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Uniform bounds for solutions of a degenerate diffusion equation with nonlinear boundary conditions

JÁN FILO

Dedicated to the memory of Svatopluk Fučík

Abstract. This paper deals with solutions \( u(x,t) \) of the degenerate parabolic equation
\[
(\beta(u))_t = \Delta u \text{ in the cylinder } D \times (0,T), D \subset \mathbb{R}^N \text{ bounded}, \quad \beta(u) = |u|^{m} \text{ sign } u
\]
under the assumption, that on the lateral boundary nonlinear boundary conditions of the form
\[
\frac{\partial u}{\partial \nu} = f(u), \quad f(u)u \leq L(|u|^{\alpha+1} + 1), \quad \alpha \geq 1, \quad L > 0,
\]
are imposed. It is shown that the value of the integral
\[
\sup \int_0^T \int_\Gamma |u(x,t)|^{(N-1)(\alpha-1)+\epsilon} \, ds
\]
for positive \( \epsilon \) is crucial for obtaining the \( L^\infty \)-estimate of the solution.

Keywords: Parabolic equations, nonlinear boundary conditions, \( L^\infty \)-estimate

Classification: 35K55, 35K60

Let \( u(x,t) \) be a smooth function satisfying the heat equation \( u_t = u_{xx} \) in the rectangle \( 0 < x < 1, \ 0 < t < T \) and assume that \( u(x,0) = u_0(x)(0 \leq x \leq 1), \ u_x(0,t) = 0, \ u_x(1,t) = f(u,(1,t))(0 \leq t \leq T), f \in C(\mathbb{R}) \). Multiplying the equation by \( u^r \) for positive odd \( r \) and integrating we immediately derive
\[
\int_0^1 |u(x,t)|^{r+1} \, dx \leq \int_0^1 |u_0(x)|^{r+1} \, dx + (r+1)(\sup |f(x)|)C^r t
\]
for \( C = \max \max \max |u(1,\tau)| \). Taking the \( (r+1)\)-th root of both sides and passing to the limit as \( r \to \infty \) we obtain
\[
|u(x,t)| \leq \max |u_0(x)| + \max \max |u(1,\tau)|
\]
for all \( (x,t) \in [0,1] \times [0,T] \), i.e., the solution \( u \) can be pointwise estimated by its maximum value at the beginning (\( t = 0 \)) and on the boundary (\( x = 1 \)).

In this note we shall prove a result similar to (1) for more general parabolic equations in several space variables. To begin with, let us consider the problem
\[
\begin{align*}
\frac{\partial u}{\partial t} &= \Delta u \quad \text{for } x \in D, \quad t > 0 \\
\frac{\partial u}{\partial \nu} &= f(u) \quad \text{for } x \in \Gamma, \quad t > 0 \\
u(x,0) &= u_0(x), \quad u_0 \in C^2(\overline{D}),
\end{align*}
\]
where \( D \subset \mathbb{R}^N \) is a bounded domain with a smooth boundary \( \Gamma \), \( \partial u/\partial \nu \) denotes the outward directed normal derivative of \( u \) on \( \Gamma \) and let \( f \) be a smooth function satisfying

\[
f(u)u \leq L(|u|^{\alpha+1} + 1)
\]

for fixed constants \( \alpha \geq 1, L > 0 \). For \( u_0 \) satisfying the compatibility conditions \( \partial u_0/\partial \nu = f(u_0) \) on \( \Gamma \) there exists a unique classical solution \( u(x,t) \in C^{2,1}([\overline{D} \times [0,T]) \) for some positive \( T \) (see, e.g., [2], [7]).

We prove that for

\[
p > (N-1)(\alpha-1)
\]

there exist positive constants \( M, \nu \), independent of \( T \), such that

\[
|u(x,t)| \leq M(1 + \sup_{x \in \overline{D}}|u_0(x)|)(1 + \sup_{0 \leq t \leq T} \int_{\Gamma} |u(s, \tau)|^\nu d\tau)^\nu
\]

for all \((x,t) \in \overline{D} \times [0,T] \). The constants \( M, \nu \) depend solely on the data \( D, f \) and on \( p \).

Similar results for problems in which the nonlinearity occurs in the equation rather than in the boundary conditions, e.g.

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \Delta u + f(u) \quad (x,t) \in D \times (0,T), \\
u(x,t) &= 0 \quad (x,t) \in \Gamma \times (0,T), \\
u(x,0) &= u_0(x) \quad x \in D,
\end{align*}
\]

have been obtained using the same Moser type method by Alikakos [1], Rothe [12], Nakao [9], [10], Filo [5] (the list is surely not complete). To point out the difference between the value of the "critical" exponent \((N-1)(\alpha-1)\) for Problem (2) and the analytical one for Problem (4), let us recall the result of [12]. Let

\[
\begin{align*}
r &> \frac{N}{2}(\alpha-1) & \text{for } N \geq 3, \\
r &> \alpha-1 & \text{for } N = 1, 2
\end{align*}
\]

and let \( u \) be a solution, say classical, to Problem (4). Then there exist positive constants \( K, \rho, \sigma \), independent of \( T \), such that

\[
|u(x,t)| \leq K((1 + \sup_{x \in \overline{D}}|u_0(x)|) + \sup_{0 \leq t \leq T} \left( \frac{1}{|D|} \int_{D} |u(x,t)|^\rho d\tau \right)^{1/r})^{\rho/\sigma})\sigma
\]

for all \((x,t) \in \overline{D} \times [0,T] \).

As follows from results of Friedman and McLeod [6], this result is sharp (except for \( N = 1 \)) in the following sense. For special choice of the domain \( D \) (\( D \) being a ball) and initial states it may occur that

\[
\sup_{0 \leq t \leq T} \int_{D} |u(x,t)|^r dx < \infty \quad \text{for } r < \frac{N}{2}(\alpha-1),
\]
but
\[ \limsup_{t \to T} \| u(\cdot, t) \|_{L^\infty(D)} = \infty. \]

However, as far as we know, no similar results have been obtained for Problem (2).

In [8] Levine and Payne considered Problem (2) for \( f \) satisfying \( f(u) = |u|^\alpha h(u) \), \( h \) increasing, \( \alpha > 1 \). They proved that if \( u_0 \) is sufficiently large, the corresponding classical solution \( u \) breaks down by becoming unbounded in finite time, say \( T_0 \). Applying our result (3) we may conclude that also
\[ \limsup_{t \to T_0} \int_\Gamma |u(s, t)|^p \, ds = \infty \]
for all \( p > (N - 1)(\alpha - 1) \), whenever \( h \) is bounded.

In [4] it is shown that any global classical solution of Problem (2) with \( 1 < \alpha < N/(N - 2) \) (if \( N \geq 3 \)) is bounded in \( H^1(D) \) uniformly with respect to \( t \geq 0 \). (By a global solution we mean one which exists on \( \overline{D} \times (0, \infty) \).) From our result it follows that it is also bounded in \( C(\overline{D}) \) for \( t \geq 0 \).

As in the several past years nonlinear diffusion problems have been intensively studied, we shall consider Problem (2) in which the heat equation is replaced by \( (\beta(u))_t = \Delta u \) for the exact power law nonlinearity \( \beta(u) = |u|^m \text{sign} u \), \( m > 0 \). This equation is for \( 0 < m < 1 \) well known as the porous medium equation and for \( m > 1 \) as the fast diffusion equation. (see, e.g. [3] and references therein). Nevertheless, this nonlinearity does not change the value of "critical" exponent and we shall prove the analogy to (3) whenever \( m \) is sufficiently small for \( N \geq 3 \).

The method of our proof consists in modifying suitably the Moser type technic [1], as appearing in [12], [9], [10].

Assumptions and Statement of Results.

We start by introducing some notation. For \( 0 < T < \infty \) let \( Q = D \times (0, T) \), \( S = \Gamma \times (0, T) \). The norms in the spaces \( L^\infty(D), H^1(D) \) will be denoted by \( \| \cdot \|_\infty, \| \cdot \|_{1,2} \) and we shall write \( u^* := |u|^\alpha \text{sign} u \), \( \int_D u(t)\varphi(t) := \int_D u(x, t)\varphi(x, t) \, dx, \int_\Gamma |u(t)|^\rho := \int_\Gamma |u(s, t)|^\rho \, ds \).

Now we consider the initial and boundary value problem
\begin{align}
(\beta(u))_t &= \Delta u & (x, t) & \in Q, \\
\frac{\partial u}{\partial \nu} &= f(u) & (x, t) & \in S, \\
u(x, 0) &= u_0(x), & u_0 & \in L^\infty(D) \cap H^1(D). \tag{5}
\end{align}

Throughout the paper we will make the following assumptions, on \( \beta \) and the boundary datum \( f \).

\begin{enumerate}
\item[(H1)] \( \beta(u) = |u|^m \text{sign} u \), where \( m \) is a positive constant, which may be arbitrary if \( N = 1, 2 \), but must satisfy
\[ 0 < m < (N + 2)/(N - 2) \quad \text{for } N \geq 3. \]
\item[(H2)] \( f \in C(R) \) is a given function such that
\[ f(u)u \leq L(|u|^\alpha+1 + 1) \]
\end{enumerate}
for some $L > 0$ and $\alpha \geq 1$.

It is known that in general we cannot expect Problem (5) to be solvable in the classical sense even if the data are arbitrarily smooth. Therefore, it is necessary to deal with a suitable class of weak solutions.

**Definition.** By a weak solution of Problem (5) we mean a function $u \in L^\infty(0, T; H^1(D)) \cap L^\infty(Q)$, such that $(u^{(m+1)/2})_t \in L^2(Q)$, satisfying

\[
\int_D \beta(u(\tau))\varphi(\tau) - \int_0^\tau \int_D (\beta(u)\varphi_t - \nabla u \nabla \varphi) = \\
\int_0^\tau \oint_\Gamma f(u)\varphi + \int_D \beta(u_0)\varphi(0)
\]

for all $\varphi \in H^1(Q)$ and a.e. $\tau \in (0, T)$.

We note that if $u$ is a weak solution of Problem (5) with $f \in C^1(\mathbb{R})$, then from the results of DiBenedetto [3] it follows that $u \in C(\bar{D} \times (0, T])$. If in addition $u_0(x)$ is continuous in $\bar{D}$, then $u \in C(\bar{Q})$. We can now state our main result.

**Theorem 1.** Let $u$ be a weak solution of Problem (5) and assume that $(H1)$, $(H2)$ hold. Let

\[
\mathcal{B}(u) := \sup_{0 \leq \tau \leq T} \oint_{\Gamma} |u(\tau)|^{(N-1)(\alpha-1)+\varepsilon}
\]

for some $\varepsilon > 0$.

Then there exist positive constants $M, \nu$ depending solely on the data $m, D, f$ and on $\varepsilon$ such that

\[
\|u(\cdot, t)\|_{\infty} \leq M(1 + \|u_0\|_{\infty})(1 + \mathcal{B}(u))^\nu \quad \text{for all } 0 \leq t \leq T.
\]

**Remark 1.** One can prove an analogous statement if $\partial u / \partial \nu = f(u)$ holds only over a part $\Gamma_1$ of the boundary with positive $(N-1)$ dimensional Lebesgue measure and if we require, e.g. $u \equiv 0$ on $\Gamma_2, \Gamma_2 = \Gamma \setminus \Gamma_1$.

Let us now state a series of assertions, which contain all elements for the proof of Theorem 1 and say more about the dependence of $M, \nu$ on the data and on $\varepsilon$.

**Proposition 1.** Assume that $(H1)$ holds, and suppose that the (appropriately smooth) function $u(x, t)$ satisfies the inequality

\[
\frac{d}{dt} \int_D |u(t)|^{m+r} + L_0\|u^{(1+r)/2}(t)\|_{1,2}^2 \leq \\
\leq L_1(m+1)^\xi(\int_D |u(t)|^{m+r})^{\frac{1+r}{m+r}} + L_2(m+r)
\]

for all $r \geq 2$ and a.e. $t \in (0, T)$ with some constants $L_0 > 0, L_1, L_2, \xi \geq 0$. Let

\[
U_0 := \sup_{0 \leq t \leq T} \left( \frac{1}{|D|} \int_D |u(t)|^{m+1})^{1/(m+1)} \right).
\]
Then there exist positive constants $C, \theta$ depending on $D, m, \xi, L_0, L_2$, independent of $T$, such that

$$\sup_{0 \leq t \leq T} \|u(\cdot, t)\|_\infty \leq (1 + L_1)^\theta C \max(1, \|u(\cdot, 0)\|_\infty, U_0).$$  

(9)

**Proposition 2.** Let $u$ be a weak solution of Problem (5) and let (H1), (H2) hold.

Then there exist positive constants $\Theta, \mu, \xi$, depending only on the data $D, f, m$ and on $\epsilon$ such that $u$ satisfies (8) with $L_0 = 1/4m$, $L_2 = |\Gamma|N-1(L + 4L_0)$ and

$$L_1 = \Theta(1 + B(u))^{m/2}.$$  

(10)

**Proposition 3.** Under the hypotheses of Proposition 2 there exist positive constants $K, \zeta$ such that

$$U_0 \leq \max(\|u_0\|_\infty, K(1 + B(u))^\zeta).$$  

(11)

The constants $K$ and $\zeta$ depend only on the data and not on $T$.

Let us start with

**Proof of Proposition 2:** Putting $\varphi = u^r$ for $r \geq 1$ into (6) we obtain, with the assistance of (H1), (H2),

$$\frac{m}{m + r} \frac{d}{dt} \int_D |u(t)|^{m+r} + \frac{4r}{(1 + r)^2} \int_D |\nabla u^{(1+r)/2}(t)|^2 \leq$$

$$2L \int_\Gamma |u(t)|^{\alpha+r} + L|\Gamma|N-1,$$

(12)

for a.e. $t \in (0, T)$. We note that it is possible to take $u^r$ as the test function also in the case of $m > 1$, in which $u_1$ does not always exist. However, in this case $(\beta(u))_t$ exists and (6) yields (12). First of all we estimate the first term on the right hand side of (12). We shall distinguish two cases. If $N \geq 3$ then for positive $\epsilon$ we obtain

$$\int_\Gamma |u|^{\alpha+r} \leq (\int_\Gamma |u|^{(1+r)(N-1)/(N-2)})^p (\int_\Gamma |u|^{(N-1)(\alpha-1)+\epsilon})^q,$$

(13)

where

$$P = \frac{(N-2)(\alpha-1)}{(N-1)(\alpha-1)+\epsilon}, \quad Q = \frac{\alpha-1}{(N-1)(\alpha-1)+\epsilon}$$

and

$$R = \frac{\epsilon}{(N-1)(\alpha-1)+\epsilon}.$$  

If $N = 2$ it holds that

$$\int_\Gamma |u|^{\alpha+r} \leq (\int_\Gamma |u|^{2(\alpha-1)+\epsilon})^{(1+r)/\epsilon} (\int_\Gamma |u|^{(\alpha-1)+\epsilon})^q.$$

(14)
where
\[
P = \frac{\varepsilon(\alpha - 1)}{(2(\alpha - 1) + \varepsilon)(\alpha - 1 + \varepsilon)},
\]
\[
Q = \frac{\alpha - 1}{\alpha - 1 + \varepsilon}
\quad \text{and} \quad
R = \frac{\varepsilon}{2(\alpha - 1) + \varepsilon}.
\]

The above inequalities play a key role in our considerations. Now, let us come back to (13). Put
\[
p = (N-2)/(N-1)P \quad \text{and} \quad q = 1/R.
\]

By the embedding theorem, there exists a \( C_\varepsilon > 0 \) such that
\[
\int_{\Gamma} |\varphi|^2 \frac{(N-1)}{(N-2)} \leq (C_\varepsilon \|\varphi\|_{1,2}^2)^\frac{(N-1)}{(N-2)}
\]
for all \( \varphi \in H^1(D) \),

and as \( p^{-1} + q^{-1} = 1 \), we arrive at
\[
\int_{\Gamma} |u|^{\alpha+r} \leq \eta \|u^{(1+r)/2}\|_{1,2}^2 +
\]
\[
+ \left(\frac{C_\varepsilon}{\eta}\right)^{(N-1)(\alpha-1)/\varepsilon} \left(\int_{\Gamma} |u|^{(N-1)(\alpha-1)+\varepsilon} \right)^{(\alpha-1)/\varepsilon} \int_{\Gamma} |u|^{1+r}
\]
for any \( \eta > 0 \), where Young's inequality has been used. Let \( L_0 \) be a constant such that \( 0 < L_0 \leq r(m+r)/m(1+r)^2 \) for all \( r \geq 1 \). Specifying \( \eta \) as \( mL_0/2L(m+r) \), (12) and (15) then yield
\[
\frac{d}{dt} \int_{D} |u(t)|^{m+r} + 3L_0 \|u^{(1+r)/2}(t)\|_{1,2}^2 \leq
\]
\[
\leq C_1(B(u))^{(\alpha-1)/\varepsilon}(m+r)^\sigma \int_{\Gamma} |u(t)|^{1+r} + L_2(m+r)
\]
where
\[
\sigma = 1 + \frac{(N-1)(\alpha-1)}{\varepsilon}
\]
and the nonnegative constant \( C_1 \) depends only on the data \( D, m, f \) and on \( \varepsilon \).

Next, the following inequality is very useful
\[
\int_{\Gamma} |\varphi|^2 \leq \delta \int_{D} |\nabla \varphi|^2 + \frac{C}{\delta} \int_{D} |\varphi|^2
\]
for all \( \varphi \in H^1(D) \) and all sufficiently small \( \delta \), say \( 0 < \delta \leq \delta_0 \), \( \delta_0 \) being given, where the positive constant \( C \) does not depend on \( \delta \) (see, for example, [11, page 15]).

Now, with the assistance of (17), (16) gives
\[
\frac{d}{dt} \int_{D} |u(t)|^{m+r} + 2L_0 \|u^{(1+r)/2}(t)\|_{1,2}^2 \leq
\]
\[
\leq C_2(B(u))^{2(\alpha-1)/\varepsilon}(m+r)^{2\sigma} \int_{D} |u(t)|^{1+r} + L_2(m+r)
\]
for a.e. $t \in (0, T)$ and all $r \geq 1 \; (C_2 = C C_1^2 / L_0)$. If $m \geq 1 \; (8)$ and $(10)$ follow easily.

Thus, let $0 < m < 1$. In this case, Hölder's inequality and Sobolev embedding theorem immediately yield

$$
\int_D |u|^{1+r} \leq (C_2 \|u^{(1+r)/2}\|_{1,2}^2)^P (\int_D |u|^{m+r})^Q
$$

for

$$
P = \frac{N(1-m)}{N(1-m) + 2(m+r)} \quad \text{and} \quad Q = \frac{2(1+r)}{N(1-m) + 2(m+r)}.
$$

Now, applying Young's inequality, we arrive at

$$
\int_D |u|^{1+r} \leq \eta \|u^{(1+r)/2}\|_{1,2}^2 + \left( \frac{C_2}{\eta} \right)^\frac{N(1-m)}{2(m+r)} (\int_D |u|^{m+r})^{1+r

}^\frac{1}{m+r}
$$

for $\eta > 0$ and $(8)$ follows.

If $N = 2$, considering $(14)$, the proof is essentially the same.

PROOF of Proposition 1: Many, but not all, of the technical details used in our proof were established by Alikakos in [1]. However, to make our work self-contained, we include the proof of Proposition 1 for $0 < m \leq 1$ here. We note that the case $m > 1$ can be proved using the same procedure. To simplify the notation, put

$$
r_k = 2^k \; \text{and} \; q_k = L_1(m + r_k)^{2} \; \text{for} \; k = 0, 1, 2, \ldots
$$

Consider first the case $N \geq 3$. Using Hölder's inequality and Sobolev embedding theorem, the integral in the first term on the right hand side of $(8)$ can be estimated as follows,

$$
\int_D |u|^{m+r_k} \leq (C_2 \|u^{(1+r_k)/2}\|_{1,2}^2)^P (\int_D |u|^{m+r_{k-1}})^Q
$$

where

$$
P = \frac{N r_{k-1}}{N(1-m) + 2(m + r_{k-1}) + N r_{k-1}}
$$

and

$$
Q = \frac{N(1-m) + 2(m + r_k)}{N(1-m) + 2(m + r_{k-1}) + N r_{k-1}}.
$$

Now, by Young's inequality, $(19)$ yields

$$
(\int_D |u|^{m+r_k})^{\frac{1+r_k}{m+r_k}} \leq \epsilon_k \|u^{(1+r_k)/2}\|_{1,2}^2 + \delta_k (\int_D |u|^{m+r_{k-1}})^{s_k}
$$

where

$$
s_k = \frac{1 + r_k}{m + r_{k-1}} \; \text{and} \; \delta_k = \left( \frac{C_2}{\epsilon_k} \right)^{P s_k/Q}.
$$
The positive number \( \varepsilon_k \) will be determined later. Next, due to (20) and (8), we arrive at

\[
\frac{d}{dt} \int_D |u(t)|^{m+ra} + (L_0 - q_k \varepsilon_k - \varepsilon_k^2)\|u^{(1+ra)/2}(t)\|_{1,2}^2 \leq
\]

\[
-\varepsilon_k \left( \int_D |u(t)|^{m+ra}(1+ra)/(m+ra) + L_2 (m+r_k) + \right.
\]

\[
+ \varepsilon_k \delta_k (1 + \frac{q_k}{\varepsilon_k}) \left( \int_D |u(t)|^{m+ra-1} \right)_{ra}
\]

for all \( k = 1, 2, \ldots \) and a.e. \( t \in (0,T) \).

Now, if we choose

\[
\varepsilon_k = \varepsilon_0 / (m + r_k)^{\xi}(1 + L_1)
\]

for \( \varepsilon_0 \) sufficiently small we obtain \( L_0 - q_k \varepsilon_k - \varepsilon_k^2 \geq 0 \) for all \( k = 1, 2, \ldots \). Solving the differential inequality

\[
y' + \epsilon y^{\nu} \leq \epsilon P \quad \text{a.e. on } (0,T),
\]

for \( \epsilon > 0, \nu \geq 1 \), we obtain

\[
y(t) \leq \max(y(0), P^{1/\nu}) \quad \text{for all } t \in [0,T],
\]

and thus

\[
y_k(t) \leq \max(y_k(0), (\delta_k (1 + \frac{q_k}{\varepsilon_k}) \frac{m+ra}{1+ra} |D|^{\frac{m+ra-1}{m+ra}} U_{k-1}^{m+ra} + \right.
\]

\[
+ L_2 \left( \varepsilon_k (m + r_k) \right)^\frac{m+ra}{1+ra} / |D|)
\]

for all \( t \in [0,T] \) and \( k = 1, 2, \ldots \), where

\[
y_k(t) = \frac{1}{|D|} \int_D |u(t)|^{m+ra}
\]

and

\[
U_k = \sup_{0 \leq t \leq T} \left( \frac{1}{|D|} \int_D |u(t)|^{m+ra} \right)^{1/(m+ra)}.
\]

Taking \( \varepsilon_0 \) small, it is not difficult to verify that

\[
1 < d_k := \max((\delta_k (1 + \frac{q_k}{\varepsilon_k}) \frac{m+ra}{1+ra} |D|^{\frac{m+ra-1}{m+ra}}(\frac{L_2}{\varepsilon_k} (m + r_k) \frac{m+ra}{1+ra} \frac{1}{|D|}) \leq
\]

\[
\leq (1 + L_1)^{\kappa} a r_k^{\sigma}
\]

for some positive \( \kappa, a, \sigma \) (independent of \( k \)) and all \( k = 1, 2, \ldots \). We note that the constants \( \kappa, \sigma \) depend only on \( N, m, \xi \) and \( a \) on \( L_0, L_2, D, m \). Therefore, (21) implies

\[
y_k(t) \leq \max(\|u_0\|^{m+ra}_\infty, d_k(U_{k-1}^{m+ra} + 1))
\]
for all $t \in [0,T]$. Put $K = \max(1, \|u_0\|_\infty, U_0)$, then (23) yields

\[ U_i \leq \left( \prod_{j=1}^{i} (2d_j)^{1/(m+r_j)} \right) K \]

for all $i = 1, 2, \ldots$. Now, with the assistance of (24) and (22), (23) gives

\[ \left( \int_D |u(t)|^{m+r} \right)^{1/(m+r)} \leq |D|^{1/(m+r)} (2(1+L)^{\kappa_0})^{S_1} 2^\sigma S_2 K \]

for all $k = 1, 2, \ldots$ and $t \in [0,T]$, where

\[ S_1 = \sum_{i=1}^{\infty} \frac{1}{m+r_i}, \quad S_2 = \sum_{i=1}^{\infty} \frac{i}{m+r_i}. \]

Finally, passing to the limit as $k \to \infty$ we obtain (9).

If $N = 2$ we obtain (19) with $P = r_{k-1}/(1-m+r_{k-1})$ and (20) with $s_k$ as above. The rest of the proof is then the same. \( \blacksquare \)

**Proof of Proposition 3:** To prove Proposition 3, we shall deal with the differential inequality (16) for $r = 1$. Let us suppose that $(N-1)(\alpha-1) + \varepsilon < 2$ as otherwise the proof is straightforward. In this case, using Hölder's inequality, the embedding theorem and Young's inequality, we arrive at

\[ \int_{\Gamma} |u|^2 \leq \eta \|u\|_{1,2}^2 + \left( \frac{C_\varepsilon}{\eta} \right) P \left( \int_{\Gamma} |u|^{(N-1)(\alpha-1) + \varepsilon} \right) \]

for any $\eta > 0$, where

\[ P = \frac{(N-1)(2-(N-1)(\alpha-1) - \varepsilon)}{(N-1)(\alpha-1) + \varepsilon} \]

for $N \geq 3$,

\[ P = \frac{p(3-\alpha-\varepsilon)}{(p-2)(\alpha-1+\varepsilon)} \]

for $N = 2$, $p > 2$ being arbitrary, and $C_\varepsilon > 0$ originates from the embedding theorem. Thus, (16) yields

\[ \frac{d}{dt} \int_D |u(t)|^{m+1} + 2L_0 \|u(t)\|_{1,2}^2 \leq C_3 (1 + B(u))^{\omega} \]

a.e. on $(0,T)$, where the constants $C_3, \omega$ depend solely on the data $D, m, f$ and on $\varepsilon$. According to (H1), $L^{m+1}(D)$ is embedded into $H^1(D)$, hence

\[ u_0 + C_4 \varepsilon_0^{2/(m+1)}(t) \leq C_3 (1 + B(u))^{\omega}/|D|, \]

\[ C_4 = 2L_0 |D|^{(1-m)/(m+1)}/C_\varepsilon. \]

Solving this differential inequality we obtain

\[ U_0 \leq \max(\|u_0\|_\infty, \frac{C_4 C_3}{2L_0 |D|^{2(m+1)}})^{1/2} (1 + B(u))^{\omega/2}, \]

hence (11).

The proof of Theorem 1 is completed. \( \blacksquare \)
Remark 2. The preceding theorem can be extended to problems, where the reaction term occurs also in the equation, i.e.

\[
\begin{align*}
(\beta(u))_t &= \Delta u + g(u) & \text{in } Q, \\
\partial u / \partial v &= f(u) & \text{on } S, \\
u(x, 0) &= u_0(x) & \text{in } D,
\end{align*}
\]

where \( g \) is sufficiently smooth, satisfying

\[ g(\lambda) \lambda \leq C(|\lambda|^{\gamma+1} + 1) \]

for some \( C > 0, \gamma \geq 1 \).

Theorem 2. Let \( u \) be a weak solution of Problem (25). Put

\[ \mathcal{F}(u) := \sup_{0 \leq t \leq T} \int_D |u(t)|^r \]

for

\[ r > \frac{N}{2}(\gamma m - 1) \text{ if } N \geq 3, \quad r > \gamma m - 1 \text{ if } N = 1, 2, \quad r > 0. \]

Under the preceding hypotheses on \( u_0, \beta, f \) and \( g \), there exists a constant \( M \), independent of \( T \), such that

\[ \|u(\cdot, t)\|_\infty \leq M \]

for all \( t \in [0, T] \). The constant \( M \) depends on \( D, f, g, \beta, \|u_0\|_\infty, \varepsilon, r, B(u) \) and \( \mathcal{F}(u) \).

Theorem 2 can be proved in a manner similar to that of Theorem 1 and analogical one in [5].

REFERENCES

[5] Filo J., \( L^{00} \)-estimate for nonlinear diffusion equations, manuscript.

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