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Abstract. It is shown that a series of compact operators between Banach spaces is subseries convergent with respect to the weak operator topology iff it is subseries convergent with respect to the norm topology provided the domain space does not contain a copy of ℓ^∞.

Keywords: compact operator, unconditional convergence

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In two recent papers in the Proceedings of the American Mathematical Society, it has been shown that if a series of compact operators on a Hilbert space is subseries (s.s.) convergent in the weak operator topology to a compact operator, then it is actually norm s.s. convergent ([**5**], [**7**]). In this note, we would like to point out and present an essentially self-contained proof of a much more general result due to Kalton ([**6**] Theorem 5). The recent appearance of [**5**] and [**7**] suggest that Kalton's result is not as well-known as it deserves to be.

Let X and Y be Banach spaces with $K(X,Y)$ the space of compact operators from X into Y. The weak operator topology on $K(X,Y)$ is the locally convex topology induced by the semi-norms $T \to |\langle y', Tx\rangle|$ for $x \in X, y' \in Y'$. We say that a B-space X has property (DF) if every series $\sum x'_k$ in X' which is weak* s.s. convergent is norm s.s. convergent. By a result of Diestel and Faires, this property is equivalent to X' containing no subspace which is topologically isomorphic to ℓ^∞ ([**3**]). A reflexive space and, in particular, a Hilbert space has this property so the following theorem of Kalton gives a generalization of the results of [**5**] and [**7**] concerning series of compact operators which were described above.

Theorem 1. *(Kalton) Let X have property (DF). If $\sum T_i$ is a series in $K(X,Y)$ which is s.s. convergent in $K(X,Y)$ with respect to the weak operator topology, then $\sum T_i$ is norm s.s. convergent.*

PROOF : Since each T_i has separable range, we may assume that Y is separable.

By [**2**], Theorem IV.1.1, it suffices to show that $\|T_i\| \to 0$. For each i, pick $y'_i \in Y'$ with $\|y'_i\| = 1$ and $\|T_i\| = \|T'_i\| \le \|T'_i y'_i\| + 1/i$. Thus, it suffices to show that $\|T'_i y'_i\| \to 0$. By the Banach–Alaoglu Theorem and the separability of Y, there is a subsequence of $\{y'_i\}$ which is weak* convergent to some $y' \in Y'$; in order to keep the notation simple, assume that $\{y'_i\}$ is weak* convergent to y'.

Since the series $\sum T_i$ is s.s. convergent in the weak operator topology, for each $z' \in Y'$ the series $\sum T'_i z'$ is weak* s.s. convergent in X' and, therefore, norm s.s. convergent by hypothesis. In particular, $\|T'_i z'\| \to 0$ for each z'. Also, since each T_j is compact, $\lim_i \|T'_j(y'_i - y')\| = 0$ ([**4**] VI.5.6). Hence, the rows and columns of the matrix $M = [T_j'(y_i' - y')]$ are norm convergent to 0. Therefore, if $p_1 = 1$, there exists $p_2 > p_1$ such that $\|T_{p_1}'(y_{p_2}' - y')\| < 2^{-3}$ and $\|T_{p_2}'(y_{p_1}' - y')\| < 2^{-3}$. Continuing by induction produces an increasing sequence of positive integers $\{p_i\}$ such that $\|T_{p_j}'(y_{p_i}' - y')\| < 2^{-i-j}$ for $i \neq j$. If we let $T \in K(X,Y)$ be the sum of the subseries $\sum_{j=1}^{\infty} T_{p_j}$ in the weak operator topology, we have

$$\begin{aligned}(1)\qquad \|T_{p_i}' y_{p_i}'\| &\leq \|T_{p_i}'(y_{p_i}' - y')\| + \|T_{p_i}' y'\| \leq \\ &\leq \sum_{\substack{j=1\\ j\neq i}}^{\infty} \|T_{p_j}'(y_{p_i}' - y')\| + \|T'(y_{p_i}' - y')\| + \|T_{p_i}' y'\| \leq \\ &\leq 2^{-i} + \|T'(y_{p_i}' - y')\| + \|T_{p_i}' y'\|.\end{aligned}$$

The second term of the right hand side of (1) is norm convergent to 0 ([**4**] VI.5.6) and the last term is norm convergent to 0 since the series $\sum T_{p_i}' y'$ is norm s.s. convergent. Thus, $\|T_{p_i}' y_{p_i}'\| \to 0$, and since the same argument can be applied to any subsequence, it follows that $\|T_i' y_i'\| \to 0$. ■

For other proofs and remarks concerning Kalton's result see [**6**], [**1**], [**8**].

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