is not homeomorphic to ω_1^* ?

The box product problem [5] is really many problems: which infinite box products are normal; which are paracompact ; is there a difference? Lawrence [6] has recently shown that a box product of countably many copies of the rationals is consistently paracompact; this has long been known for locally compact metric spaces [7]. There are countable box products of compact spaces which are not normal [8] and a nontrivial infinite box product having the irrationals as one factor is not normal [9]. The whole question remains a mystery except for isolated, mostly inconclusive, results.

Another rather set theoretic topological problem which has haunted me for many years and one on which there are no partial results is the question, is there a normal space with a σ-disjoint basis which is not paracompact? (We assume all spaces are Hausdorff.) A related problem is partially solved in [10].

Most recently I have been working with metrizability in manifolds. A basic open question here is,are all normal manifolds collectionwise Hausdorff? We know that it is consistent that normal manifolds not be collectionwise normal [11], but there are no partial results on the collectionwise Hausdorff problem. A less basic problem of interest to me concerns the fact that the Continuum Hypothesis can be used to construct a normal nonmetrizable manifold with a countable, point-separating open cover [12]; can one construct such an example without any special assumption? If so, Balogh has shown [13] that the Lindelöf number of the space must be cardinality of the continuum.

The next group of questions all have to do with normality in products. Such questions are tied to the shrinkability of open covers. (An open cover $\{V_\alpha \mid \alpha \in A\}$ shrinks an open cover $\{U_\alpha \mid \alpha \in A\}$ if $\overline{V}_\alpha \subset U_\alpha$ for all $\alpha \in A$.) Dowker spaces are normal spaces with countable open covers which cannot be shrunk. Is there a small Dowker space, say one of cardinality or weight ω_1? We only know that the existence of such a Dowker space is consistent with ZFC. In fact any Fowker space other than that found in [14] would constitute a major break through. Let us call a space a $\varkappa$-Dowker space if it is normal and $\varkappa$ is the minimal cardinality of an open cover which cannot be shrunk. For $\varkappa > \omega$ the situation is similar to the case $\varkappa = \omega$: we know essentialy only one real example of a $\varkappa$-Dowker space. Many questions depend on the existence of further examples. For instance, is there an ω_1-Dowker P-space? (A P-space being one whose product with every metric space is normal.) The existence of such a space would prove two old conjectures of Morita [15]; as of now we only know that the existence is consistent with ZFC [16].

Bešlagič [17] has a number of interesting normality questions. For instance:

(a) If every monotone open cover of a normal space X has a countable refinement, is X Lindelöf?

(b) If X is normal and Y is a collectionwise normal perfect image of X, is X collectionwise normal?

(c) Suppose every open cover of X is shrinkable. If Y is compact and $X \times Y$ is normal, is every open cover of $X \times Y$ shrinkable?

Let $C_p(X)$ be the space of all continuous real valued functions on a completely regular space X, using the topology of pointwise convergence. A number of Soviet mathematicians, notably Arhangelskii [18] have been studying properties of $C_p(X)$. Two of many open problems in this area which attract me are: For normal $C_p(X)$ is $C_p(X) \times C_p(X)$ always normal? For Lindelöf $C_p(X)$ is $C_p(X) \times C_p(X)$ always Lindelöf?

Michael's conjecture that X^{ω} is Lindelöf if $X \times Y$ is Lindelöf for all Lindelöf Y also remains unproved.

Junilla has the rather technical conjecture that a space must be θ-refinable if every directed open cover of the space has a σ-cushioned refinement. (The related strict p-space problem has recently been solved [19].)

These remind me of the better known conjecture that all M_3 spaces are M_1 spaces, that is, that every stratifiable space has a σ-closure preserving open base. (See [20] Chapter 10 for background on this and Junilla's problem.)

Z. Balogh has recently asked if there can be a closed discrete set of cardinality ω_1 in a normal, first countable space, which is not a G_δ-set. (By a theorem of W. Fleissner [21] the answer is consistently no.)

Caryn Navy [22] has shown us normal spaces which are para-Lindelöf but not paracompact. Could such a space be collectionwise normal?

In mathematics, when we answer one question, it leads us to another one.

Bibliography

[1] C.H. DOWKER: A problem in set theory, J. London Math. Soc. 27(1952), 371-374.

[2] M.E. RUDIN: Two questions of Dowker, Proceedings Amer. Math. Soc., Vol. 91(1984), 155-158.

[3] S. TODORČEVIĆ: Forcing positive partition relations, Trans. Amer. Math. Soc. 280(1983), 703-720.

[4] V. PONOMAREV and L. ŠAPIRO: Absolutes of topological spaces and their continuous maps, Uspehi Math. 31(1976), 121-136 (Sov. Math. Surv. 21(1976), 138-154).

[5] C.J. KNIGHT: Box topologies, Quart. J. Math., Oxford 15(1964), 41-54.

[6] L.B. LAWRENCE: The box product of countably many copies of the rationals is consistently paracompact (to appear).

[7] M.E. RUDIN: The box product of countably many compact metric spaces, Gen. Top. Appl. 2(1972), 293-298.

[8] K. KUNEN: Paracompactness of box products of compact spaces, Trans. Amer. Math. Soc. 240(1978), 307-316.

[9] E. van DOUWEN: The box products of countably many metrizable spaces need not be normal, Fund. Math. 88(1975), 127-132.

[10] M.E. RUDIN: A normal screenable nonparacompact space, Top.Appl. 15(1983), 313-322.

[11] M.E. RUDIN: Two nonmetrizable manifolds (to appear).

[12] M.E. RUDIN: Countable point separating open covers of manifolds, Houston J. (to appear).

[13] Z. BALOGH: Locally nice spaces under Martin's Axiom, Comment. Math. Univ. Carolinae 24(1983), 63-87.

[14] M.E. RUDIN: A normal space X for which $X \times I$ is not normal, Fund. Math. LXXIII(1971), 179-186.

[15] K. CHIBA, T. PRZYMUSINSKI, M. RUDIN: Normality of products and Morita's conjectures, Top. Appl. 22(1986), 19-32.

[16] A. BEŠLAGIČ and M. RUDIN: Set theoretic constructions of non-shrinking open covers, Top. Appl. 20(1985), 167-177.

[17] A. BEŠLAGIČ: Normality in products, Top. Appl. 22(1986),71-82.

[18] A.V. ARHANGEL'SKII: A survey of C_p-theory, Q&A in Gen. Top. 5(1978), 1-109.

[19] Jiang SHOULI: Every strict p-space is θ-refinable, Top. Proc. 12(1987) (to appear).

[20] Handbook of Set Theoretic Topology. K.Kunen and J.E. Vaughan, editors, Elsevier Sci. Pub. BV(1984).

[21] W. FLEISSNER: Normal Moore spaces in the constructible universe, Proc. Amer. Math. Soc. 46(1974), 294-298.

[22] C. NAVY: Paracompactness in para-Lindelöf spaces, Thesis, University of Wisconsin, Madison, Wis. (1981).

Dept. of Mathematics, University of Wisconsin, Madison, Wisconsin 53706, U.S.A.

(Oblatum 16.3. 1988)