Jarosław Górnicki

Weak convergence theorems for asymptotically nonexpansive mappings in uniformly convex Banach spaces

Commentationes Mathematicae Universitatis Carolinae, Vol. 30 (1989), No. 2, 249--252

Persistent URL: http://dml.cz/dmlcz/106742

Weak convergence theorems for asymptotically nonexpansive mappings in uniformly convex Banach spaces

JAROSŁAW GÓRNICKI

Abstract. In the present note, using specific uniformly convex Banach spaces techniques of asymptotic center we consider a necessary and sufficient condition for the weak convergence of trajectories of asymptotically nonexpansive mappings. The main result of this paper is contained in the following Theorem: Let E be a uniformly convex Banach space satisfying the Opial's condition, C a closed convex subset of $E, T : C \to C$ an asymptotically nonexpansive mapping and $x \in C$. Then $\{T^n x\}$ converges weakly to a fixed point of T iff $T^{n+1}x - T^n x \rightharpoonup 0$ as $n \to +\infty$

Keywords: Uniformly convex Banach space, asymptotic center, Opial's condition, asymptotically nonexpansive mapping, fixed point, asymptotic regularity.

Classification: 47H09, 47H10

1. Preliminaries and notations. Let E be a uniformly convex Banach space (see e.g. [5]), $\{x_n\}$ be a bounded sequence in E and let C be a closed convex subset of E. Consider the functional

$$r : E \to [0, +\infty)$$

defined by

$$r(x) = \overline{\lim_{n\to\infty}} \|x_n - x\|, \quad x \in E.$$

The infimum of $r(\cdot)$ over C is called asymptotic radius of $\{x_n\}$ with respect to C and is denoted by $r(C, \{x_n\})$. A point z in C is said to be an <u>asymptotic center</u> of the sequence $\{x_n\}$ with respect to C if

$$r(z) = \min[r(x) : x \in C].$$

The set of all asymptotic centers is denoted by $A(C, \{x_n\})$.

Lemma 1. *[5]. Every bounded sequence $\{x_n\}$ in a uniformly convex Banach space E has a unique asymptotic center with respect to any closed convex subset C of E, i.e. $A(C, \{x_n\}) = \{z\}$ and*

$$\bigwedge_{x\neq z} \overline{\lim_{n\to\infty}} \|x_n - x\| < \overline{\lim_{n\to\infty}} \|x_n - x\|.$$

Lemma 2. *[2]. Let $\{x_n\}$ be a bounded sequence in a closed convex subset C of a uniformly convex Banach space E, and $A(C, \{x_n\}) = \{z\}$. Then*

$$(\{y_m\} \subset C \text{ and } r(y_m) \to r(C, \{x_n\}) \text{ as } m \to +\infty) \Rightarrow (y_m \to z \text{ as } m \to \infty).$$

The weak convergence of sequence will be denoted by $x_n \rightharpoonup x$, while the strong convergence by $x_n \to x$. The set of fixed points of a mapping T will be denoted by $F(T)$.

2. A fixed point theorem. Let E be a Banach space and $C \subset E$. A mapping $T : C \to C$ is called asymptotically nonexpansive on C [4] if there exists a sequence $\{k_i\}$ of real constants such that $k_i \downarrow 1$ as $i \to +\infty$ and for which

$$\|T^i x - T^i y\| \leq k_i \cdot \|x - y\|, \quad x, y \in C, i = 1, 2, \ldots$$

Thus every nonexpansive mapping is asymptotically nonexpansive and the class of asymptotically nonexpansive mappings is essentially wider than the class of nonexpansive mappings [4].

Theorem 1. *Let C be a closed convex (but not necessarily bounded) subset of a uniformly convex Banach space. If an asymptotically nonexpansive mapping, then the following statements are equivalent:*

(a) *T has a fixed point;*
(b) *There is a point $x_0 \in C$ such that the sequence of iterates $\{T^n x_0\}$ is bounded;*
(c) *There is a bounded sequence $\{y_n\} \subset C$ such that*

$$\lim_{n \to \infty} \|y_n - T y_n\| = 0.$$

PROOF : (a)$\Rightarrow$ (b) and (a)$\Rightarrow$ (c) follow easily. (b)$\Rightarrow$(a). Assume $x_0 \in C$ is such that the sequence $\{x_n = T^n x_0\}$ is bounded, and let $A(C, \{x_n\}) = \{z\}$. Let $\{y_m = T^m z\}$. We shall show

$$r(y_m) = \overline{\lim_{n \to \infty}} \|x_n - y_m\| \to r(C, \{x_n\}) \quad \text{as } m \to +\infty.$$

By Lemma 2, this would imply $y_m \to z$ as $m \to +\infty$, and because T is continuous

$$Tz = T(\lim_{n \to \infty} T^m z) = \lim_{n \to \infty} T^{m+1} z = z.$$

For two integers $n > m \geq 1$ we have

$$\begin{aligned} \|x_n - y_m\| = \|T^m x_{n-m} - T^m z\| \leq \\ \leq k_m \cdot \|x_{n-m} - z\|, \end{aligned}$$

and

$$r(y_m) \leq k_m \cdot r(z), \text{ where } k_m \downarrow 1 \text{ as } m \to +\infty.$$

This shows that $r(y_m) \to r(C, \{x_n\})$ as $m \to +\infty$. (c)$\Rightarrow$(a). Let $\{y_n\}$ be a bounded sequence such that $\lim_{n \to \infty} \|y_n - Ty_n\| = 0$ and $A(C, \{y_n\}) = \{z\}$. We consider a sequence $\{x_m = T^m z\}$. For two integers $m, n \geq 1$ we have

$$\begin{aligned} \|x_m - y_n\| \leq \|T^m z - T^m y_n\| = \|T^m y_n - T^{m-1} y_n\| + \cdots + \|T y_n - y_n\| \leq \\ \leq k_m \cdot \|z - y_n\| + \|T y_n - y_n\| \cdot (k_{m-1} + k_{m-2} + \cdots + k_1 + 1). \end{aligned}$$

Thus

$$r(x_m) \leq k_m \cdot r(z), \text{ where } k_m \downarrow 1 \text{ as } m \to +\infty.$$

This shows that $r(x_m) \to r(C, \{y_n\})$ as $m \to +\infty$. By Lemma 2 and by the continuity of T, z will be the fixed point of T. ∎

Remark 1. Recently the present author and M.Krüppel proved (b)$\Rightarrow$(a) in a more general situation, see the paper: Fixed points of uniformly Lipschitzian mappings, Bull.Polish Acad.Sci., Math.(in Print).

3. Banach spaces satisfying the Opial's condition. We say that a Banach space E satisfies the Opial's condition [8] if for each sequence $\{x_n\} \subset E$ weakly convergent to a point x, and for all $y \neq x$

$$\underline{\lim}_{n\to\infty} \|x_n - x\| < \underline{\lim}_{n\to\infty} \|x_n - y\|. \tag{1}$$

It is known that (1) is equivalent to the analogous condition obtained by replacing $\underline{\lim}$ with $\overline{\lim}$.

Examples of Banach spaces which satisfy the Opial's condition are Hilbert spaces and all spaces $l^p (1 < p < +\infty)$. On the other hand $L^p[0, 2\pi]$ with $1 < p \neq 2$ fails to satisfy the Opial's condition [8].

Lemma 3. *Let C be a closed convex subset of a uniformly convex Banach space satisfying the Opial's condition. If a sequence $\{x_n\} \subset C$ converges weakly to a point x, then x is the asymptotic center of $\{x_n\}$ in C.*

Lemma 4. *(Demiclosedness principle). Let E be a uniformly convex Banach space satisfying the Opial's condition, C a closed convex subset of E, $T : C \to C$ an asymptotically nonexpansive mapping. If $\{x_n\} \subset C$ is a weakly convergent sequence with the weak limit x and if $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then $Tx = x$.*

PROOF : From Lemma 3 the asymptotic center of $\{x_n\}$ in C is x. We consider a sequence $\{y_m = T^m x\}$ and analogously as above $x \in F(T)$. ■

Remark 2. Recently M.Krüppel [7], using a more complicate method, proved that the demiclosedness principle is true in any uniformly convex Banach space for asymptotically nonexpansive mappings.

Lemma 5. *[1]. Let C be a closed convex subset of a uniformly convex Banach space satisfying the Opial's condition and $T : C \to C$ an asymptotically nonexpansive mapping. Suppose z is the asymptotic center of the bounded sequence $\{T^n x\}$ for some $x \in C$. If the weak limit x_0 of a subsequence $\{T^{n_i} x\}$ is a fixed point of T, then x_0 coincides with z.*

The concept of asymptotic regularity is due to Browder and Petryshyn [3]: a mapping $T : C \to C$ is said to be <u>(weakly) asymptotically regular</u> at $x \in C$ if $T^{n+1}x - T^n x \to 0$ (weakly) as $n \to +\infty$.

The next Theorem generalizes the result of I.Miyadera (see [6, Theorem 3.1.]).

Theorem 2. *Let E be a uniformly convex Banach space satisfying the Opial's condition and C be a closed convex (but not necessarily bounded) subset of E, and $T : C \to C$ is a asymptotically nonexpansive mapping, $x \in C$. Then $\{T^n x\}$ converges weakly to a fixed point of T iff T is weakly asymptotically regular at x.*

PROOF : Let us assume that $T^n x \rightharpoonup p$ as $n \to \infty$. We can show that $p \in F(T)$. By Lemma 3, $A(C, \{T^n x\}) = \{p\}$ and analogously as in Theorem 1, $p \in F(T)$. From

$T^n x \rightharpoonup p$ as $n \to +\infty$ we get $T^{n+1}x - T^n x \rightharpoonup 0$ as $n \to +\infty$. Now we are going to show the implication in the opposite way. From the assumption $T^{n+1}x - T^n x \rightharpoonup 0$ as $n \to +\infty$ we have $T^{(n_i+m)}x \rightharpoonup y$ as $i \to +\infty$ for $m = 0, 1, \ldots$. By Lemma 3, $A(C, \{T^{(n_i+m)}x\} = \{y\}$ for $m = 0, 1, 2, \ldots$. Let $\{y_s = T^s y\}$. For integers $m > s \geq 1$ we have

$$\|y_s - T^{n_i+m}x\| = \|T^s y - T^s(T^{n_i+m-s}x)\| \leq$$
$$\leq k_s\|y - T^{n_i+m-s}x\|$$

and

$$r(y_s) \leq k_s \cdot r(y), \text{ where } k_s \downarrow 1 \text{ as } s \to +\infty.$$

By Lemma 2, $T^s y \to y$ as $s \to +\infty$ and by the continuity of T, $T_y = y$. Let $\omega_w(x)$ denote the set of weak limits of subsequences of a sequence $\{T^n x\}$. From this part of the proof we get $\omega_w(x) \subset F(T)$. By Lemma 5, the point $y \in \omega_w(x)$. This proves that $\omega_w(x) = \{z\}$, so $T^n x \rightharpoonup z$ as $n \to +\infty$. Thus the proof is complete. ∎

Theorem 2 gives the following Corollary analogously to the result of Opial [8, Theorem 2] and Bose [**1**].

Corollary 1. *Let C be a closed convex subset of a uniformly convex Banach space E satisfying the Opial's condition. Assume that $T : C \to C$ is an asymptotically nonexpansive, weakly asymptotically regular and $F(T) \neq \emptyset$. Then for any $x \in C$, the sequence of iterates $\{T^n x\}$ is weakly convergent to a fixed point of T.*

Problem. Is Theorem 2 true in all Banach spaces satisfying the Opial's condition? (For a nonexpansive mapping see: T.Kuczumov, Proc.Amer.Math.Soc. 93(1985), 430-432).

References

[1] Bose S.C., *Weak convergence to the fixed point of an asymptotically nonexpansive map*, Proc.Amer.Math.Soc. **68** (1978), 305-308.

[2] Bose S.C., Laskar S.K., *Fixed point theorems for certain class of mappings*, Journ.Math.Phys. Sci. **19** (1985), 503-509.

[3] Browder F.E., Petryshyn W.V., *The solution by iteration of nonlinear functional equations in Banach spaces*, Bull.Amer.Math.Soc. **72** (1966), 671-575.

[4] Goebel K., Kirk W.A., *A fixed point theorem for asymptotically nonexpansive mappings*, Proc.Amer.Math.Soc. **35** (1972), 171-174.

[5] Goebel K., Reich S., "Uniform convexity, Hyperbolic geometry and Nonexpansive mappings," Marcel Dekker, INC., New York, Basel, 1984.

[6] Hirano N., *Nonlinear ergodic theorems and weak convergence theorems*, J.Math.Soc.Japan **34** (1982), 35-46.

[7] Krüppel M., "Mean ergodic theorems for asymptotically nonexpansive mappings in uniformly convex Banach spaces," [unpublished manuscript].

[8] Opial Z., *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull.Amer.Math.Soc. **73** (1967), 591-597.

Institute of Mathematics, Pedagogical University of Rzeszów, Rejtana 16 A, 35-310 Rzeszów, Poland

(*Received* January 30,1989)