Giovanni Rotondaro

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GIOVANNI ROTONDARO

Abstract. Let $f : M^m \to \mathbf{R}^{m+1}$ be an immersion of a closed orientable smooth m–manifold, $m \geq 2$. Denote by H, r, p the first mean curvature, distance and support functions of f. We prove that, if $Hp = 1$, then M is embedded as a standard m–sphere. Furthermore we derive an integral formula, which also implies this theorem. Finally we point out an extrinsic inequality for H^2.

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Let $f : M^m \to \mathbf{R}^{m+1}$ be an immersion of a connected orientable m–manifold M into Euclidean $(m+1)$–space, $m \geq 2$. (o) Denote by $n, r = |f|, p = -f \cdot n$, respectively the Gauss normal field, the distance function and the support function with respect to the origin 0 which is supposed not lying in $f(M)$. Let H be the first mean curvature, i.e. the arithmetic mean of principal curvatures. The classical Hp–theorem [2] [4] says that a convex (hence embedded) closed surface of $\mathbf{R}^3$ with $Hp = 1$ is a standard sphere. In [1] we have shown that the same result holds if the surface is merely immersed, without the strong hypothesis of convexity. In this note we want to extend our theorem to higher–dimensional hypersurfaces.

Let us adopt all customary conventions of index notation. In some local coordinate system $(u^1, u^2, \dots, u^m)$ on M the fundamental forms of the immersion can be written as

$$I = g_{ij} du^i \otimes du^j \qquad II = l_{ij} du^i \otimes du^j.$$

From the identity $r^2 = |f|^2$ we derive

$$r r_{ij} + r_i r_j = g_{ij} - l_{ij} p + r \Gamma_{ij}^k r_k.$$

Then

$$\Delta \log r = \frac{m - mHp - 2\,\Delta_1(r)}{r^2}. \tag{1}$$

Here, Δ is the Laplacian of the Riemannian metric induced on M by f and Δ_1 is the first Beltrami differential parameter, i.e. the square norm of the gradient.

(o) All the manifolds and maps are supposed sufficiently smooth.

Lemma. *If $Hp \geq 1 - (2/m)\,\Delta_1(r)$ and r has a relative minimum, then $f(M)$ is a piece of a standard m-sphere.*

PROOF : In fact $\log r$ has a relative minimum, and because of (1) $\Delta \log r$ is non-positive. Therefore, by E.Hopf's principle [3,v.V, 181] $\log r$ must be a constant. ■

From this lemma we deduce the high-dimensional Hp-theorem.

Theorem. *Let $f : M^m \to \mathbf{R}^{m+1}$ be an immersion, M a connected closed orientable m-manifold, $m \geq 2$. Suppose that $Hp = 1$. Then M is embedded by f as a standard m-sphere.*

PROOF : Of course $Hp \geq 1 - (2/m)\,\Delta_1(r)$ and r has a relative minimum. Then r is a constant. Thus $f(M)$ is a subset of the m-sphere U with centre 0 and radius r. By standard connectedness arguments, we must have actually $f(M) = U$. On the other hand (changing orientation, if necessary), the principal curvatures satisfy $k_1 = k_2 = \cdots = k_m = 1/r$. Therefore, at every point of M, the Weingarten map is positive definite. Then, by Hadamard's theorem on ovaloids [3,v.IV,121], f must be an embedding. ■

Remark 1. The proof of 2-dimensional Hp-theorem in [1] is based on the integral formula

$$\int_M \frac{p^2 - Hpr^2}{r^4}\,dV = 0,$$

which holds for closed immersed surface. (dV is the volume element.) We can generalize this formula as follows. First observe that $f = rg^{ij}r_i f_j - pn$. Then $r^2 = |f|^2 = r^2 g^{ij} r_i g^{ab} r_a f_j f_b + p^2 = r^2\,\Delta_1(r) + p^2$, i.e. $\Delta_1(r) = (r^2 - p^2)/r^2$. Substituting into (1),

$$\Delta \log r = \frac{mr^2(1 - Hp) - 2(r^2 - p^2)}{r^4}. \tag{2}$$

On integration we have, for compact M,

$$\int_M \frac{mr^2(1 - Hp) - 2(r^2 - p^2)}{r^4}\,dV = 0. \tag{3}$$

Notice that, by this formula, $Hp = 1$ implies immediately $r = |p| =$ constant. Thus Hp-theorem is also a consequence of (3) (or (2)).

Remark 2. By considering (2) as a quadratic equation for p, we have the inequality

$$H^2 \geq \frac{8}{m^2}\left(\frac{m-2}{r^2} - \Delta \log r\right) \quad \text{for all immersed hypersurfaces.}$$

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Università degli Studi di Napoli – Dipartimento di Matematica e Applicazioni "R.Caccioppoli" – Via Mezzocannone, 8 – 80134 Napoli–Italy

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