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Abstract. We give a short proof of the fact that a connected, simply connected and complete Riemannian 3–manifold which is curvature–homogeneous up to order 1 is a homogeneous Riemannian space

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Let (M,g) be a connected Riemannian manifold, and denote by $R, \nabla R, \dots, \nabla^k R, \dots$ the curvature tensor of M and its successive covariant derivatives. I.M. Singer [5] has considered the following condition $P(n)$ for each integer $n \geq 0$.

P(n): For every $x, y \in M$, there is a linear isometry Φ of T_xM onto T_yM such that $\Phi^*(\nabla^k R)_x = (\nabla^k R)_y$ for $k = 0, 1, \dots, n$.

Further, for any point $x \in M$, and any $s \geq 0$, he defines the Lie algebra $\underline{G}^x_s$ of all skew–symmetric endomorphisms of the tangent space T_xM which annihilate, as derivations of the tensor algebra, all tensors $R_x, (\nabla R)_x, \dots, (\nabla^s R)_x$. Then there exists the first integer k_x such that $\underline{G}^x_{k_x} = \underline{G}^x_{k_x+1}$. If now the condition $P(k_x+1)$ holds, then the number k_x is independent of the choice of $x \in M$, and one can put $k_M = k_x$. The main result by I.M.Singer is then the following

Theorem. *If (M,g) is a connected, simply connected, complete Riemannian manifold which satisfies the condition $P(k_M+1)$, then it is Riemannian homogeneous.*

The general estimate for the number k_M from above is given by $k_M \leq \frac{n(n-1)}{2} - 1$, where $n = \dim M$.

On the other hand, non–homogeneous Riemannian manifolds are known which satisfy the condition $P(0)$ (so–called curvature homogeneous spaces). Such non–homogenous examples (in dimensions $n = 3, 4$) have been first discovered in subsequent papers by K.Sekigawa [3] and H.Takagi [6], and many new examples have been found since that time. Yet, non–homogeneous examples satisfying the next condition $P(1)$ are not known, so far. K.Sekigawa has proved in another paper [4] that such examples do not exist in the dimension $n = 3$:

Theorem. *Let (M,g) be a 3-dimensional connected, simply connected and complete Riemannian manifold satisfying the condition $P(1)$. Then* a) *(M,g) is homogeneous,* b) *(M,g) is either a symmetric space, or (M,g) is a group space with a left–invariant metric.*

Here the general Singer's estimate only says that the condition $P(3)$ implies homogeneity. Thus, the Sekigawa's theorem provides a strengthening of the Singer's theorem in a special situation.

The original proof by Sekigawa is rather long, because the proof of the homogeneity is closely connected with the classification. The purpose of this Note is to give a short and direct proof of the homogeneity part only. (The higher dimensions $n \geq 4$ are also discussed in this context).

Let (M, g) be given as in the Sekigawa's theorem. Because the Weyl curvature tensor C vanishes identically for $\dim M = 3$, the condition $P(1)$ is equivalent to the following condition

P'(1): For every $x, y \in M$, there is a linear isometry Φ of T_xM onto T_yM such that $\Phi^*(\rho_x) = \rho_y, \Phi^*(\nabla\rho)_x = (\nabla\rho)_y$, where ρ denotes the Ricci tensor and $\nabla\rho$ its covariant derivative.

Using paragraph 2 in [5], we obtain easily the following

Lemma. *If $P'(1)$ is satisfied, then there exists a maximal principal subbundle F^b of the orthogonal frame bundle $0(M, g) \to M$ on which the functions $\rho_{ij}, \nabla_k\rho_{ij} (i, j, k = 1, \dots, n)$ are constants and which contains a given frame $b \in 0(M, g)$. Moreover, the structure group of F^b is a connected Lie group with the Lie algebra $\underline{G}_1^x$ ($x \in M$ being arbitrary).*

Let $x \in M$ be fixed and let $b = (e_1, e_2, e_3)$ be an orthonormal frame at x consisting of eigenvectors of the Ricci tensor. This means that all frames $c \in F^b$ consist of eigenvectors of the Ricci tensor as well, and that the Ricci roots $\lambda_1, \lambda_2, \lambda_3$ are constant on F^b and hence on M.

Now, we shall distinguish 3 cases:

1) All Ricci roots $\lambda_1, \lambda_2, \lambda_3$ are different. Then for each $x \in M$ we see that $\underline{G}_0^x = (0) = \underline{G}_1^x$, i.e., $k_M = 0$. Because $P(1)$ is satisfied, (M, g) is homogeneous according to the Singer's theorem.
2) All Ricci roots are equal. Then (M, g) is Einsteinian and hence a space form.
3) We have $\lambda_1 = \lambda_2 \neq \lambda_3$. Then $\underline{G}_0^x = \underline{so}(2)$ for each $x \in M$, and we can distinguish two cases.

3a): $\underline{G}_0^x = \underline{G}_1^x$ and $k_M = 0$. Here we can use the Singer's theorem once more. It remains to settle the only non-trivial case

3b): $\underline{G}_0^x = \underline{so}(2), \underline{G}_1^x = (0)$ for all $x \in M$, i.e., $k_M = 1$.

In the last case, the fibre bundle F^b is just a global section of $0(M, g)$ over M (because its connected structure group has the Lie algebra $\underline{G}_1^x = (0)$). We put $F^b = (E_1, E_2, E_3)$ on M, and we shall use this global orthonormal frame in the rest of the proof.

Next, let us introduce the functions B_{ij}^k on M by

$$\nabla_{E_i} E_j = \sum_k B_{ij}^k E_k \qquad (i, j = 1, 2, 3). \tag{1}$$

Using the obvious skew-symmetry

$$B_{ri}^j + B_{rj}^i = 0 \tag{2}$$

we get easily

$$\nabla_r \rho_{ij} = (\lambda_j - \lambda_i) B_{ri}^j \qquad (i, j, k = 1, 2, 3). \tag{3}$$

From (3) we see that B^3_{r1} and B^3_{r2} are constant functions on M for $r = 1,2,3$. We want to show that the remaining functions B^2_{r1} are also constant on M.

According to the classical formula for the curvature (see [2]) we have

$$\sum_u [B^u_{jk}B^v_{iu} - B^u_{ik}B^v_{ju} + (B^u_{ji} - B^u_{ij})B^v_{uk}] + E_i(B^v_{jk}) - E_j(B^v_{ik}) = R_{jivk}.$$

For $v = 3$, the functions B^v_{jk} and B^v_{ik} are constant and our formula is reduced to

$$\sum_u [B^u_{jk}B^3_{iu} - B^u_{ik}B^3_{ju} + (B^u_{ji} - B^u_{ij})B^3_{uk}] = R_{jik3}. \tag{4}$$

Now, for $(i,j,k) = (1,2,1)$ and $(i,j,k) = (1,2,2)$ we get

$$B^2_{21}(B^3_{12} + B^3_{21}) - B^2_{11}(B^3_{22} - B^3_{11}) = B^3_{31}(B^3_{12} - B^3_{21}) + R_{2123}, \tag{5}$$

$$B^2_{21}(B^3_{22} - B^3_{11}) + B^2_{11}(B^3_{12} + B^3_{21}) = B^3_{32}(B^3_{12} - B^3_{21}) + R_{2123}. \tag{6}$$

For $(i,j,k) = (1,3,1)$ and $(i,j,k) = (2,3,1)$ we get

$$B^2_{31}(B^3_{12} + B^3_{21}) = -B^2_{11}B^2_{33} - (B^1_{33})^2 - (B^3_{11})^2 - B^3_{12}B^3_{21} + R_{3113}, \tag{7}$$

$$B^2_{31}(B^3_{22} - B^3_{11}) = B^2_{33}(B^1_{22} - B^1_{33}) + B^1_{23}(B^3_{11} + B^3_{22}) + R_{3213}. \tag{8}$$

Now, the condition $\underline{G}^x_1 = (0)$ means that the tensor $\nabla_i\rho_{jk}$ is not invariant with respect to the group SO(2) (acting on the subspace $\mathrm{span}(E_1,E_2)_x \subset T_xM$ at each $x \in M$). This implies

$$(\nabla_1\rho_{23} + \nabla_2\rho_{13} \neq 0) \vee (\nabla_2\rho_{23} - \nabla_1\rho_{13} \neq 0), \tag{9}$$

and from (3) we get

$$(B^3_{12} + B^3_{21} \neq 0) \vee (B^3_{22} - B^3_{11} \neq 0). \tag{10}$$

This means that the system of non-homogeneous linear equations (5), (6) with constant coefficients for the unknowns B^2_{21} and B^2_{11} has a non-zero determinant, and hence we derive that B^2_{21} and B^2_{11} are constant. But then the right-hand sides of (7) and (8) are constant and the function B^2_{31} can be calculated from one of these equations as a constant, as well.

Now, define a tensor field T of type $(1,2)$ on M by the formula

$$T_{E_i}E_j = \sum_k B^k_{ij}E_k, \tag{11}$$

and define a new connection $\tilde{\nabla} = \nabla - T$ on M. Then $\hat{\nabla}_{E_i}E_j = 0$ for $i,j = 1,2,3$, and hence, because B^k_{ij} are constants, we get $\hat{\nabla}T = 0$. Also $\hat{\nabla}R = 0$ holds because $R(E_i,E_j)E_k$ are constants. Now, the homogeneity of (M,g) follows from the Ambrose–Singer theorem (see [1] or [8]).

Note. A natural generalization of the Sekigawa's result could be expected for spaces of higher dimensions with the vanishing Weyl tensor, $C = 0$. For $n \geq 4$, the space (M, g) is then conformally flat. A theorem by H.Takagi (see[7], Theorem A) implies that any homogeneous conformally flat Riemannian manifold is locally symmetric. A careful but routine inspection of the Takagi's proof shows that it remains valid for those conformally flat spaces which are only curvature homogeneous. Hence we obtain the following result:

Let (M, g) be a connected and simply connected complete Riemannian manifold which is conformally flat and satisfies the condition $P(0)$. Then (M, g) is homogeneous.

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