Věra Trnková; Miroslav Hušek
Non-constant continuous maps of modifications of topological spaces

Commentationes Mathematicae Universitatis Carolinae, Vol. 29 (1988), No. 4, 747--765

Persistent URL: http://dml.cz/dmlcz/106693

Terms of use:
© Charles University in Prague, Faculty of Mathematics and Physics, 1988

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project DML-CZ: The Czech Digital Mathematics Library http://project.dml.cz
Abstract: For every pair of monoids $M_1 \leq M_2$ there exists a regular $T_1$-space $X$ such that all the non-constant continuous endomaps of $X$ form a monoid isomorphic to $M_1$ and of its completely regular modification form a monoid isomorphic to $M_2$. An analogous statement is true also for compactly generated modification and sequential modification. A more general setting of simultaneous representations of small categories is investigated and stronger and more complex results are presented.

Key words: Representations, modifications.

Classification: 54H10, 54B30, 18B30

I. Introduction. In [G], J. de Groot proved that every group can be represented as the group of all homeomorphisms of a (suitable) topological space onto itself. He put the question (at the conference in Tihany in 1964, see [He]), whether every monoid (=semigroup with a unit) can be represented by all the non-constant continuous maps of a topological space into itself (i.e. whether for every monoid $M$ there exists a topological space $X$ such that for every non-constant continuous $f, g: X \to X$, $g \circ f$ is non-constant again and the monoid of all these maps is isomorphic to $M$). This was solved in [T], where a metrizable space $X$ representing a given monoid $M$ in the above sense was constructed.

In [KR], V. Kannan and M. Rajagopalan proved that for every pair of groups $G \leq H$, there exists a metric space $X$ such that all the isometries of $X$ form a group isomorphic to $G$ and all the homeomorphisms of $X$ onto itself a group isomorphic to $H$. All the above results are strengthened in [T], where the following statement is proved: for every triple of monoids $M_1 \leq M_2 \leq M_3$
there exists a complete metric space $X$ such that all the non-constant maps of $X$ into itself which are
continuous, form a monoid $\simeq M_3$,
unif. continuous, form a monoid $\simeq M_2$,
non-expanding, form a monoid $\simeq M_1$.

Another result of this kind is presented in $\mathbb{T}_4$ : for every pair of monoids $A \leq B$ there exists a Tichonov space $X$ such that all the non-constant continuous maps of $X$ into itself form a monoid isomorphic to $A$ and all the non-constant continuous maps of $\beta X$ into itself form a monoid isomorphic to $B$. (However, in general it is not true that for every quadruple of monoids $M_1 \leq M_2 \leq M_3 \leq A \leq B$ there exists a metric space $X$ such that both these statements are valid for $X$.)

The method developed in $\mathbb{T}_3$ can be used (after suitable modifications which unfortunately make it more involved) also for simultaneous representation of a pair of monoids $M_1 \leq M_2$ by a topological space and by its (suitable) modification. Three of these results are mentioned in the Abstract. Some further results are mentioned in the part III of this paper. A more general setting of almost full embeddings of categories, which is investigated in the present paper, admits also to obtain results of another kind than the mere representing of pairs of monoids. For example, for every cardinal number $\kappa$ there exists a stiff set $\mathcal{K}$ of paracompact spaces (stiff in the sense that if $X, Y \in \mathcal{K}$ and $f : X \to Y$ is a continuous map, then either $f$ is constant or $X=Y$ and $f$ is the identity) such that all the spaces from $\mathcal{K}$ have the same compactly generated modification (and the obtained $k$-space is rigid).

Our notation is a standard one. If $\mathcal{C}$ is a category, then obj $\mathcal{C}$ denotes the class of all its objects and, for $a, b \in$ obj $\mathcal{C}$, $\mathcal{C}(a, b)$ denotes the set of all morphisms of $\mathcal{C}$ with the domain $a$ and codomain $b$. For $a \xrightarrow{f} b \xrightarrow{g} c$, the composition is written $\beta \circ \alpha$. The category of all sets and maps is denoted by Set. A "concrete category" means always concrete over Set. If $\mathcal{C}$ is a concrete category and $a \in$ obj $\mathcal{C}$, then $|a|$ denotes the underlying set of $a$ (but speaking about a topological space, we often do not distinguish between the space and its underlying set if there is no danger of confusion). If $\mathcal{C}$ is a concrete category, then $a \trianglelefteq b$ for $a, b \in$ obj $\mathcal{C}$ means that the identity map of $|a| \trianglelefteq |b|$ in $\text{Set}$ carries a morphism from $\mathcal{C}(a, b)$ (and we say that $a$ is finer than $b$ or $b$ coarser than $a$). For concrete functors $F, G : \mathcal{C} \to \mathcal{D}$, we denote by $F \triangleleft G$ the fact that $Fa \trianglelefteq Ga$ for all $a \in$ obj $\mathcal{C}$ . In a topological space $X$, the closure of a set $A$ is denoted by $\bar{A}$ or only by $\bar{A}$ if there is no danger of confusion.
II. Preliminaries and some negative results

II.1. Let us denote by Top the category of all topological spaces and all their continuous maps. Let us recall that a functor

$$\Phi: \mathcal{X} \to \text{Top}$$

is called an almost full embedding (see e.g. [PT]) if it is one-to-one and for every pair of objects $a, b$ of $\mathcal{X}$:

- a continuous map $f: \Phi(a) \to \Phi(b)$ is non-constant,
- iff $f=\Phi(g)$ for a (unique!) $\mathcal{X}$-morphism $g:a \to b$.

If $\mathcal{X}$ has precisely one object, say $a$, then the existence of an almost full embedding $\Phi$ of $\mathcal{X}$ into Top is precisely the representability of the endomorphism monoid $M:=\mathcal{X}(a, a)$ as the monoid of all non-constant continuous maps of the topological space $X=\Phi(a)$ into itself. If $\mathcal{X}$ is a discrete category (i.e. it has no morphisms except the unities), then $\{\Phi(c)|c\in \text{obj } \mathcal{X}\}$ is a stiff class of topological spaces. In $\llbracket T \rrbracket$, it was proved that every small category can be almost fully embedded into the category of all metrizable spaces.

II.2. In what follows, we investigate the situation, when a functor

$$m: \text{Top} \to \text{Top}$$

is given such that

(i) $m$ is idempotent (i.e. $m \circ m = m$) and

(ii) $m$ preserves the underlying sets and maps, i.e. the following diagram commutes,

$$\begin{array}{ccc}
\text{Top} & \xrightarrow{m} & \text{Top} \\
\downarrow & & \downarrow \\
\text{Set} & & \\
\end{array}$$

where the unnamed arrows denote the forgetful functor. Let us call any such functor $m$ shortly a modification.

Given a modification $m$, we ask, for which small categories $k_1, k_2$ and for which functors $\Upsilon:k_1 \to k_2$ there exist almost full embeddings $\Phi_1:k_1 \to \text{Top}$ and $\Phi_2:k_2 \to \text{Top}$ such that the square

$$\begin{array}{ccc}
k_1 & \xrightarrow{\Phi_1} & k_2 \\
\downarrow \Phi_1 & & \downarrow \Phi_2 \\
\text{Top} & \xrightarrow{m} & \text{Top} \\
\end{array}$$

commutes (i.e. $\Phi_2 \circ \Upsilon = m \circ \Phi_1$). In such a case, we say that $\Upsilon$ has a simul-
taneous representation by \( m \).

If \( \mathcal{Y} \) has a simultaneous representation by \( m \), then \( \mathcal{Y} \) must be faithful. In fact, \( \mathcal{Y} \) and \( m \) are faithful so that \( m \cdot \mathcal{Y} = \mathcal{Y} \) is also faithful, hence \( \mathcal{Y} \) must be faithful. We prove that for some modifications used in topology, the faithfulness of \( \mathcal{Y} \) is also sufficient. Let us say that a modification \( m: \text{Top} \to \text{Top} \) is comprehensive if every faithful \( \mathcal{Y} \) has a simultaneous representation by \( m \).

Clearly, if \( m \) has to be comprehensive, then its image \( m(\text{Top}) \) must be big enough, every small category \( k_2 \) has to be almost fully embedded into \( m(\text{Top}) \). This eliminates e.g. the discrete or the indiscrete modifications. On the other hand, if \( m(\text{Top}) \) is too big, then \( m \) fails to be comprehensive again - the trivial example is the identity functor \( m: \text{Top} \to \text{Top} \), then only full embeddings \( \mathcal{Y}: k_1 \to k_2 \) have simultaneous representations by \( m \). Below, we discuss less trivial examples of modifications which fail to be comprehensive.

II.3. First, let us recall that a modification \( m \) is called an upper modification (or a lower modification) if \( m \times X \) (or \( m \times X \)) for all topological spaces \( X \) (in a little wider sense, these terms are used in [\( \mathcal{C} \)]). In \( \text{Top} \), upper modifications coincide with the bireflections, lower modifications with all the coreflections distinct from the functor sending every space to the void space (i.e. the concrete coreflections), the corresponding bireflective (or coreflective) subcategory of \( \text{Top} \) is determined by the class \( \{X|mX=X\} \). Let us also recall that every class \( \mathcal{C} \) of topological spaces determines

\[
\text{a lower modification } m_\mathcal{C}
\]

by the rule that for every space \( X \),

\( \mathcal{C} \text{ is open in } m_\mathcal{C} X \text{ iff } f^{-1}(\mathcal{C}) \text{ is open in } Y \text{ for all } Y \in \mathcal{C} \text{ and all continuous } f: Y \to X \)

and an upper modification \( m_\mathcal{C} \)

by the rule that for every space \( X \),

\( \{f^{-1}(\mathcal{C})|Y \in \mathcal{C}, \mathcal{C} \text{ is open in } Y, f: X \to Y \text{ is continuous}\} \)

forms a subbasis of open sets in \( m_\mathcal{C} X \).

(Thus, the discrete and the indiscrete modifications are \( m_{\text{discrete}} \) and \( m_{\text{indiscrete}} \) where \( \mathcal{C} \) consists of a one-point space.) The class \( b\mathcal{C} = \{X|m_\mathcal{C} X = X\} \) (or \( c\mathcal{C} = \{X|m_\mathcal{C} X = X\} \) determines the bireflective hull (or the coreflective hull) of the class \( \mathcal{C} \) and the modification \( m_\mathcal{C} = m_{b\mathcal{C}} \) (or \( m_\mathcal{C} = m_{c\mathcal{C}} \)) is the corresponding bireflection (or coreflection).

II.4. Let us discuss the comprehension of the upper modifications \( m \) from the point of view of the size of \( m(\text{Top}) \). The indiscrete modification has
the smallest possible image, it is the last modification in the order of concrete functors \( \text{Top} \to \text{Top} \). The next upper modification in \( \text{Top} \) smaller than the indiscrete one in the zerodimensional modification \( z \). The class \( z(\text{Top}) \) is still too small, no zerodimensional space \( X \) with more than one point has the property that every non-constant continuous map of \( X \) into itself is already a homeomorphism of \( X \) onto itself, hence no non-trivial group has a representation by all the non-constant continuous maps of a space in \( z(\text{Top}) \). The next smaller upper modifications are generated by continua. As it follows from our Main Theorem, some of these modifications are already comprehensive.

Now, we try to approximate the comprehension of upper modifications from the opposite side - when \( m(\text{Top}) \) is big (equivalently, if the functor \( m \) is close to \( \text{Id}_{\text{Top}} \)). As already mentioned, \( m \) cannot be the identity, but it cannot be either the nearest upper modification, namely the symmetric modification (\( X \) is said to be symmetric if \( x \in Y \) implies \( y \in X \)). We shall show that in a more general context. The class of symmetric spaces is the bireflective hull of all \( T_1 \)-spaces, and the class of \( T_1 \)-spaces is an extremal epireflective subcategory of \( \text{Top} \) (extremal means that the reflective maps are quotient, i.e., if \( X \) belongs to the subcategory then any finer space belongs to it, too).

**Proposition 1.** If \( \mathcal{K} \) is the bireflective hull in \( \text{Top} \) of an extremal epireflective subcategory \( \mathcal{L} \) of \( \text{Top} \), then the reflection \( m \) onto \( \mathcal{K} \) is not comprehensive.

**Proof.** Take for \( k_1 \) the trivial category with a unique morphism and for \( k_2 \) a category with a unique object \( a \) and such that \( k_2(a,a) \) is an infinite group (the functor \( \mathcal{Y} \) is the unique possible). Suppose that a simultaneous representation \( (\Phi_1, \Phi_2) \) exist, then \( \Phi_2 a \) must be a \( T_1 \)-space, \( \Phi_2 a \in \mathcal{K} \), hence \( \Phi_2 a \in \mathcal{L} \) and consequently, \( \Phi_1 a \in \mathcal{L} \) (since \( \Phi_1 a \leq \Phi_2 a \)), which is impossible.

The last Proposition applies e.g. to the bireflective hull of Hausdorff spaces, Uryson spaces (every two points have disjoint closed neighborhoods), functionally separated spaces (the completely regular modification is Hausdorff), totally disconnected spaces, hereditarily disconnected spaces; the last two examples can be also treated similarly as zerodimensional spaces. So, these subcategories are too big as targets of upper modifications \( m \) which can be used for simultaneous representations.

**II.5.** In the case of lower modifications, \( m \) cannot be the least one, neither the last but one (the corresponding subcategory consists of sums of indiscrete spaces). That is very easy to show. We may prove an assertion si-
similar to Proposition 1.

**Proposition 2.** If \( C \) is a class in Top such that \( Y \in C \) provided \( Y \) is coarser than a connected \( X \in C \), then \( m_C \) is not comprehensive.

**Proof.** If \( Y, k_1, k_2 \) are the same as in the proof of Proposition 1, then \( \Phi_2 a \) must be connected, belongs to \( C \), thus \( \Phi_1 a \) belongs to \( C \) as well, hence \( \Phi_1 a = \Phi_2 a \).

Thus every comprehensive lower modification \( m \) is finer than a non-trivial lower modification \( m' \) which is not comprehensive (take the coreflective hull of \( m(\text{Top}) \cup \{\text{connected spaces}\} \)).

II.6. Thus, we have seen that there are bounds for the comprehension of modifications, bounds both from "above" and "below". We do not know conditions necessary and sufficient for the comprehension of modifications. However, the construction presented in this paper is rather general and gives the proof of the comprehension of some current modifications.

III. The Main Theorem and its applications

III.1. Like above, the properties needed for our construction are of two kinds: one group of properties says that the image \( m(\text{Top}) \) cannot be too small, the other says that \( m(\text{Top}) \) cannot be too big.

**Definition.** We say that a modification \( m: \text{Top} \to \text{Top} \) is stabilized by \( \gamma \)-complete metrizability if, for every space \( X \), the following statements are fulfilled:

a) if \( A \subseteq X \) is a \( \gamma \)-embedded regularly closed completely metrizable subspace of \( X \), then the topologies of \( X \) and of \( mX \) coincide on \( A \).

b) If \( A \subseteq X \) is a \( \gamma \)-embedded zero set, \( B \subseteq X \) is cozero set containing \( A \) and such that \( B \setminus A \) is completely metrizable, then \( mA \) (or \( mB \)) is a closed (or open, resp.) subspace of \( mX \) and the topologies of \( X \) and of \( mX \) coincide on \( B \setminus A \).

III.2. The conditions from the preceding definition are needed in our construction (in fact, a little less is needed - see the construction in IV.2, 3, 4). By taking \( A=X \) in (a) we get that every completely metrizable space is a fixed object of every modification \( m \) stabilized by complete metrizability. If \( m \) is an upper modification, it follows that \( m \) is finer than the completely regular (=uniformizable) modification \( u \) (i.e. \( u=m \) where \( \gamma \) is the class of all completely regular spaces, or, equivalently, \( \gamma \) consists of an
arc). And conversely, if an upper modification \( m \) is finer than \( u \), then \( m \) is "almost" stabilized by complete metrizability. Indeed, the condition (a) is trivially satisfied and in (b), one can see easily that \( m^A \) is a closed set and \( m^B \) an open set in \( mX \) and that the topologies of \( X \) and \( mX \) coincide on \( B \setminus A \) (use the fact that any \( x \in B \setminus A \) has a neighborhood \( \mathcal{V} \subseteq B \setminus A \) in \( X \) which is determined by a continuous function being zero outside \( B \setminus A \)). So, for a given upper modification \( m \) finer than the completely regular one, it suffices to show that both \( m^A, m^B \) are subspaces of \( mX \).

III.3. If \( m \) is a lower modification stabilized by complete metrizability, then \( m \) must be coarser than the sequential modification \( s \) (i.e. \( s = m^\omega \), where \( \mathcal{G} \) consists of all finite spaces and all convergent sequences). Again, if a lower modification is coarser than \( s \), then \( m \) is "almost" stabilized by complete metrizability: (a) is trivially satisfied and similarly all the conditions of (b) except for those conditions that \( m^A, m^B \) are subspaces of \( mX \).

III.4. If \( m \) is an arbitrary modification stabilized by complete metrizability, then \( m \) must be coarser than the sequential modification \( s \) and finer than the completely regular modification \( u \) (since \( uX \) [or \( sX \)] is the coarsest [or the finest] space rendering all the continuous maps on \( X \) into a metrizable space \( M \) [or on a metrizable space \( M \) into \( X \)] continuous as a mapping \( uX \to M \) [or \( M \to sX \), resp.]). It follows from the two preceding paragraphs that for such a modification \( m \) to be stabilized by complete metrizability, it suffices and is necessary that \( m^A, m^B \) are subspaces of \( mX \) in the condition (b).

III.5. The conditions describing the stabilization by complete metrizability ensure that \( m(\text{Top}) \) is not too small (since \( m(\text{Top}) \) contains all metrizable spaces; Hence, by \( \mathcal{T}_1 \), any small category can be almost fully embedded in it). Now we add the condition ensuring that \( m(\text{Top}) \) is not too big:

**Definition.** We say that a modification \( m: \text{Top} \to \text{Top} \) is **essentially non-identical** if there exists a Hausdorff space \( X \) such that \( mX \) is Hausdorff, either \( X \not\subseteq mX \) or \( mX \not\subseteq X \) and neither \( X \) nor \( mX \) contains a metrizable continuum. Any such space is called a **distinguishing space** of \( m \).

III.6. Every non-identical lower modification \( m \) is essentially non-identical. That assertion follows from the fact that \( \text{Top} \) is the coreflective hull of \( \mathcal{T}_1 \)-spaces with a unique accumulation point (such spaces are zerodimensional); if \( m \not\subseteq \mathcal{T}_1 \), then \( mX \not\subseteq X \) for some of those spaces \( X \) and, hence, this
space \( X \) is a distinguishing space for \( m \).

For upper modifications \( m \), the situation is more delicate. It is easy to show that if a totally disconnected space does not belong to \( m(\text{Top}) \) then \( m \) is essentially non-identical (such a situation occurs if \( m(\text{Top}) \) is contained in the class of regular spaces). This situation can be given a more general setting. In the case that \( m(\text{Top}) \) contains any Hausdorff space \( X \) not containing metrizable continua all finer spaces than \( X \), one must proceed individually for every such \( m \).

III.7. Main Theorem. Every essentially non-identical modification \( m \) which is stabilized by the complete metrizability is comprehensive and all the representing spaces (i.e. all the spaces \( \Phi_i(\sigma), \sigma \in \text{obj } k_i, i=1,2 \), in the notation of II.2) can be always chosen to be Hausdorff spaces. If, moreover, there is a distinguishing space \( X \) of \( m \) such that both \( X \) and \( mX \) are regular (or completely regular or normal or paracompact), then all the representing spaces can be chosen with the same property.

III.8. Remark. The choice of the categories \( k_1, k_2 \) and \( \Psi \) in the Main Theorem is rather free. If we choose \( k_1 \) and \( k_2 \) with precisely one object, say \( a, M_1=k_1(a,a), M_2=k_2(a,a) \) are their endomorphism monoids, a faithful functor \( \Psi: k_1 \to k_2 \) is precisely an embedding of \( M_1 \) into \( M_2 \), we obtain a representation of the pair of monoids \( M_1 \subseteq M_2 \) by \( \Phi_1(a) \) and its modification \( m \Phi_1(a) \). If we choose \( k_2 \) with precisely one object \( a \) and one morphism \( 1_a \), and \( k_1 \) is a discrete category with \( \text{card } \text{obj } k_1 = \infty \), where \( \infty \) is a prescribed cardinal number, and \( \Psi \) sends all the objects of \( k_1 \) to the unique object of \( k_2 \), then \( X' = \{ \Phi_1(\sigma) | \sigma \in \text{obj } k_1 \} \) is a stiff set of spaces and \( m \Phi_1(\sigma) = \Phi_2(a) \), hence all the spaces \( X, a \in X' \) have the same modification \( mX \) (which is a rigid space because \( k_2(a,a) = 1_{a,a} \)).

III.9. Let us present some examples of modifications \( m: \text{Top} \to \text{Top} \) which fulfil the presumption of the Main Theorem, so that the Main Theorem can be applied on them.

a) Completely regular modification. As follows easily from III.2, the completely regular modification \( u \) is stabilized by the complete metrizability. The classical example of a regular \( T_1 \)-space which is not completely regular \( [EJ] \), is a distinguishing space of \( u \) such that both \( X \) and \( uX \) are regular \( T_1 \). Hence every faithful \( \Psi: k_1 \to k_2 \) has a simultaneous representation by \( u \) such that all the representing spaces are regular \( T_1 \)-spaces.

b) Sequential modification. The sequential modification \( s = m_{\Phi} \) (see
III.3) is stabilized by the complete metrizability. (In fact, s\(A\) is a closed subspace of s\(X\) for every closed \(A \subseteq X\); s\(B\) is an open subspace of s\(X\) for every cozero set \(B \subseteq X\); the proof of (a) and of the other requirements in (b) in III.1 is trivial, see III.3). It has a distinguishing space \(X\) such that both \(X\) and \(sX\) are paracompact, see III.6. Hence every faithful :\(k_1 \rightarrow k_2\) has a simultaneous representation by \(s\) such that all the representing spaces are paracompact.

c) Further lower modifications. If \(\mathcal{C}\) is a class of topological spaces which is closed with respect to continuous images, then for every space \(X\)

\[ A \subseteq X \text{ is closed in } m_{\mathcal{C}}X \iff A \cap K \text{ is closed in } K \text{ for each subspace } K \text{ of } X \]

which belongs to \(\mathcal{C}\).

This description implies easily that \(m_{\mathcal{C}}A\) is a closed subspace of \(m_{\mathcal{C}}X\) for every closed \(A \subseteq X\) and \(m_{\mathcal{C}}B\) is an open subspace of \(m_{\mathcal{C}}X\) for every cozero set \(B \subseteq X\) provided that \(\mathcal{C}\) is closed also with respect to closed subspaces. Thus, if \(\mathcal{F} \subseteq \mathcal{C}\) and \(\mathcal{C}\) is closed with respect to continuous images and closed subspaces, then \(m_{\mathcal{F}}\) is stabilized by the complete metrizability, see III.3. Moreover, if \(m_{\mathcal{F}}\) is not identical, then it has a distinguishing space \(X\) such that both \(X\) and \(m_{\mathcal{F}}X\) are paracompact, see III.6. Hence the Main Theorem can be applied e.g. on the following classes \(\mathcal{C}\):

- \(\mathcal{C} = \text{all compact spaces, i.e. } m_{\mathcal{C}}\text{ is a compactly generated modification;}\)
- \(\mathcal{C} = \text{all the spaces of the cardinality } \leq \omega, \text{ where } \omega \text{ is a given infinite cardinal, i.e. } m_{\mathcal{C}}\text{ is the coreflection on the subcategory of all the spaces with the tightness } \leq \omega;\)
- \(\mathcal{C} = \text{all the compact spaces of the cardinality } \leq \omega.\)

For any class \(\mathcal{C}\) containing \(\mathcal{F}\), we can form its closure with respect to closed subspaces and then with respect to continuous images. If the obtained class \(\mathcal{D}\) is still not so large that \(m_{\mathcal{D}}\) is already the identity, then \(m_{\mathcal{D}}\) is comprehensive and all the representing spaces can be chosen to be paracompact.

d) Composition of modifications. If \(m\) is an upper modification and \(m'\) a lower one, then both \(m \circ m'\) and \(m' \circ m\) are modifications again, but it is neither an upper nor a lower modification in general. For example, the modification \(s \circ u\) sends the space \(Y=Z \uplus T\), where \(Z\) is a non-compact Lindelöf space with a unique non-isolated point and \(T\) is the real line, the open subsbasis of which is formed by all open intervals and the set of all irrational numbers (and \(\uplus\) denotes the coproduct, i.e. the sum), on the space \(s (uY)=D \uplus R\), where \(D\) is discrete and \(R\) is the real line with the usual topology, so that neither \(s \circ u Y\) nor \(Y \circ s \circ u\). Let us notice that still \(s \circ u\) is stabilized by
the complete metrizability and the space \( Z \) is a distinguishing space for \( s \circ u \). However, since the distinguishing space \( X \) of a modification \( m \) has to satisfy either \( X \subseteq mX \) or \( mX \subseteq X \) (and this is necessary for the construction in the proof of the Main Theorem), all the representing spaces fulfill the same inequality for any faithful \( \Psi \) (this can be seen from the construction), so that we work "in essence" only with lower or upper modifications.

**IV. The proof of the Main Theorem**

IV.1. Let us denote by \( G \) the category of all directed connected graphs without loops (i.e. the objects of \( G \) are all \((V,R)\), where \( V \) is a set and \( R \subseteq V \times V \) such that never \((v,v) \in R\) and for every \( v,v' \in V \) [not necessarily distinct] there exist \( v_0=v,v_1,\ldots,v_n=v' \) in \( V \) with \((v_{i-1},v_i) \in R \cup R^{-1} \); \( h:(V,R) \longrightarrow (V',R') \) is a morphism of \( G \) iff it is an \( RR' \)-compatible map, i.e. it maps \( V \) into \( V' \) such that \((v,v') \in R \implies (h(v),h(v')) \in R' \). Let \( H \) be a category, the objects of which are all triples \((V,R,S)\), where \((V,R)\) is an object of \( G \) and \( S \subseteq R \); \( h: (V,R,S) \longrightarrow (V',R',S') \) is a morphism of \( H \) iff it is both \( RR' \)-compatible and \( SS' \)-compatible. There is a natural forgetful functor

\[ \Gamma:H \longrightarrow G \]

which forgets the second relation, i.e. \( \Gamma(V,R,S)=(V,R), \Gamma(h)=h \). In [17], for any faithful functor \( \Psi:k_1 \longrightarrow k_2 \), where \( k_1 \) and \( k_2 \) are small categories, full embeddings (= full one-to-one functors) \( \Lambda_1:k_1 \longrightarrow H, \Lambda_2:k_2 \longrightarrow G \) are constructed such that the square

\[ \Lambda_1 \] \[ \Lambda_2 \]
\[ \downarrow \] \[ \downarrow \]
\[ H \] \[ G \]
\[ \Gamma \] \[ \Psi \]

commutes. Hence to prove our Main Theorem, it is sufficient to construct, for a given essentially non-identical modification \( m \) stabilized by the complete metrizability, almost full embeddings \( \Phi_1:H \longrightarrow \text{Top}, \Phi_2:G \longrightarrow \text{Top} \) such that the square

\[ \Phi_1 \] \[ \Phi_2 \]
\[ \downarrow \] \[ \downarrow \]
\[ \text{Top} \] \[ \text{Top} \]
\[ m \] \[ \Gamma \]

commutes. Then \( \Phi_1 \circ \Lambda_1 \) and \( \Phi_2 \circ \Lambda_2 \) give a simultaneous representation of \( \Psi \) by \( m \).
IV.2. We will construct the functors $f_1, f_2$ in IV.3 - 4 below. First, let us introduce some notation and show some auxiliary statements. If $P$ is a space metrized by a complete metric $\rho$, $A \subseteq P$ its subspace with $\rho(a,a') \geq 1$ for all distinct $a,a' \in A$ and $X$ is a space with $|X| = |A|$, we denote by

\[ P_A^X \]

the space on $|P|$ with the topology $\text{sup}(t,d)$, where $t$ is the topology given by $\rho$ and $d$ is the topology of $X$ extended on $|P|$ such that any point $x \in |P| \setminus |X|$ is isolated. Hence $X$ is a zero set and $C^\infty$-embedded subspace of $P_A^X$ and $P_A^X \setminus X$ is metrizable (by a complete metric).

Observation. If $m$ is a modification stabilized by the complete metrizability, then $mX$ is a closed and $P \setminus A$ an open subspace of $mP_A^X$. Moreover, if $X \preceq mX$ (or $X \succeq mX$), then $P_A \preceq mP_A^X$ (or $P_A \succeq mP_A^X$).

IV.3. In the rest of IV, we suppose that an essentially nonidentical modification $m$ stabilized by the complete metrizability is given and $X$ is its distinguishing space.

Let $P$ be a space metrized by a complete metric $\rho$, $A$ its subspace with $\rho(a,a') \geq 1$ for all distinct $a,a' \in P$ and $|A| = |X|$, $p_1, p_2, p_3$ are three distinguished points of $P$ such that $\rho(p_1, p_j) \geq 1$ and $\rho(p_1, A) \geq 1$ for $i,j \in \{1,2,3\}$, $i \neq j$. Depending on it, we construct the functor $\Phi_2: \mathcal{G} \rightarrow \text{Top}$.

The construction of $\Phi_2$ is just the arrow construction, described in a general setting e.g. in [PTj]: each arrow $r \in R$ in a connected graph $(V,R) \in \text{obj } \mathcal{G}$ is replaced by a copy of $mP_A^X$. More in detail, we take a copy $(mP_A^X)_r$ of the space $mP_A^X$ (all the points, subspaces, ... of $(mP_A^X)_r$ are denoted as in $mP_A^X$, only the letter $r$ is added) for each $r \in R$ and, in the coproduct $\coprod (mP_A^X)_r$, we identify, for each $r = (v_1, v_2) \in R$, $p_{1,r}$ with $p_{1,r}$ iff $r' = (v_1, v_2) \in R$, we denote the obtained point by $v_1$, $p_{1,r}$ with $p_{1,r}$ iff $r' = (v_1, v_1) \in R$, $p_{2,r}$ with $p_{2,r}$ iff $r' = (v_1, v_1) \in R$, $p_{2,r}$ with $p_{2,r}$ iff $r' = (v_1, v_1) \in R$, $p_{3,r}$ with $p_{3,r}$ for all $r' \in R$, $p_{3,r}$ with $p_{3,r}$ for all $r' \in R$, $p_{3,r}$ with $p_{3,r}$ for all $r' \in R$, taking all the sets $\bigcup_{r \in R} \{ y_r | \mathcal{F}_r(y_r, p_{1,r}) \leq \epsilon \}$ with $\epsilon > 0$ as a local basis of $c$ and all the sets $\{ y_r | y_r(y_r, p_{1,r}) \leq \epsilon \} \cup \{ y_r | y_r(y_r, p_{1,r}) \leq \epsilon \}$ with $\epsilon > 0$ as a local basis of $V$, where $R_1$ is the set of all $r \in R$ such that $v$ is its $i$-th member. Hence the obtained space $\Phi_2(V,R)$ contains $V$ (as its
Observation. \( m \Phi_2(V,R) = \Phi_2(V,R) \).
If \( h : (V,R) \to (V',R') \) is a morphism of \( \mathcal{G} \), we define \( f = \Phi_2(h) \) such that it maps each \( (m^P_X)_r \) onto \( (m^P_X)_r \) as the identity, for all \( r = (v_1,v_2) \in R \) and \( r' = (h(v_1),h(v_2)) \in R' \), i.e., in our convention, \( f(x_r) = x_{r'} \).

Observation. \( \Phi_2 : \mathcal{G} \to \text{Top} \) is a correctly defined one-to-one functor. Every \( \Phi_2(V,R) \) is regular or ... or paracompact whenever \( mX \) has this property.

IV.4. The functor \( \Phi_1 : \mathcal{H} \to \text{Top} \) is also constructed by the arrow-construction; given \( (V,R,S) \in \text{obj} \mathcal{H} \), then,
- if \( X \notin mX \), the arrows in \( S \) are replaced by copies of \( m^P_A X \) and the arrows in \( R \setminus S \) are replaced by copies of \( P_A X \);
- if \( mX \subseteq X \), the arrows in \( S \) are replaced by copies of \( P_A X \) and the arrows in \( R \setminus S \) are replaced by copies of \( m^P_A X \).

We do not describe the arrow-construction with all details as in IV.3 because the identifications are as in IV.3 and the local basis of the glueing-points is also as in IV.3.

Observation. \( m \Phi_1(V,R,S) = \Phi_2(V,R) \); \( \Phi_1(V,R,S) \) is regular or ... or paracompact whenever both \( X \) and \( mX \) have this property.

If \( h : (V,R,S) \to (V',R',S') \) is a morphism of \( \mathcal{H} \), we define \( g = \Phi_1(h) \) similarly as in IV.3, i.e. \( g \) maps the space replacing an arrow \( r = (v_1,v_2) \) onto the space replacing the arrow \( r' = (h(v_1),h(v_2)) \) as the identity. Here, we have to mention that if \( r \in S \), then \( r' \in S' \) so that \( r \) and \( r' \) both are replaced by copies of \( m^P_A X \) (if \( X \notin mX \)) or both are replaced by copies of \( P_A X \) (if \( mX \subseteq X \)), hence the identity map is continuous; if \( r \in R \setminus S \), then either \( r' \in R \setminus S' \) and then \( r \) and \( r' \) are replaced by copies of the same space again, or \( r' \in S' \); in this last case, the identity map is continuous again, being a map \( (m^P_A X)_r \to (m^P_A X')_{r'} \), if \( X \notin mX \) or \( (m^P_A X)_r \to (P_A X)'_{r'} \), if \( mX \subseteq X \).

Observation. \( \Phi_1 : \mathcal{H} \to \text{Top} \) is a correctly defined one-to-one functor and \( m \circ \Phi_1 = \Phi_2 \circ \Gamma \).

IV.5. The parts IV.6 - 9 are devoted to the construction of such space \( P \), its subspace \( A \) and the distinguished points \( p_1, p_2, p_3 \), that the functors \( \Phi_1 \) and \( \Phi_2 \), constructed from them as described in IV.3 - 4, are almost full. First, let us show that it is sufficient to construct them such that the following statements a), b) are true.
a) if \((V,R)\in \text{obj } G\) and \(\mathcal{L}:\mathbb{P}_A\to \Phi_2(V,R)\) is continuous, then either \(\mathcal{L}\) is constant or there exists \(r\in R\) such that \(\mathcal{L}\) is the identity map of \(\mathbb{P}_A\) onto its \(r\)-th copy in \(\Phi_2(V,R)\), i.e. \(\mathcal{L}(x) = x_r\) for all \(x\in \mathbb{P}_A\):

b) if \((V,R,S)\in \text{obj } H\) and \(\mathcal{L}\) is a continuous map of \(\mathbb{P}_A\) (or \(\mathbb{P}_A\)) into \(\Phi_1(V,R,S)\), then either \(\mathcal{L}\) is constant or there exists \(r\in R\) such that \(\mathcal{L}(x) = x_r\) for all \(x\in \mathbb{P}_A\) (or for all \(x\in \mathbb{P}_A\)).

Thus, let us suppose that a) is valid and that \(f: \mathbb{J}_2(V,R) \to \Phi_2(V',R')\) is a non-constant continuous map. For each \(r\in R\), we investigate the domain-restriction \(\mathbb{P}_A\cdot f \to \Phi_2(V',R')\) of \(f\), analogously as in [PT], pp. 105–6. If one of these restrictions is constant, say the \(r\)-th one, necessarily \(f(x) = f(p_{1,r}) = f(p_{2,r})\), hence all these restrictions must be constant, so that \(f\) must be constant, which is a contradiction. Thus, by a), for every \(r\in R\) there exists \(r'\in R'\) such that \(f(x_r) = x_{r'}\) for all \(x\in \mathbb{P}_A\). Since \((V,R)\) is connected, for every \(v\in V\) there exists \(r\in R\) and \(i \in \{1,2\}\) such that \(v = p_{i,r}\). Then \(h(v) = p_{1,r}'\) give a \(G\)-morphism \(h:(V,R) \to (V',R')\) such that \(f = \Phi_2(h)\).

Let us suppose that b) is satisfied and that \(g: \Phi_1(V,R,S) \to \mathbb{J}_1(V',R',S')\) is a continuous non-constant map. We find \(h:(V,R) \to (V',R')\) similarly as in the previous case. However, b) implies that if \(r = (v_1,v_2)\in S\), then \(r' = (h(v_1),h(v_2))\) must be in \(S'\) because \(X^m = X\), so that the identity map

\[
\mathbb{P}_A \to \mathbb{P}_A \text{ is not continuous whenever } mX \not\subseteq X.
\]

Thus, the constructed \(h\) is also \(SS'\)-compatible, hence it is a \(H\)-morphism and \(g = \Phi_1(h)\).

IV.6. The construction of \(P, A\) and \(p_1, p_2, p_3\) such that a), b) in IV.5 are fulfilled, heavily depends on the existence of a Cook continuum. Let us recall that a Cook continuum is a non-degenerate metrizable continuum \(Q\) such that

- \(K\) is a continuum with these properties was constructed by H. Cook in [C]. A more detailed version of the construction is contained in Appendix A in [PT]. We use its non-degenerate subcontinua (in what follows, continuum always means a non-degenerate continuum).
- We choose pairwise disjoint subcontinua of \(Q\), denoted by \(A_0, B_0, C_0, A_1, B_1, A_2, B_2, C_2, \ldots\) (there is a countable collection of them); we will call them building blocks. We metrize them such that the diameter of \(A_0, B_0, C_0\) is 1, that of \(A_1, B_1, C_1\) is \(\frac{1}{2}\) etc., in general the space with single index \(i\) has
diameter $2^{-1}$ and spaces with indices $i, j$ have diameter $2^{-\min(i,j)}$. Furthermore, in each of these subcontinua we choose two points of distance equal to

the diameter of the subcontinuum; we call the points in $A_0, a_0$ and $a_0'$, in $B_0$, $b_0$ and $b_0'$, in $A_0, 1$, $a_0, 1$ and $a_0', 1$ etc. We now glue the spaces into a connected metric as indicated by Figure 1. Finally, we form the completion of the resulting metric space by adding the points $a, b$ and $c$ as indicated (i.e. $a=\lim a_n$). The resulting space will be called a triangle space (the construction of the triangle space is also, in another notation, in [PT], pp. 223-4).

IV.7. We need four triangle spaces, say $T_1, T_2, T_3, T_4, T_5$ is created from a collection $Z_1=\{A_0(i), B_0(i), C_0(i), A_1(i), ...\}$ of subcontinua of $\mathbb{Q}$ as in IV.6.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Figure 1}
\end{figure}
such that the collection $\mathcal{Z} = \bigcup_{i=1}^{4} \mathcal{Z}_i$ is stiff (any pairwise disjoint collection of subcontinua of $\mathcal{Q}$ is stiff). The subspaces $A^{(1)}_0, B^{(1)}_0, \ldots$ are called building blocks of $T_1$. The copies of building blocks of $T_1$ are called also building blocks of $Y$ whenever $Y$ is created from the triangle spaces (or also from $T_2, x$, as described below. Denote by $S_1$ the subset of all the glueing points of $T_1$, i.e., $S_1 = \{a^{(1)}_0, b^{(1)}_0, c^{(1)}_0, a^{(1)}_1, b^{(1)}_1, c^{(1)}_1, \ldots\}$ and by $a^{(1)}, b^{(1)}, c^{(1)}$ the points added in the forming of the completion.

IV.8. The role of $T_3$ and $T_4$ is to obtain sufficiently large stiff collection of spaces of a special form. For this reason, we choose a rigid collection

$$\Psi_3 = \{(v_x, R_x) \mid x \in X\}$$

of objects of $G$, where $X$ is the distinguishing space of $m$ (we recall that each $(V_x, R_x)$ has no non-identical endomorphism and if $x \neq x'$, there is no morphism $(V_x, R_x) \to (V_{x'}, R_{x'})$, such a collection does exist, see [PT]). In $T_3 \sqcup T_4$, we identify $a^{(3)}$ with $a^{(4)}$ and $b^{(3)}$ with $b^{(4)}$, the obtained space is denoted by $T$, the obtained points by $a$ and $b$. We use the arrow construction again: in $(V_x, R_x)$, we replace each $r \in R_x$ by a copy of $T$; more in detail, in the coproduct $\amalg_t (T)_t$ (where $(T)_t$ are copies of the space $T$; we use the convention of IV.3 that points, subspaces, ... of $(T)_t$ are denoted as in $T$, only the index $t$ is added) we make the following identifications for each $r=(v_1, v_2) \in R_x$:

- $a_t$ with $a_t$ iff $r'=(v_1, v_2) \in R_x$, we denote the obtained point by $v_1$,
- $a_t$ with $b_t$ iff $r'=(v_1, v_2) \in R_x$, we denote the obtained point by $v_1$,
- $b_t$ with $c_t$ iff $r'=(v_1, v_2) \in R_x$, we denote the obtained point by $v_2$,
- $c_t$ with $c_t$ for all $r \in R_x$, we denote the obtained point by $c_x$,
- $c_t$ with $c_t$ for all $r \in R_x$, we denote the obtained point by $c_x$,
- the local basis of the glued points is defined similarly as in IV.3, so that we obtain a complete metric space; it is denoted by $C_x$.

IV.9. For every $x \in X$, we form the space $T_{2, x}$ such that we replace the subcontinuum $C^{(2)}_0$ in the triangle space $T_2$ by the space $C_x$. More in detail: in $C_x \sqcup (T_2 \setminus (C_0^{(2)} \cup C_0^{(2)} \cup C_0^{(2)}))$, we identify $c_x$ with $c_0^{(2)}$ and $c_x$ with $c_0^{(2)}$. Let us denote points, subspaces etc. of $T_{2, x}$ which are outside $C_x$, as in $T_2$, only the index $x$ is added. Our desired space $P$ is obtained from

$$T_1 \sqcup \amalg_{x \in X} T_{2, x}$$

- 761 -
by the identification of 
\[ a^{(1)}_x \] with \[ a^{(2)}_x \] for all \( x \in X \), the obtained point is \( p_1 \), 
\[ b^{(1)}_x \] with \[ b^{(2)}_x \] for all \( x \in X \), the obtained point is \( p_2 \);
the local basis of the glued points is defined as in IV.3, so that \( P \) is really metrizable by a complete metric. Its points \( p_1, p_2 \) are already defined, we put \( p_3 \equiv c^{(1)}_x \). Finally, the \( C^\infty \) -embedded discrete zero set \( A \) with the same underlying set as \( X \) is formed by all \( c^{(2)}_x, x \in X \).

IV.10. It remains to prove that \( P, p_1, p_2, p_3 \) and \( A \) satisfy the statements a), b) in IV.5. First, we prove several auxiliary lemmas.

Lemma. Let \( Z \) be a collection of all building blocks of all \( T_1, \ldots, T_4 \). Let \( Y \) be a space containing \( Z \subseteq Z \) such that the boundary of \( Z \) in \( Y \) consists of two points \( z_1, z_2 \). Let \( Z' \subseteq Z \) and let \( f:Z' \to Y \) be a continuous non-constant map. Then either \( Z' = Z \) and \( f \) is the inclusion (i.e. \( f(z) = z \) for all \( z \in Z \)) or \( f(Z') \cap Z \neq \emptyset \{ z_1, z_2 \} \).

Proof. Put \( \mathcal{O} = f^{-1}(Z \setminus \{ z_1, z_2 \}) \). Suppose that \( \mathcal{O} \neq \emptyset \). If \( Z' \setminus \mathcal{O} = \emptyset \), then \( f(Z') \subseteq Z \setminus \{ z_1, z_2 \} \), hence \( f \) must be constant, which is a contradiction. Hence \( Z' \setminus \mathcal{O} \neq \emptyset \). Choose \( z \in \mathcal{O} \) and denote by \( K \) the component of \( z \) in \( \mathcal{O} \). Since \( Z' \setminus \mathcal{O} \neq \emptyset \), \( K \) intersects the boundary of \( \mathcal{O} \) (see [K], § 42,III); hence \( f(K) \) is a subcontinuum of \( Z \) (it is non-degenerate because \( f(K) \cap z_1, z_2 \neq \emptyset \) and \( f(K) \cap (Z \setminus \{ z_1, z_2 \}) \neq \emptyset \). Since a subcontinuum of \( Z \) is mapped continuously onto a subcontinuum of \( Z \), necessarily \( Z' = Z \) and \( f(y) = y \) for all \( y \in K \). Hence \( f(z) = z \); but \( z \) was an arbitrarily chosen point of \( \mathcal{O} \), so that \( f(z) = z \) for all \( z \in Z \).

IV.11. Let \( T_1, \ldots, T_4 \) be as in IV.7.

Lemma. Let \( i, j \in \{ 1, \ldots, 4 \} \), let \( Y \) be a space containing \( T_i \) such that the boundary of \( T_i \) in \( Y \) consists of \( a^{(1)}_i, b^{(1)}_i, c^{(1)}_i \). Let \( f:T_j \to Y \) be a non-constant continuous map. Then either \( i = j \) and \( f(x) = x \) for all \( x \in T_j \) or \( f(T_j) \cap T_i \subseteq \{ a^{(1)}_i, b^{(1)}_i, c^{(1)}_i \} \).

Proof. Put \( \mathcal{U} = T_i \setminus \{ a^{(1)}_i, b^{(1)}_i, c^{(1)}_i \} \), \( \mathcal{O} = f^{-1}(\mathcal{U}) \).

(1) Let \( j \neq i \): If there is a building block \( Z \) of \( T_j \) with \( Z \cap \mathcal{O} \neq \emptyset \), \( f/Z \) must be constant, by IV.10; hence its image is a point \( y \in \mathcal{U} \). Then \( f \) maps all the building blocks, intersecting \( Z \), on \( y \) again; hence it maps all the building blocks, intersecting them, on \( y \) again. We conclude \( f \) maps the whole \( T_j \) on \( y \), which is a contradiction. Hence \( f(T_j) \cap \mathcal{U} = \emptyset \).

- 762 -
Let $i=j$: Let there be a building block $Z$ of $T_1$ with $Z \cap \mathcal{O} \neq \emptyset$. Then either $Z \subseteq \mathcal{O}$ and $f(z) = z$ for all $z \in Z$ or $f/Z$ is constant. Let us suppose that $f/Z$ is constant for a building block $Z$ with $Z \cap \mathcal{O} \neq \emptyset$, $f$ maps $Z$ on a point $y \in \mathcal{U}$. If $y$ is an interior point of a building block of $T_1$, we can repeat the argument of $(\alpha)$ and conclude that $f$ must be constant on $T_1$. Thus, let us suppose that $y$ is in $S_1$ (see IV.7). However, every building block $Z'$ of $T_1$ can be joined with $Z$ by a finite sequence $Z = Z_0, Z_1, \ldots , Z_n = Z'$ of building blocks such that $Z_{i-1}, Z_i \equiv \{z_1, \ldots , z_{n-1}\}$ is equal to $y$, so that all $Z_1, \ldots , Z_n = Z'$ must be mapped on $y$ again. We conclude that if $Z$ is a building block of $T_1$ such that $Z \cap \mathcal{O} \neq \emptyset$, then necessarily $Z \subseteq \mathcal{O}$ and $f(z) = z$ for all $z \in Z$. But if $\mathcal{O} \neq \emptyset$, it contains at least one building block of $T_1$, hence it contains all the building blocks which intersect it, etc. Thus, in this case, $f(z) = z$ for all $z \in T_1$.

IV.12. Let $T_{2,x}$, $x \in X$, be as in IV.9.

Lemma. Let $x, x' \in X$, let $Y$ be a space containing $T_{2,x}$ such that the boundary of $T_{2,x}$ in $Y$ consists of $a_x^{(2)}, b_x^{(2)}, c_x^{(2)}$. Let $f: T_{2,x} \to Y$ be a non-constant continuous map, then either $x = x'$ and $f(z) = z$ for all $z \in T_{2,x}$ or $f(T_{2,x}) \cap T_{2,x} \equiv \{a_x^{(2)}, b_x^{(2)}, c_x^{(2)}\}$.

Proof. Put $\mathcal{U} = Y \setminus \{a_x^{(2)}, b_x^{(2)}, c_x^{(2)}\}$, $\mathcal{O} = f^{-1}(\mathcal{U})$. By IV.11, any copy of $T_3$ or $T_4$ in $T_{2,x}$ which intersects $\mathcal{O}$, is mapped by $f$ either onto some of its copies in $T_{2,x}$ "as the identity" or $f$ is constant on it. However, if it is constant on it, it must be constant on the whole $C_x$ (the graph $(V_x, R_x)$ is connected!). Then it must be constant on the whole $T_{2,x}$ - the proof is analogous as in IV.11. Let us suppose that $f$ maps any copy of $T_3$ and of $T_4$ which intersects $\mathcal{O}$, onto some of its copies in $\mathcal{U}$ as the identity and that there is a copy of $T_3$ or $T_4$ in $T_{2,x}$ which intersects $\mathcal{O}$. Then every copy of $T_3$ and $T_4$ is in $\mathcal{O}$ (and $f$ maps them on some of their copies as the identity). Then necessarily there is a morphism $h:(V_x, R_x) \to (V_x, R_x)$ such that $f$ maps the $r$-th copy of $T_3$ (or $T_4$) onto its $r'$-th copy in $\mathcal{U}$, where $r(v_1, v_2) \in R_x$ and $r' = (h(v_1), h(v_2))$. Since $\mathcal{O}$ is a rigid collection of graphs, then necessarily $x = x'$ and $h$ is the identity, i.e. $f$ maps $C_x$ onto itself as the identity. Then necessarily $f(z) = z$ for all $z \in T_{2,x}$ - the rest of the proof is analogous as in IV.11.

IV.13. Lemma. Let $(V, R) \in \text{obj } G$ (or $(V, R, S) \in \text{obj } H$), let $Y = \Phi_2(V, R)$ (or $Y = \Phi_1(V, R, S)$). Let $f:P \to Y$ be a non-constant continuous map. Then there exists $r \in R$ such that $f(z) = z_r$ for all $z \in P$.

- 763 -
Proof. Any building block of $Y$ has the boundary in $Y$ consisting of two points. Hence, by IV.10, $f$ maps any building block $Z$ of $P$ on some of its copies in $Y$ as the identity or $f$ is constant on $Z$ or $f$ maps $Z$ into $Y \setminus \text{Int } Z$ for any copy $Z$ of $Z$ in $Y$. In the last case, $f/Z$ must be constant again. In fact, if we subtract from $Y$ all the interiors of all the building blocks of $Y$, the remaining subspace consists of components homeomorphic to the components of $mX$ (or of $X$ and $mX$) and one-point components. Since neither $X$ nor $mX$ contains a non-degenerate metrizable continuum and both $X$ and $mX$ are Hausdorff spaces (see II.4), $f(Z)$ must be a one-point set. Hence $f$ maps any building block $Z$ of $P$ either onto some of its copies in $Y$ as the identity or $f/Z$ is constant. If $f/Z$ is constant for some building block $Z$, then $f/T$ is constant for the triangle space $T$ (or the copy $T_{x,x}$) containing $Z$, by IV.11 (or by IV.12). Hence it is constant on the whole $P$. If $f$ maps every building block $Z$ of $P$ on some of its copies in $Y$ as the identity, then $f$ maps any triangle space on some of its copies as the identity, by IV.11, and it maps any $T_{x,x}$ in $P$ on some of its copies as the identity, by IV.12. Consequently there exists $r \in \mathbb{R}$ such that $f(z) = z$ for all $z \in P$.

Corollary. The statements a) and b) in IV.5 follow immediately from IV.13 because the identity maps $P \rightarrow P_A$ and $P \rightarrow mP_A$ are continuous.

References


[C] H. COOK: Continua which admit only the identity mapping onto non-degenerate subcontinua, Fund. Math. 60(1967), 241-249.


Mathematical Institute of Charles University, Sokolovská 83, 186 00 Praha 8, Czechoslovakia

(Oblatum 28.9. 1988)