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Hereditary κ -separability and the hereditary κ -Lindelöf property in function spaces

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Abstract. This paper is concerned with the smallest linear subspace $L_p(X)$ of $C_p C_p(X)$ containing the Tychonoff space X . It is proved that $L_p(X)$ is hereditarily κ -Lindelöf (hereditarily κ -separable, resp.) if and only if X^ω is hereditarily κ -Lindelöf (hereditarily κ -separable, resp.). Moreover, it is shown that a certain cardinal function of $L_p(X)$ called the weak pseudonet weight of $L_p(X)$ equals the net weight of X .

Keywords: Tychonoff space, function space, pointwise convergence, hereditary κ -separability, hereditary κ -Lindelöf property, weak pseudonet weight, net weight

Classification: 54A25, 54C35

Throughout this article, X denotes a Tychonoff space. The symbol $C_p(X)$ stands for the algebra of all continuous real-valued functions on X , with the topology of pointwise convergence. One easily sees that the formula

$$(1) \quad e(x)(f) = f(x),$$

where $x \in X$ and $f \in C_p(X)$, defines a homeomorphic embedding of X into $C_p C_p(X)$ (cf. [1; Proposition 3.5 (β), p.16]). Denote by $L_p(X)$ the smallest linear subspace of $C_p C_p(X)$ which contains $e(X)$ (cf. [1; p.17]).

It is known that $C_p C_p(X)$ is hereditary κ -Lindelöf (hereditary κ -separable, resp.) if and only if X^ω has that property (cf. [1; Corollary 3.28]). Our purpose is to prove that $C_p C_p(X)$ can be replaced by $L_p(X)$ in the preceding statement. Moreover, we shall show that the weak pseudonet weight of $L_p(X)$ is equal to the net weight of X .

Before proceeding to the body of the paper, let us introduce some notation and establish some useful facts.

In what follows, κ denotes an infinite cardinal number, and for simplicity, all cardinal functions will be infinite.

The smallest (infinite) cardinal number κ such that X is hereditarily κ -Lindelöf (hereditarily κ -separable, resp.) is denoted by $hl(X)$ ($hd(X)$, resp.).

As usual, $nw(X)$ denotes the net weight of X and $w(X)$ denotes the weight of X .

For each positive integer $n \geq 2$, let

$$\Delta_n = \{(x_1, \dots, x_n) \in X^n : x_i = x_j \text{ for some } i \neq j, j = 1, \dots, n\}.$$

Obviously,

$$(2) \quad \Delta_n = \cup \{\Delta_{ij}^n : 1 \leq i < j \leq n\},$$

where $\Delta_{ij}^n = \{(x_1, \dots, x_n) \in X^n : x_i = x_j\}$ (cf. [2; Definition 10.1]). One can readily observe that for $n \geq 2$, we have

$$(3) \quad \Delta_{ij}^n \text{ is homeomorphic to } X^{n-1}.$$

Let \sum_n be the set of all permutations of the numbers $1, \dots, n$. For $(x_1, \dots, x_n) \in X^n$, we define $q_n(x_1, \dots, x_n) = \{(x_{\sigma(1)}, \dots, x_{\sigma(n)}) : \sigma \in \sum_n\}$. Let

$$X_n^* = \{q_n(x_1, \dots, x_n) : (x_1, \dots, x_n) \in X^n\}$$

be considered with the quotient topology generated by q_n . Because

$$q_n^{-1}[q_n(U_1 \times \dots \times U_n)] = \cup\{U_{\sigma(1)} \times \dots \times U_{\sigma(n)} : \sigma \in \sum_n\},$$

where $U_i \subset X$ for $i = 1, \dots, n$, then

$$(4) \quad q_n : X^n \rightarrow X_n^* \text{ is a continuous open surjection.}$$

For $n \geq 2$, we define

$$(5) \quad Y_n = X_n^* \setminus q_n(\Delta_n),$$

$$(6) \quad Z_n = \{e(x_1) + \dots + e(x_n) \in L_p(X) : (x_1, \dots, x_n) \in X^n \setminus \Delta_n\},$$

$$(7) \quad \phi_n(e(x_1) + \dots + e(x_n)) = q_n(x_1, \dots, x_n), \text{ where } e(x_1) + \dots + e(x_n) \in Z_n.$$

Then since $e(X)$ is an algebraic base for $L_p(X)$ (cf. [1; Remark 3.14]), ϕ_n is a well-defined one-to-one mapping from Z_n onto Y_n . We shall show the following

8.Lemma. *The mapping $\phi_n : Z_n \rightarrow Y_n$ is a homeomorphism for each $n \geq 2$.*

PROOF: To begin with, we shall show that the mapping ϕ_n is continuous. Let us take $z = e(x_1) + \dots + e(x_n) \in Z_n$ and an open neighborhood V of $\phi_n(z)$ in Y_n . Because Y_n is an open subspace of X_n^* , then V is open in X_n^* . Since $(x_1, \dots, x_n) \notin \Delta_n$, it follows from the continuity of q_n that there exist open subsets U_i of X such that $x_i \in U_i$, where $U_i \cap U_j = \emptyset$ for $i \neq j$, $i, j = 1, \dots, n$ and $q_n(U_1 \times \dots \times U_n) \subset V$. Choose functions $f_i \in C_p(X)$ such that $f_i(x_i) = 1$ and $f_i(X \setminus U_i) = \{0\}$ for $i = 1, \dots, n$. From Proposition 3.10* given in [1;p.18], it follows that there exist continuous linear functionals $h_i : L_p(X) \rightarrow \mathbb{R}$ such that $h_i|_{e(X)} = f_i \circ e^{-1}$ for $i = 1, \dots, n$. Put

$$W = \cap\{h_i^{-1}(\mathbb{R} \setminus \{0\}) : i = 1, \dots, n\}.$$

Because $h_i(z) = h_i[e(x_1)] + \dots + h_i[e(x_n)] = f_i(x_1) + \dots + f_i(x_n) = 1$ for $i = 1, \dots, n$, then $z \in W$. To complete the proof of continuity of ϕ_n it suffices to check that $\phi_n(W \cap Z_n) \subset q_n(U_1 \times \dots \times U_n)$.

Let $z' \in W \cap Z_n$ and $z' = e(x'_1) + \dots + e(x'_n)$, where $(x'_1, \dots, x'_n) \in X^n \setminus \Delta_n$. Then $f_i(x'_1) + \dots + f_i(x'_n) = h_i(z') \neq 0$, so $f_i(x'_j) \neq 0$ for some $j = 1, \dots, n$. Hence, for each $i = 1, \dots, n$ there exists $j = 1, \dots, n$ such that $x'_j \in U_i$ so that $(x'_1, \dots, x'_n) \in \cup\{U_{\sigma(1)} \times \dots \times U_{\sigma(n)} : \sigma \in \sum_n\}$. Thus, $\phi_n(z') = q_n(x'_1, \dots, x'_n) \in q_n(U_1 \times \dots \times U_n)$.

Now we shall show that ϕ_n is an open mapping. Let U be an open set in $L_p(X)$ and $y \in \phi_n(U \cap Z_n)$. Take a point $z = e(x_1) + \dots + e(x_n) \in U \cap Z_n$ such that $y = \phi_n(z)$. We can choose open sets $U_i \subset X$ such that $x_i \in U_i$, where $U_i \cap U_j = \emptyset$ for $i \neq j$ ($i, j = 1, \dots, n$) and $e(U_1) + \dots + e(U_n) \subset U \cap Z_n$. Put $V = \phi_n(e(U_1) + \dots + e(U_n))$. Because $V = q_n(U_1 \times \dots \times U_n)$, then by (4), V is open in Y_n . Moreover, $y \in V \subset \phi_n(U \cap Z_n)$, so ϕ_n is an open mapping. ■

Now we are a position to prove the main theorems of this paper.

9.Theorem. *If X is a Tychonoff space, then $hl[L_p(X)] = hl(X^\omega)$.*

PROOF: Denote $\kappa = hl[L_p(X)]$. By virtue of [6; Theorem 3, p.177], to prove that $hl(X^\omega) \leq \kappa$, it suffices to show that $hl(X^n) \leq \kappa$ for each positive integer n .

Clearly, $hl(X) \leq \kappa$. Assume that $hl(X^{n-1}) \leq \kappa$ for some $n \geq 2$. Because $Z_n \subset L_p(X)$, then $hl(Z_n) \leq \kappa$; hence, Lemma 8 yields that $hl(Y_n) \leq \kappa$. Let U be an open subset of $X^n \setminus \Delta_n$. Then, by (4), the mapping $p_n = q_n|_U$ is open, continuous and, moreover, all fibers of p_n have cardinalities less than or equal to $n!$. For $i = 1, 2, \dots, n!$, put $V_i = \{y \in p_n(U) : p_n^{-1}(y) = i\}$ and $U_i = p_n^{-1}(V_i)$. Applying [3; Proposition 2.1.4, p.95] and the theorem given in [3; Problem 4.4D(b), p.358], we deduce that $p_n|_{U_i}$ is a perfect mapping from U_i onto V_i , so U_i is κ -Lindelöf for $i = 1, \dots, n!$ (cf. [3; Theorem 3.8.9, p.248]). As $U = \bigcup_{i=1}^{n!} U_i$, then U_i is κ -Lindelöf. Hence, $hl(X^n \setminus \Delta_n) \leq \kappa$. Now, using (2)–(3) and our assumption that $hl(X^{n-1}) \leq \kappa$, we deduce that $hl(\Delta_n) \leq \kappa$; thus $hl(X^n) \leq \kappa$.

As $L_p(X) \subset C_p C_p(X)$, the inequality $hl[L_p(X)] \leq hl(X^\omega)$ is an immediate consequence of [6; Theorem 6, p.178] ■

10.Theorem. *If X is a Tychonoff space, then $hd[L_p(X)] = hd(X^\omega)$.*

PROOF: Put $\kappa = hd[L_p(X)]$ and fix a positive integer $n \geq 2$. Similarly as in the proof of Theorem 9, we show that $hd(Y_n) \leq \kappa$. Let $M \subset X^n \setminus \Delta_n$ and let A_0 be a dense subset of $q_n(M) \subset Y_n$ such that $|A_0| \leq \kappa$; then by (4) we have that

$$\overline{q_n^{-1}(A_0)} \cap q_n^{-1}[q_n(M)] = q_n^{-1}(\overline{A_0}) \cap q_n^{-1}[q_n(M)] = q_n^{-1}[q_n(M)]$$

(cf. [3; Exercise 1.4.C, p.57]); so $q_n^{-1}(A_0)$ is dense subset of $q_n^{-1}[q_n(M)]$. Put $D_0 = q_n^{-1}(A_0)$ and $M_1 = \overline{D_0} \cap M \cap M$. Let us choose a set $A_1 \subset q_n(M \setminus M_1)$ such that A_1 is dense in $q_n(M \setminus M_1)$ and such that $|A_1| \leq \kappa$. Put $D_1 = D_0 \cup q_n^{-1}(A_1)$ and $M_2 = \overline{D_1} \cap M \cap M$. We can inductively define sets A_i, D_i, M_i such that $|A_i| \leq \kappa$, A_i is a dense subset of $q_n(M \setminus M_i)$, $D_i = D_{i-1} \cup q_n^{-1}(A_i)$ and $M_i = \overline{D_{i-1}} \cap M \cap M$. The set $D = D_{n-1} \cap M$ is of cardinality $\leq \kappa$. To show that D is dense in M , we need the following notation:

For $x = (x_1, \dots, x_n) \in X^n$, $U = U_1 \times \dots \times U_n \subset X^n$ and $\sigma \in \sum_n$, let $x^\sigma = (x_{\sigma(1)}, \dots, x_{\sigma(n)})$ and $U^\sigma = U_{\sigma(1)} \times \dots \times U_{\sigma(n)}$. Denote by $\mathcal{B}(x)$ the family of all open sets of the form $U_1 \times \dots \times U_n \subset X^n$ which contain x . Finally, let $id \in \sum_n$ be the identity permutation.

Suppose that the set D is not dense in M . Then there exists $x \in M$ and $U \in \mathcal{B}(x)$ such that

$$(d_0) \quad U \cap D = \emptyset.$$

Because $x \in M \setminus M_{n!-1}$ and the set $q_n^{-1}(A_{n!-1})$ is dense in $q_n^{-1}[q_n(M \setminus M_{n!-1})]$, then there exist ${}_1y, \sigma_1 \in \sum_n$ and ${}_1V \in \mathcal{B}({}_1y)$ such that

- (a₁) ${}_1V \subset U,$
- (b₁) ${}_1y \in q_n^{-1}(A_{n!-1}),$
- (c₁) ${}_1y^{\sigma_1} \in M \setminus M_{n!-1} \subset M \setminus M_{n!-2},$
- (d₁) ${}_1V^{\sigma_1} \cap D_{n!-2} \cap M = \emptyset.$

It follows from (a₁), (b₁) and (d₀) that $\sigma_1 \neq id$. As the set $q_n^{-1}(A_{n!-2})$ is dense in $q_n^{-1}[q_n(M \setminus M_{n!-2})]$, then we obtain from (c₁) that there are ${}_2y, \sigma_2 \in \sum_n$ and ${}_2V \in \mathcal{B}({}_2y)$ such that

- (a₂) ${}_2V \subset {}_1V,$
- (b₂) ${}_2y^{\sigma_1} \in q_n^{-1}(A_{n!-2}),$
- (c₂) ${}_2y^{\sigma_2} \in M \setminus M_{n!-2} \subset M \setminus M_{n!-3},$
- (d₂) ${}_2V^{\sigma_2} \cap D_{n!-3} \cap M = \emptyset.$

It follows from (d₀), (a₁), (d₁), (a₂), (b₂) that $\sigma_2 \in \sum_n \setminus \{id, \sigma_1\}$. Arguing similarly, in the i -th step, we obtain ${}_iy, \sigma_i \in \sum_n$ and ${}_iV \in \mathcal{B}({}_iy)$ such that

- (a_i) ${}_iV \subset {}_{i-1}V,$
- (b_i) ${}_iy^{\sigma_{i-1}} \in q_n^{-1}(A_{n!-i}),$
- (c_i) ${}_iy^{\sigma_i} \in M \setminus M_{n!-i} \subset M \setminus M_{n!-i-1},$
- (d_i) ${}_iV^{\sigma_i} \cap D_{n!-i-1} \cap M = \emptyset.$

From all the conditions (d₀), (a_j), (d_j) ($1 \leq j \leq i$) and from (a_i), (b_i) it follows that $\sigma_i \in \sum_n \setminus \{id, \sigma_1, \dots, \sigma_{i-1}\}$. As ${}_{n!-1}y^{\sigma_{n!-1}} \in M$ and the set $q_n^{-1}(A_0)$ is dense in $q_n^{-1}[q_n(M)]$, then there is ${}_ny$ such that ${}_ny^{\sigma_{n!-1}} \in {}_{n!-1}V^{\sigma_{n!-1}} \cap q_n^{-1}(A_0)$. For some $\sigma_{n!} \in \sum_n$, we have that ${}_ny^{\sigma_{n!}} \in M$. From the conditions (a_i), (d_i) it follows that $\sigma_{n!} \neq \sigma_i$ for $1 \leq i \leq n!-1$. Because $\sigma_i \neq \sigma_j$ for $i \neq j$ ($1 \leq i, j \leq n!-1$), then $\sigma_{n!} = id$ which contradicts (d₀). Hence, the set D is dense in M and $hd(X^n \setminus \Delta_n) \leq \kappa$. Arguing as in the proof of Theorem 9, we can inductively show that $hd(X^n) \leq \kappa$ for each positive integer n . By applying Theorem 3* of [6;p.177], we obtain that $hd(X^\omega) \leq \kappa$.

The inequality $hd[L_p(X)] \leq hd(X^\omega)$ follows from [6; Theorem 6, p.179] ■

The preceding two theorems imply

11. Corollary. *If X is a Tychonoff space, then $hl[C_p C_p(X)] = hl(X^\omega)$ and $hd[C_p C_p(X)] = hd(X^\omega)$.*

The above corollary is a slight modification of Corollary 3.28 given in [1; p.38], but with a different proof.

12. Corollary. *If X is a Tychonoff space, then $hl[L_p(X)] = hl[C_p C_p(X)]$ and $hd[L_p(X)] = hd[C_p C_p(X)]$.*

13. Corollary. *If X is a locally compact Hausdorff space, then $hl[L_p(X)] = w(X)$.*

PROOF: If ωX is the one-point compactification on X , then $hl(X \times X) = hl(\omega X \times \omega X) \geq \Delta(\omega X) = w(\omega X) = w(X)$ where $\Delta(\omega X)$ denotes the diagonal degree of ωX (cf. [4; p.16 and Corollary 7.6, p.27]). This implies that $hl(X \times X) = w(X)$, so that $hl(X^\omega) = w(X)$. Theorem 9 completes the proof. ■

14. Corollary. *If X is a locally compact Hausdorff space, then $hl[C_p C_p(X)] = w(X)$.*

15. Corollary. *If X is a locally compact Hausdorff space, then $hd[C_p(X)] = w(X)$.*

PROOF: Inasmuch as $nw[C_p(X)] = nw(X) = w(X)$ (cf. [1; Theorem 1.1, p.24] and [3; Theorem 3.3.5, p.197]), then $hd[C_p(X)] \leq w(X)$.

The inequality $w(X) \leq hd[C_p(X)]$ follows from Corollary 3.26 given in [1; p.37] and the fact that $hl(X^2) = w(X)$. ■

16. Definition. A family ε of subsets of a space X will be called a *weak κ -pseudonet* in X if and only if $|\varepsilon| \leq \kappa$. and for each open set $U \subset X$, there exists a set $A \subset U$ such that $|A| \leq \kappa$ and for each $x \in U \setminus A$, there exists $E \in \varepsilon$ such that $x \in E$ and $|E \setminus U| \leq \kappa$. The cardinal number.

$$wpn(X) = \min\{\kappa \geq \omega : \text{there exists a weak } \kappa\text{-pseudonet in } X\}$$

will be called the weak pseudonet weight of X .

17. Theorem. *If X is a Tychonoff space, then $wpn[L_p(X)] = nw(X)$.*

PROOF: Because $nw(X) = nw[C_p C_p(X)]$ (cf. [Theorem 1.1, p.24]), it follows that $wpn[L_p(X)] \leq nw(X)$.

Denote $\kappa = wpn[L_p(X)]$ and consider the spaces Y_2, Z_2 defined in (5)–(6). By Lemma 8, Y_2 is homeomorphic to Z_2 , so $wpn(Y_2) = wpn(Z_2) \leq \kappa$. Let ε be a weak κ -pseudonet in Y_2 . It is easy to observe that $|q_2^{-1}(B)| \leq \kappa$ for each $B \subset Y_2$ such that $|B| \leq \kappa$. This—together with (4)—implies that the family $\{q_2^{-1}(E) : E \in \varepsilon\}$ is a weak κ -pseudonet in $X^2 \setminus \Delta_2$; hence $wpn(X^2 \setminus \Delta_2) \leq \kappa$. As Δ_2 is homeomorphic to X , we have $wpn(\Delta_2) = wpn(X) \leq wpn[L_p(X)] = \kappa$ so that $wpn(X^2) \leq \kappa$. As $nw(X) = wpn(X^2)$ (cf. [5; Corollary 1.12]), we have $nw(X) \leq \kappa$. ■

18. Corollary. *If X is a Tychonoff space, then $wpn[C_p C_p(X)] = nw(X)$.*

We do not know if Corollary 18 can be strengthened to say that $wpn[C_p(X)] = nw(X)$.

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