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The Mal'tsev operation on countably compact spaces

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Abstract. Let $X$ be a countably compact topological space and $f : X^3 \to X$ be a continuous mapping such that the identity $f(x,y,y) = f(y,y,x) = x$ holds. Then $\beta X$ is Dugundji. It follows that compact retracts of topological groups are dyadic.

Keywords: Dugundji space, retract, pseudocompact

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We say that a topological space $X$ is Mal'tsev if there exists a continuous mapping $f : X^3 \to X$ (called a Mal'tsev operation [M]) such that the identity $f(x,y,y) = f(y,y,x) = x$ holds for all $x, y \in X$. Every topological group $G$ is a Mal'tsev space, since the mapping $(x,y,z) \mapsto xy^{-1}z$ is a Mal'tsev operation on $G$. M.G.Tkačenko proved in 1981 that compactly-generated topological groups have the Suslin property [T1]. This result was extended by the author in [U1]: every $\sigma$-compact Mal'tsev space has the Suslin property. In the present paper we show that compact Mal'tsev spaces are Dugundji (Theorem 1). Moreover, if $X$ is a countably compact Mal'tsev space, then $\beta X$ is Dugundji (Theorem 2). Clearly any retract of a Mal'tsev space is Mal'tsev. It follows that compact retracts of topological groups are Dugundji. This is an extension of the Ivanovskij – Kuzminov theorem (its proof can be found in [U3]) which says that compact groups are dyadic.

Let $X$ be a subspace of $Y$. A topological space $Z$ is said to be injective with respect to the pair $(X,Y)$ iff every continuous function $f : X \to Z$ has a continuous extension $\bar{f} : Y \to Z$. For a compact space $X$ the following are equivalent [H]:
1) if $Y$ is a zerodimensional compact space and $Z$ is closed in $Y$, then $X$ is injective with respect to the pair $(Z,Y)$; 2) if $X$ is a subspace of a compact space $Y$, then every compact convex subset of a locally convex topological vector space is injective with respect to the pair $(X,Y)$. A compact space $X$ is called Dugundji [P] if one of these conditions holds. Let $X$ be Dugundji. There exist a zero–dimensional compact space $Y$ and an onto mapping $f : Y \to X$. We may assume that $Y$ is a subspace of a Cantor cube $2^m$. Since $X$ is injective with respect to $(Y,2^m)$, there is an extension $f : 2^m \to X$ of $f$. This shows that Dugundji spaces are dyadic [H].

An equivalence relation $R$ on a space $X$ is open if the quotient mapping $X \to X/R$ is open.

Ščepin's theorem S1 [Š]. A compact space $X$ is Dugundji iff there exists a family $\Phi$ of equivalence relations on $X$ such that:
1) for every $R \in \Phi$ the quotient space $X/R$ is metrizable
2) $\Phi$ is closed under countable intersections
3) all $R \in \Phi$ are open
4) $\Phi$ separates points of $X$ (that means the intersection $\cap \Phi$ equals the diagonal of $X^2$).

Let $S = \{X_\alpha, p_\alpha^\beta, A\}$ be an inverse system of compact metric spaces $X_\alpha$ and onto bonding mappings $p_\alpha^\beta : X_\beta \to X_\alpha$. We call $S$ a Ščepin system if for every countable subset $B \subseteq A$ there exists a least upper bound $\alpha = \sup B \in A$ and the family $\{p_\beta^\alpha : \beta \in B\}$ separates points of $X_\alpha$. Theorem S1 means that a compact space $X$ is Dugundji iff it is the inverse limit of a Ščepin system $S = \{X_\alpha, p_\alpha^\beta, A\}$ such that the projections $p_\alpha : X \to X_\alpha$ are open (or, equivalently, the bonding mappings $p_\alpha^\beta$ are open).

Ščepin's theorem S2 [S]. Let $S = \{X_\alpha, p_\alpha^\beta, A\}$ and $T = \{Y_\alpha, q_\alpha^\beta, A\}$ be two Ščepin systems (the directed index set $A$ is the same for $S$ and $T$), $X = \lim S, Y = \lim T$. Let $f$ be a mapping of $X$ to $Y$. Let $A_f$ be the set of all $\alpha \in A$ with the following property: there exists $f_\alpha : X_\alpha \to Y_\alpha$ such that $f_\alpha \circ p_\alpha = q_\alpha \circ f$ (here $p_\alpha : X \to X_\alpha$ and $q_\alpha : Y \to Y_\alpha$ are the projections). Then $A_f$ is cofinal in $A$.

Let $f : X^3 \to X$ be a Mal'tsev operation on a space $X$. An equivalence relation $R$ on $X$ is called a congruence (or an $f$-congruence) if $f(x_1, x_2, x_3)$ is $R$-equivalent to $f(y_1, y_2, y_3)$ whenever $x_i$ is $R$-equivalent to $y_i$, $i = 1, 2, 3$.

Mal'tsev's theorem [M]. If $f$ is a Mal'tsev operation on a space $X$, then every $f$-congruence $R$ on $X$ is open.

**Proof:** Let $U$ be open in $X$ and $V = \{y \in X : (x, y) \in R$ for some $x \in U\}$. We have to show that $V$ is open. Let $y \in V$. Choose $x \in U$ so that $(x, y) \in R$. If $z$ is close enough to $y$, then $f(x, y, z)$ is in $U$, since $f(x, y, y) = x$ is in $U$. On the other hand, $f(x, y, z)$ is $R$-equivalent to $f(x, x, z) = z$. Hence $V$ is a neighbourhood of $y$.

**Theorem 1.** Every compact Mal'tsev space is Dugundji.

**Proof:** Let $f : X^3 \to X$ be a Mal'tsev operation on a compact space $X$. Let $\Phi$ be the family of all $f$-congruences $R$ on $X$ such that the quotient space $X/R$ is metrizable. Clearly $\Phi$ is closed under countable intersections, so conditions 1) and 2) of Ščepin's theorem S1 are satisfied. Mal'tsev's theorem shows that condition 3) holds, too. It remains to prove that $\Phi$ separates points of $X$. Let $S = \{X_\alpha, p_\alpha^\beta, A\}$ be a Ščepin system whose limit is $X$. Then $X^3$ is the limit of Ščepin's system $S^3 = \{(X_\alpha)^3, (p_\alpha^\beta)^3, A\}$. Let $p_\alpha : X \to X_\alpha$ be the projections of $S$. Call $\alpha \in A$ nice if there exists $f_\alpha : X_\alpha^3 \to X_\alpha$ such that $f_\alpha \circ p_\alpha^3 = p_\alpha \circ f$. The relation $R_\alpha = \{(x, y) \in X^2 : p_\alpha(x) = p_\alpha(y)\}$ is an $f$-congruence whenever $\alpha$ is nice. Theorem S2 implies that the set $A_f$ of all nice $\alpha$ is cofinal in $A$. Hence the subfamily $\{R_\alpha : \alpha \in A_f\}$ of $\Phi$ separates points of $X$.

If $\beta X$ is Dugundji, then $X$ is pseudocompact. We give a characterization of spaces $X$ for which $\beta X$ is Dugundji.
Proposition 1. Let $X$ be pseudocompact. Then $\beta X$ is Dugundji if and only if there exists a family $\Phi$ of equivalence relations on $X$ such that:

1) for every $R \in \Phi$ the quotient space $X/R$ is submetrizable
2) $\Phi$ is closed under countable intersections
3) every $R \in \Phi$ is open
4) the topology of $X$ is determined by the quotient mappings $f_R : X \to X/R$, $R \in \Phi$.

We need some lemmas. Recall that a mapping $p : Y \to Z$ is $d$-open if one of the equivalent conditions holds:

1) $\text{cl}\, p^{-1}(U) = p^{-1}(\text{cl}\, U)$ for every $U$ open in $Z$
2) $p(V) \subseteq \text{int}\, \text{cl}\, p(V)$ for every $V$ open in $Y$.

Lemma 1. If $X$ is pseudocompact and $M$ is metric, then every $d$-open mapping $f : X \to M$ is open.

**Proof:** Let $U$ be open in $X$ and $x \in U$. Choose an open set $V$ so that $x \in V \subseteq \text{cl}\, V \subseteq U$. Since $\text{cl}\, V$ is pseudocompact, $f(\text{cl}\, V)$ is closed in $M$. It follows that $\text{cl}\, f(V) = f(\text{cl}\, V)$, hence $f(x) \in \text{int}\, \text{cl}\, f(V) = \text{int}\, f(\text{cl}\, V) \subseteq \text{int}\, f(U)$. This means that $f(U)$ is open. ■

Lemma 2. Let $X$ be pseudocompact. If $R$ is an open equivalence relation on $X$ such that the quotient space $X/R$ is submetrizable, then $X/R$ is metrizable (and hence compact).

**Proof:** Let $f : X \to X/R$ be the quotient mapping, $g : X/R \to K$ be a one-to-one mapping of $X/R$ onto a compact metric space $K$, and $h = g \circ f$. We have to show that $g$ is a homeomorphism, or, equivalently, that $h$ is open. In virtue of Lemma 1 it suffices to show that $h$ is $d$-open. Let $U$ be open in $K$ and $F = \text{cl}\, h^{-1}U$. Since $f$ is open, $F = \text{cl}\, f^{-1}g^{-1}(U) = f^{-1}\text{cl}\, g^{-1}U = h^{-1}E$ for some $E \subseteq K$. Now $F$ is pseudocompact, being a regular closed set in $X$, so $E = h(F)$ is closed in $K$. It follows that $E = \text{cl}\, U$ and $\text{cl}\, h^{-1}U = h^{-1}\text{cl}\, U$. This means that $h$ is $d$-open. ■

Lemma 3. Let $X$ be $G_\delta$-dense in $Y$ (= every non-empty $G_\delta$-subset of $Y$ meets $X$). If $f$ maps $Y$ to a metric space $M$ and the restriction $f|X : X \to M$ is open, then $f : Y \to M$ is open.

**Proof:** If $U$ is open in $Y$, then $f(U) = f(U \cap X)$. ■

Lemma 4. If $Y$ is Dugundji-compact and $X$ is a dense subspace of $Y$, then the following are equivalent:

1) $Y = \beta X$
2) $X$ is pseudocompact
3) $X$ is $G_\delta$-dense in $Y$.

**Proof:** Dugundji spaces are perfectly $\kappa$-normal (= closures of open sets are zero-sets), and $G_\delta$-dense subspaces of perfectly $\kappa$-normal spaces are $C$-embedded [T2]. This gives 3) $\Rightarrow$ 1). If $\beta X$ is dyadic, then $X$ is pseudocompact [EP]. Since Dugundji spaces are dyadic, 1) $\Rightarrow$ 2) follows; and 2) $\Rightarrow$ 3) is obvious. ■
PROOF of Proposition 1: The "only if" part follows from theorem S1 and Lemma 1. Conversely, let \( \Phi \) be such a family as described in Proposition 1. The spaces \( X/R, R \in \Phi \), are compact and constitute a Štepin system \( S \). Let \( Y = \lim S \). Then \( X \) can be regarded as a \( G_\delta \)-dense subspace of \( Y \). The open mappings \( f_R : X \to X/R \) are restrictions of the projections of \( S \), so Lemma 3 shows that the projections of \( S \) are open. Theorem S1 implies that \( Y \) is Dugundji, and Lemma 4 shows that \( Y = \beta X \).

Suppose \( G \) is a subgroup of the product of metrizable groups, \( H \) is a closed subgroup of \( G \). Let \( X = G/H \) be the quotient space. If \( X \) is pseudocompact, then \( \beta X \) is Dugundji. This follows from Proposition 1, combined with Lemma 7 in [U3]. Theorem 1 in [U3] also can be extended to the pseudocompact case: if \( G \) is a subgroup of the product of groups with a countable base and \( G \) acts transitively on a pseudocompact space \( X \), then \( \beta X \) is Dugundji. If a pseudocompact space \( X \) is a retract of a topological group, then \( \beta X \) is Dugundji [U4].

Conjecture 1. If \( X \) is pseudocompact and Mal'tsev, then \( \beta X \) is Dugundji.

Conjecture 2. The product of any two pseudocompact Mal'tsev spaces is pseudocompact.

Proposition 2. Conjectures 1 and 2 are equivalent.

PROOF: Suppose \( \{X_\alpha : \alpha \in A \} \) is a family of pseudocompact spaces such that \( \beta X_\alpha \) is Dugundji for every \( \alpha \in A \). Then \( Q = \Pi \beta X_\alpha \) is Dugundji and \( P = \Pi X_\alpha \) is \( G_\delta \)-dense in \( Q \). Lemma 4 implies that \( P \) is pseudocompact. Hence if Conjecture 1 is true, then the product of any family of pseudocompact Mal'tsev spaces is pseudocompact. Conversely, let \( f : X^3 \to X \) be a Mal'tsev operation on a pseudocompact space \( X \). If Conjecture 2 is true, then \( X^3 \) is pseudocompact. By virtue of Glicksberg's theorem [G], [E, problem 3.12.20], \( \beta(X^3) = (\beta X)^3 \), hence \( f \) has an extension \( \bar{f} : (\beta X)^3 \to \beta X \). Clearly \( \bar{f} \) is a Mal'tsev operation on \( \beta X \), so Theorem 1 shows that \( \beta X \) is Dugundji.

We shall prove that Conjecture 1 is true for countably compact spaces (Theorem 2).

Rezničenko's theorem R1. Suppose \( X_1, X_2, X_3 \) are countably compact and have the Suslin property, \( M \) has a countable base, \( f : X_1 \times X_2 \times X_3 \to M \) is separately continuous (i.e., if \( x_1, x_2, x_3 \) are fixed, \( x_i \in X_i \), then the functions \( f(x_1, x_2, \cdot), f(x_1, \cdot, x_3), f(\cdot, x_2, x_3) \) are continuous on \( X_3, X_2, X_1 \), respectively). Then there exist compact metric spaces \( Y_1, Y_2, Y_3 \), continuous onto mappings \( p_i : X_i \to Y_i, i = 1, 2, 3 \), and a separately continuous function \( g : Y_1 \times Y_2 \times Y_3 \to M \) such that \( f = g \circ (p_1 \times p_2 \times p_3) \).

Rezničenko's theorem R2. Suppose \( X_1 \) and \( X_2 \) are pseudocompact, \( M \) has a countable base, \( f : X_1 \times X_2 \to M \) is (jointly) continuous. Then there exist Eberlein-compact spaces \( Y_1, Y_2 \), continuous onto mappings \( p_i : X_i \to Y_i \), and a separately continuous function \( g : Y_1 \times Y_2 \to M \) such that \( f = g \circ (p_1 \times p_2) \). If \( X_1 \) and \( X_2 \) have the Suslin property, then \( Y_1 \) and \( Y_2 \) are metrizable.

Recall that a compact space \( X \) is Eberlein iff \( X \) embeds in the function space \( C_p(Y) \) for some compact \( Y \).
Proposition 3. Suppose $X$ is a Tikhonov countably compact space having the Suslin property. If there exists a separately continuous Maltsev operation $f : X^3 \to X$, then $\beta X$ is Dugundji.

Proof: Let $\Psi = \{R : R$ is an equivalence relation on $X$ and $X/R$ is submetrizable$\}$ and $\Phi = \{R \in \Psi : R$ is an $f$-congruence$\}$. Maltsev's theorem shows that every $R \in \Phi$ is open. Clearly $\Phi$ is closed under countable intersections. To check the condition 4) of Proposition 1, it suffices to prove that for every $T \in \Psi$ there exists $R \in \Psi$ such that $R \subseteq T$. Let $T \in \Psi$. Using theorem R1, construct a sequence $T_0 = T, T_1, T_2 \ldots$ such that every $T_n$ is in $\Psi$ and the conditions $(x_i, y_i) \in T_{n+1}, i = 1, 2, 3, \ldots$ imply that $f(x_1, x_2, x_3)$ and $f(y_1, y_2, y_3)$ are $T_n$-equivalent, $n = 0, 1, \ldots$ Then $R \cap T_n$ is a congruence, so $R \in \Phi$. It follows that $\Phi$ satisfies the conditions of Proposition 1. Hence $\beta X$ is Dugundji.

Definition 1. A space $X$ has the property $(T)$ iff for every family $\{x_\alpha : \alpha < \omega_1\}$ of points of $X$ and every family $\{\gamma_\alpha : \alpha < \omega_1\}$ of open covers of $X$ there exist $\alpha, \beta < 1$ such that $\alpha \neq \beta$ and $St(x_\alpha, \gamma_\beta)$ meets $St(x_\beta, \gamma_\alpha)$ (where $St(x, \gamma) = \bigcup\{U \in \gamma : U \ni x\}$).

This property was considered by M. Tkačenko [T1] who proved that every compact space has the property $(T)$. Moreover, every Lindelöf $\Sigma$-space has the property $(T)$ [U1], [U2]. A space $X$ is a $\Sigma$-space [N] iff there are two covers $F$ and $C$ of $X$ such that $F$ is $\sigma$-locally finite, $C$ consists of countably compact sets and for every $K \subseteq C$ and every neighbourhood $U$ of $K$ there exists $E \subseteq F$ such that $K \subseteq E \subseteq U$.

Now we prove that countably compact spaces have the property $(T)$.

Proposition 4. If $X$ is a $\Sigma$-space in which every closed discrete subset is countable, then $X$ has the property $(T)$.

Proof: Let $\{x_\alpha : \alpha < \omega_1\}$ be a family of points of $X$ and $\{\gamma_\alpha : \alpha < \omega_1\}$ be a family of open covers of $X$. For $\alpha < \omega_1$ put $X_\alpha = \{x_\beta : \beta < \alpha\}$ and $\lambda_\alpha = \{x_\alpha \cap St(x_\alpha, \gamma_\beta) : \beta < \alpha\}$. For $E \subseteq X$ let $S(E)$ be the set of all $\alpha < \omega_1$ such that $x_\alpha \in E$ and the family $\lambda_\alpha|E = \{L \cap E : L \in \lambda_\alpha\}$ does not have the finite intersection property. We assert that for every $E \subseteq X$ the set $S(E)$ is non-stationary (i.e. some closed unbounded subset of $\omega_1$ does not meet $S(E)$). For every infinite set $A \subseteq \omega_1$ fix an open cover $\gamma_A$ of $X$ which refines $\gamma_\alpha$ for every $\alpha \in A$. If $\alpha \in S(E)$, there is a finite set $A = f(\alpha) \subseteq \alpha$ such that $X_\alpha \cap E \cap St(x_\alpha, \gamma_A) = \emptyset$. Suppose $S(E)$ is stationary. Fodor's Lemma [K] implies there is an uncountable subset $B \subseteq S(E)$ and a finite set $A \subseteq \omega_1$ such that $f(\alpha) = A$ for every $\alpha \in B$. If $\alpha, \beta \in B$ and $\beta < \alpha$, then $x_\beta \in X_\alpha \cap E \subseteq X \setminus St(x_\alpha, \gamma_A)$. It follows that the set $\{x_\alpha : \alpha \in B\}$ is closed discrete in $X$, since every element of $\gamma_A$ contains at most one point of this set. This contradiction proves that $S(E)$ is non-stationary.

Let $F$ and $C$ be two covers of $X$ witnessing the $\Sigma$-property. Then $F$ is countable. Since the union of countably many non-stationary subsets of $\omega_1$ is non-stationary [K], there exists $\alpha \in \omega_1 \setminus \{0\}$ such that $\alpha \notin S(E)$ for every $E \in F$. Let $K$ be an element of $C$ containing $x_\alpha$. If $E \in F$ and $K \subseteq E$, then the family $\lambda_\alpha|E$ has finite intersection property. It follows that the family $\mu = \{K \cap \overline{L} : L \in \lambda_\alpha\}$ also has the finite intersection property. Since $K$ is countably compact and $\mu$ is countable,
there exists a point \( z \in \cap \mu \). Then \( z \in \overline{X_\alpha} \) and \( z \in \overline{St(x_\alpha, \gamma_\beta)} \) for every \( \beta < \alpha \). Let
\( U \) be an element of \( \gamma_\alpha \) containing \( z \). Since \( z \in \overline{X_\alpha} \), \( U \) meets \( X_\alpha \), so there is \( \beta < \alpha \) such that \( x_\beta \in U \). Since \( z \in \overline{St(x_\alpha, \gamma_\beta)} \) and \( z \in U \subseteq \overline{St(x_\beta, \gamma_\alpha)} \), the intersection \( \overline{St(x_\alpha, \gamma_\beta)} \cap \overline{St(x_\beta, \gamma_\alpha)} \) is non-empty. The proof is complete.

**Proposition 5.** If a Mal'tsev space \( X \) has the property \( (T) \) then \( X \) has the Suslin property.

**Proof:** Let \( \{O_\alpha : \alpha < \omega_1\} \) be a family of non-empty open sets in \( X \). We have to show that this family is not disjoint. For every \( \alpha < \omega_1 \) pick \( x_\alpha \in O_\alpha \). Let \( f : X^3 \to X \) be a Mal'tsev operation. Call a set \( U \subseteq X \) \( \alpha \)-small if \( f(x_\alpha, y, z) \in O_\alpha \) and \( f(y, z, x_\alpha) \in O_\alpha \) whenever \( y, z \in U \). Since \( f(x_\alpha, y, y) = f(y, y, x_\alpha) = x_\alpha \in O_\alpha \) for every \( y \in X \), the collection \( \gamma_\alpha \) of all \( \alpha \)-small open sets covers \( X \). The property \( (T) \) implies there are distinct \( \alpha, \beta < \omega_1 \) and a point \( z \in X \) such that \( z \in \overline{St(x_\alpha, \gamma_\beta)} \cap \overline{St(x_\beta, \gamma_\alpha)} \). Then \( f(x_\alpha, z, x_\beta) \in O_\alpha \cap O_\beta \neq \emptyset \).

Propositions 4 and 5 imply

**Proposition 6.** If \( X \) is a Mal'tsev \( \Sigma \)-space in which every closed discrete subset is countable, then \( X \) has the Suslin property.

Let us say that a space \( X \) is \( G_\delta \)-cellular if for every family \( \lambda \) of \( G_\delta \)-sets in \( X \) there exists a countable subfamily \( \mu \subseteq \lambda \) whose union is dense in the union of \( \lambda \). It can be shown that under the assumptions of Proposition 6, \( X \) is \( G_\delta \)-cellular. For Mal'tsev Lindelöf \( \Sigma \)-spaces this fact is proved in [U2]. In particular, \( \sigma \)-compact topological groups are \( G_\delta \)-cellular.

Propositions 3 and 6 imply the main result of the paper:

**Theorem 2.** If \( X \) is a countably compact Mal'tsev space, then \( \beta X \) is Dugundji.

Now we introduce a new class of spaces which is contained in the class of Mal'tsev spaces. For \( a \in X \) define subsets \( X_a \) and \( X^a \) of \( X^2 \) as follows: \( X_a = \{(a, x) : x \in X\} \) and \( X^a = \{(x, a) : x \in X\} \). Let \( \Delta_X = \{(x, x) : x \in X\} \) be the diagonal of \( X^2 \).

**Definition 2.** A space \( X \) has a rectifiable diagonal iff there is a homeomorphism \( f : X^2 \to X^2 \) of \( X^2 \) onto itself such that:

1. \( f(X_a) = X_a \) for every \( a \in X \);
2. \( f(\Delta_X) = X^a \) for some \( a \in X \).

Every topological group has a rectifiable diagonal, and every space with a rectifiable diagonal is homogeneous.

**Proposition 7 (M.M. Čoban).** A space \( X \) has a rectifiable diagonal if and only if there exist mappings \( p : X^2 \to X \) and \( q : X^2 \to X \) satisfying the identities \( p(x, q(x, y)) = q(x, p(x, y)) = y \) and \( p(x, x) = p(y, y) \).

**Proof:** If \( p \) and \( q \) are as above, let \( f(x, y) = (x, p(x, y)) \). Then \( f^{-1}(x, y) = (x, q(x, y)) \), so \( f \) is a homeomorphism of \( X^2 \) onto itself. If \( p(x, x) = a \) for all \( x \in X \), then \( f(\Delta_X) = X^a \). Conversely, if \( f \) is as in definition 2, the same formulas \( f(x, y) = (x, p(x, y)) \) and \( f^{-1}(x, y) = (x, q(x, y)) \) define \( p \) and \( q \).
Corollary. Every space with a rectifiable diagonal is Mal’tsev.

PROOF: If p and q are as in Proposition 7, then \((x, y, z) \rightarrow q(x, p(y, z))\) is a Mal’tsev operation. 

Theorem 3. Let \(X\) be a pseudocompact space with a rectifiable diagonal. If \(X\) has the Suslin property, then \(\beta X\) is Dugundji.

PROOF: Let \(p\) and \(q\) be as in Proposition 7. We say that an equivalence relation \(R\) on \(X\) is a congruence if the following conditions holds: if \(x_i, i = 1, 2\), then \(p(x_1, x_2)\) is \(R\)-equivalent to \(p(y_1, y_2)\) and \(q(x_1, x_2)\) is \(R\)-equivalent to \(q(y_1, y_2)\). Let \(\Phi = \{R : R\) is a congruence and \(X/R\) is submetrizable\}. Then \(\Phi\) satisfies the conditions of Proposition 1. If \(R\) is a congruence, then \(R\) is also an \(m\)-congruence, where \(m(x, y, z) = q(x, p(y, z))\) is a Mal’tsev operation, so the Mal’tsev theorem shows that \(R\) is open. It remains to show that the quotient mappings \(X \rightarrow X/R, R \in \Phi\), determine the topology of \(X\). The argument is essentially the same as in the proof of Proposition 3, only Rezničenko’s theorem R2 should be used instead of R1.

Questions. Is it true that every pseudocompact space with a rectifiable diagonal has the Suslin property? Is it true that every space with a rectifiable diagonal is a retract of a topological group?

O.V. Sipacheva has recently proved that every compact Mal’tsev space is a retract of a topological group. It can be shown that the same is true for countably compact Mal’tsev spaces.

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