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Abstract. For a non stationary α–mixing sequence of random variables it is given a necessary and sufficient condition for the central limit theorem. The condition is expressed by uniform integrability of squares of certain normalized partial sums of the process.

Keywords: Central limit theorem for weakly dependent random variables, α–mixing (strong mixing) sequence of random variables

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Let $(X_i)_{i=1}^{\infty}$ be an α–mixing (strong mixing) sequence of square integrable random variables, $EX_i = 0$ for all i. We denote $S_n = \sum_{j=1}^{n} X_j, \sigma_n^2 = ES_n^2$. In the whole of the paper conditions A and B are supposed to be fulfilled.

A. $\sigma_n \to \infty$ as $n \to \infty$.

B. $\max\limits_{1 \le j \le n} EX_j^2/\sigma_n^2 \to 0$ as $n \to \infty$.

In [2], [3] and [4] it has been shown that if (X_i) is strictly stationary, then S_n/σ_n weakly converge to the normal distribution $N(0,1)$ if and only if the random variables S_n^2/σ_n^2 are uniformly integrable. Here we give a necessary and sufficient condition for the CLT for processes which are not stationary. In proving the result we shall use ideas from [4].

We say that $J \subset \{1, \dots, n\}$ is an interval if together with any $i < k$, J contains all $j \in N$ for which $i < j < k$. By $\pi_{n,k}, k \le n$, we denote a partition $\{I_{n,1,k}, \dots, I_{n,k,k}\}$ of $\{1, \dots, n\}$ into intervals; $S_{n,j,k}$ denotes $\sum_{i \in I_{n,j,k}} X_i$, and $\sigma_{n,j,k}^2 = ES_{n,j,k}^2$.

Proposition. *Let $\mathcal{K}$ be a set of positive integers and let for each $k \in \mathcal{K}$ there exists $n(k) \in N$ so that*

$$S_{n,j,k}^2/\sigma_{n,j,k}^2, k \in \mathcal{K}, j = 1, \dots, k, n = n(k), n(k)+1, \dots,$$

are uniformly integrable,

$$\sigma_{n,j,k}^2 \to \infty \text{ as } n \to \infty \text{ for each } k \in \mathcal{K}, 1 \le j \le k.$$

Then for each $n \ge n(k)$ there exist mutually independent random variables $Z_{n,1,k}, \dots, Z_{n,k,k}$ such that

$$Z_{n,j,k}^2/\sigma_{n,j,k}^2, k \in \mathcal{K}, j = 1, \dots, k, \quad n = n(k), n(k)+1, \dots,$$

are uniformly integrable and

$$\left\|\frac{S_{n,j,k}}{\sigma_{n,j,k}} - \frac{Z_{n,j,k}}{\sigma_{n,j,k}}\right\|_2 \to 0 \text{ as } n \to \infty \text{ for each } k \in \mathcal{K}, j = 1, \dots, k.$$

Hence, $\|\frac{S_n}{\sigma_n} - \frac{1}{\sigma_n}\sum_{j=1}^{k} Z_{n,j,k}\|_2 \to 0$ *as* $n \to \infty$ *for each* $k \in \mathcal{K}$.

PROOF : The proof is based on the same idea as the proof of Theorem 1 in [4] so we shall give a sketch only.

Let q be a positive integer whose value will be specified later. By removing $(k-1)\cdot q$ numbers we replace each block $I_{n,j,k}$ by a smaller block $I'_{n,j,k} \subset I_{n,j,k}$ so that random variables $S'_{n,j,k} = \sum_{i \in I'_{n,j,k}} X_i, j = 1,\ldots,k$, are mixing with coefficient $\alpha(q)$. When considering n large enough only, random variables $S'^2_{n,j,k}/\sigma^2_{n,j,k}$ are uniformly integrable and close to $S^2_{n,j,k}/\sigma^2_{n,j,k}$ in L_1 norm. Given $K < \infty$ and $H \in N$ sufficiently large we can find random variables $\widehat{S}_{n,j,k}$ which attain only H values, so that $E\widehat{S}_{n,j,k} = 0, |\widehat{S}_{n,j,k}|/\sigma_{n,j,k} \leq K$, and $\widehat{S}_{n,j,k}/\sigma_{n,j,k}$ are sufficiently close to $S'_{n,j,k}/\sigma_{n,j,k}$ in L_2 norm (hence to $S_{n,j,k}/\sigma_{n,j,k}$, too). Given K and H fixed we can find q sufficiently large so that there exist random variables $Z_{n,j,k},\ldots,Z_{n,k,k}$ which are mutually independent and for each $j, 1 \leq j \leq k, Z_{n,j,k}/\sigma_{n,j,k}$ is close to $\widehat{S}_{n,j,k}/\sigma_{n,j,k}$ in L_2 norm and has the same distribution. From this the Proposition follows. ∎

C. For each $k \in N$ and n greater or equal than a positive integer $n(k)$ there exists a partition $\pi_{n,k}$ of $\{1,\ldots,n\}$ such that it holds

(i) $\lim_{n\to\infty} \sigma_{n,j,k} = \infty$ as $n \to \infty$, for each $k \in N, j = 1,\ldots,k$,

(ii) $\lim_{k\to\infty} \infty \to \overline{\lim_{n\to\infty}} \max_{1\leq j\leq k} \sigma_{n,j,k}/\sigma_n = 0$,

(iii) $S^2_{n,j,k}/\sigma^2_{n,j,k}, k \in N, j = 1,\ldots,k, \quad n = n(k), n(k)+1,\ldots$ are uniformly integrable.

It should be noted that according to the Proposition, from C follows

(iv) for each $k \in N, \sum_{j=1}^{k} \sigma^2_{n,j,k}/\sigma^2_n \to 1$ as $n \to \infty$.

Theorem. *Let* (X_i) *be an* α*-mixing sequence and conditions A,B are fulfilled. Then* $S_n/\sigma_n \xrightarrow[\mathcal{D}]{} N(0,1)$ *if and only if C holds.*

PROOF :

1. Let condition C hold (and A,B as well).

According to the Proposition, for each $n \in N$ there exist a positive integer $k(n)$ and mutually independent random variables $Z_{n,1},\ldots,Z_{n,k(n)}$ such that $k(n) \to \infty$ as $n \to \infty$, $\|\frac{S_n}{\sigma_n} - \frac{1}{\sigma_n}\sum_{j=1}^{k(n)} Z_{n,j}\|_2 \to 0$ as $n \to \infty$, $Z^2_{n,j}/EZ^2_{n,j}, j = 1,\ldots,k(n), n = 1,2,\ldots$ are uniformly integrable, and $\frac{1}{\sigma_n^2}E(\sum_{j=1}^{k(n)} Z_{n,j})^2 \to 1$ as $n \to \infty$, $\max_{1\leq j\leq k(n)} \frac{1}{\sigma_n^2}EZ^2_{n,j} \to 0$ as $n \to \infty$. For the triangular array of random variables $Z_{n,j}$ the Feller–Lindeberg condition is fulfilled, hence $\frac{1}{\sigma_n}\sum_{j=1}^{k(n)} Z_{n,j} \xrightarrow[\mathcal{D}]{} N(0,1)$, therefore $S_n/\sigma_n \xrightarrow[\mathcal{D}]{} N(0,1)$.

2. Let $S_n/\sigma_n \xrightarrow[\mathcal{D}]{} N(0,1)$.

From conditions A,B it follows that $\max_{1\leq j\leq n-1}(\sigma^2_{j+1} - \sigma^2_j)/\sigma^2_n \to 0$ as $n \to \infty$. Hence we can find numbers $0 = 0(n) < 1(n) < \cdots < k(n) = n, n = k, k+1,\ldots$ such that $\sigma^2_{j(n)}/\sigma^2_n \to j/k$ as $n \to \infty, 0 \leq j \leq k$. We put $I_{n,j,k} = \{(j-1)(n)+1,\ldots,j(n)\}$, and $S_{n,j,k} = \sum_{i \in I_{n,j,k}} X_i, 0 \le j \le k$. Let $k \in N$ be fixed. From [1], Theorem 5.4 it follows that S_n^2/σ_n^2 are uniformly intergrable, therefore $S_{j(n)}^2/\sigma_{j(n)}^2$ are uniformly integrable, too. From this we get that $S_{n,j,k}^2/(\sigma_n^2/k)$ are uniformly integrable. From the Proposition it follows that there exist random variables $Z_{n,j,k}, 1 \le j \le k$, such that $Z_{n,1,k}, \ldots, Z_{n,k,k}$ are mutually independent for each $n, \|\frac{S_{n,j,k}}{\sigma_n} - \frac{Z_{n,j,k}}{\sigma_n}\|_2 \to 0$ as $n \to \infty$, and $k \cdot Z_{n,j,k}^2/\sigma_n^2$ are uniformly integrable. From this it follows that $EZ_{n,j,k}^2/\sigma_n^2 \to 1/k$ as $n \to \infty$, so $\sigma_{n,j,k}^2/\sigma_n^2 \to 1/k$ as $n \to \infty, 1 \le j \le k$. In this way,conditions (i) and (ii) of C are fulfilled.

The uniform integrability of $S_{n,j,k}^2/\sigma_{n,j,k}^2$ is guaranted for each k fixed only, however. We shall show that there exist positive integers $n(k)$ such that condition (i) of C holds.

From the assumptions it follows that $S_{j(n)}/\sigma_{j(n)} \underset{\mathcal{D}}{\to} N(0,1)$ as $n \to \infty$, hence $\frac{1}{\sigma_n}\sum_{i=1}^{j} Z_{n,i,k} \underset{\mathcal{D}}{\to} N(0, \sqrt{j/k})$ as $n \to \infty$, therefore $S_{n,j,k}/\sigma_{n,j,k} \underset{\mathcal{D}}{\to} N(0,1)$ as $n \to \infty$. Let X be a random variable with distribution $N(0,1)$, for $m = 1, 2, \ldots$ let $K_1 \le K_2 \le \cdots \le K_m \le \cdots < \infty$, $E[\chi(|X| > K_m) \cdot X^2] < 1/m$. For $k = 1, 2, \ldots$ we can thus choose $k(n)$ such that for $n \ge k(n)$, $E[\chi(|S_{n,j,k}|/\sigma_{n,j,k} > K_m) \cdot S_{n,j,k}^2/\sigma_{n,j,k}^2] < 2/m, m = 1, \ldots, k, j = 1, \ldots, k$. Now, we can show that for each $\varepsilon > 0$ there exists $K < \infty$ such that $E[\chi(|S_{n,j,k}|/\sigma_{n,j,k} > K_m) \cdot S_{n,j,k}^2/\sigma_{n,j,k}^2] < \varepsilon$ for each $k = 1, 2, \ldots, j = 1, 2, \ldots, n = n(k), n(k)+1, \ldots$: For $\varepsilon > 0$ given there exists $m \in N, 2/m < \varepsilon$. For $k \ge m$ and $n \ge k(n)$ it is $E[\chi(|S_{n,j,k}|/\sigma_{n,j,k} > K_m) \cdot S_{n,j,k}^2/\sigma_{n,j,k}^2] < 2/m, j = 1, \ldots k$. There are only finitely many positive integers k smaller than m and for each k fixed, $S_{n,j,k}^2/\sigma_{n,j,k}^2$ are uniformly integrable; from this the existence of suitable K follows. ∎

Remarks. From the Theorem, the result for strictly stationary processes easily follows.

The divisions of $\{1, \ldots, n\}$ into intervals $I_{n,j,k}$ need not be equidistant; it can be seen on an example of a sequence of random variables which are mutually independent and have distributions $N(0, 1/\sqrt{n})$

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