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Non-convex perturbations of evolution equations with $m$-dissipative operators in Banach spaces

EVGENIOS P. AVGERINOS AND NIKOLAOS S. PAPAGEORGIU*

Abstract. In this paper we establish the existence of integral solutions for a nonlinear, multivalued evolution equation of the form $\dot{x}(t) \in Ax(t) + F(t,x(t))$, where $A : X \to 2^X$ is an $m$-dissipative operator and $F(\cdot,\cdot)$ a nonconvex valued perturbation. Our result generalizes a recent existence theorem of Cellina-Marchi (Israel J.Math 46 (1983), pp.1-11).

Keywords: $m$-dissipative operator, compact semigroup, lower semicontinuous multifunction, Arzela-Ascoli theorem, parabolic equation

Classification: 34G20

1. Introduction.

Evolution equations of the form $-\dot{x}(t) \in Ax(t) + f(t)$ in a Hilbert space, were first studied by Brezis [4], with $A$ a maximal monotone operator and $f(\cdot)$ an integrable perturbation. The work of Brezis was extended by Attouc-Damlamian [1], to systems of the form $-\dot{x}(t) \in Ax(t) + F(t,x(t))$, with $F(\cdot,\cdot)$ being a multivalued perturbation having convex values. Attouc-Damlamian [1] proved two existence results: one with $A$ being a general maximal monotone operator, but with the underlying state space being $\mathbb{R}^n$ and the other with $A$ being a subdifferential (i.e $A = \partial \phi$, with $\phi$ being a proper, closed, convex function) and the underlying state space being any separable Hilbert space. Recently Cellina-Marchi [6] proved an existence theorem for the case where the multivalued perturbation has nonconvex values and the state space is $\mathbb{R}^n$. The study of those evolution equations in general Banach spaces (not necessarily Hilbert), was initiated by Pazy [12], who considered the case of $A$ being a densely defined, linear, $m$-accretive operator and the perturbation was single valued. A nonlinear version of Pazy's theorem was proved by Vrabie [14], who also considered the case of multivalued perturbations with convex values, extending this way the work of Attouc-Damlamian [1]. Other interesting works in these or related issues were done by Gutman [8], Haraux [9] and Schechter [13] (he studied the dependence of the solutions on variations of the initial data).

In this note, we extend the result of Cellina-Marchi [6] to arbitrary separable Banach spaces, weakening also the hypotheses on the multivalued perturbation $F(t,x)$. Instead of assuming joint Hausdorff continuity for $F(t,x)$, we only require lower semicontinuity in the variable $x$, a more natural hypothesis in the context of applications.

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2. Preliminaries.

Let \((\Omega, \Sigma)\) be a measurable space and \(X\) a separable Banach space. By \(P_f(X)\) we will denote the collection of all nonempty, closed subsets of \(X\). A multifunction \(F : \Omega \to P_f(X)\) is said to be graph measurable, if \(GrF = \{(\omega, x) \in \Omega \times X : x \in F(\omega)\} \subseteq \Sigma \times B(X)\), where \(B(X)\) is the Borel field of \(X\). Now let \(\mu(\cdot)\) be a \(\sigma\)-finite measure on \(\Sigma\). By \(S_f^\mu\) we will denote the set of integrable selectors of \(F(\cdot)\) i.e. \(S_f^\mu = \{f \in L^1(\Omega) : f(\omega) \in F(\omega) \text{ \(\mu\)-a.e.}\}\). Using Aumann's selection theorem, it is easy to check that if \(\omega \to |F(\omega)| = \sup\{|x| : x \in F(\omega)\} \in L^1_+\) (in which case we say that \(F(\cdot)\) is integrably bounded), then \(S_f^\mu \neq \emptyset\). If \(Y, Z\) are Hausdorff topological spaces and \(G : Y \to 2^Z \setminus \{\emptyset\}\), then we say that \(F(\cdot)\) is lower semicontinuous (l.s.c.), if for all \(U \subseteq Z\) open, the set \(G^{-1}(U) = \{y \in Y : G(y) \cap U \neq \emptyset\}\) is open in \(Y\). If \(Y, Z\) are metric spaces, then the above definition is equivalent to saying that for all \(y_n \to y\) we have \(G(y) \subseteq \lim G(y_n) = \{z \in Z : z = \lim z_n, z_n \in G(y_n)\}\).

Next let \(X\) be any Banach space. Let \(J : X \to 2^{X^*}\) be the duality map of \(X\) i.e. \(J(x) = \{x^* \in X^* : (x^*, x) = \|x\|^2 = \|x^*\|^2\}\). Clearly the values of \(J(\cdot)\) are closed, convex, bounded subsets of \(X^*\), which because of the Hahn-Banach theorem are also nonempty. Recall that if \(X^*\) is strictly convex, then \(J(\cdot)\) is single valued. Using \(J(\cdot)\) we can define the upper semi-inner product (denoted by \((\cdot, \cdot)_+\)) and the lower semi-inner product (denoted by \((\cdot, \cdot)_-\)) as follows:

\[
(x, y)_+ = \sup\{(x^*, y) : x^* \in J(x)\}
\]

and

\[
(x, y)_- = \inf\{x^*, y : x^* \in J(x)\}
\]

for all \(x, y \in X\). An operator \(A : X \to 2^X\) is said to be dissipative (see Barbu [2], if \((x - x', y - y')_+ \leq 0\) for any \((x, y), (x', y') \in GrA\). We say that \(A\) is \(m\)-dissipative, if it is dissipative and in addition \(R(I - \lambda A) = X\) for all \(\lambda > 0\). It is well known that an \(m\)-dissipative operator generates a semigroup \(\{S(t)\}_{t \geq 0}\) of nonlinear contractions, via the Crandall-Liggett formula

\[
S(t)x = \lim_{n \to \infty} (I - \frac{t}{n}A)^{-n}x, \quad t \geq 0, \quad x \in \overline{D(A)}.
\]

Now let \(A\) be an \(m\)-dissipative operator, \(f \in L^1(X)\) and \(x_0 \in \overline{D(A)}\). Consider the following Cauchy problem on \(T = [0, b]\):

\[
(*) \quad \begin{cases}
\dot{x}(t) = Ax(t) + f(t) \\
x(0) = x_0
\end{cases}
\]

Following Benilan [3], we say that a function \(x \in C(T, X)\) is an "integral solution" of \((*)\), if \(x(0) = x_0\) and

\[
\|x(t) - y\|^2 \leq \|x(s) - y\|^2 + 2 \int_s^t (f(r) + z, x(r) - y)_+ \, dr
\]

for all \((y, z) \in GrA\) and all \(0 \leq s \leq t \leq b\).
It is well known that under the above hypotheses Cauchy problem (*) has a unique integral solution. Moreover this unique integral solution depends continuously on the data of the problem. In fact if \( x_1(\cdot) \) is the solution of (*) with data \((x_0, f_1) \in \overline{D(A)} \times L^1(X)\) and \( x_2(\cdot) \) the solution of (*) with data \((x_0, f_2) \in \overline{D(A)} \times L^1(X)\), then we have

\[
\|x_1(t) - x_2(t)\|^2 \leq \|x_0 - x_0^2\|^2 + 2 \int_0^t (f_1(r) - f_2(r), x_1(r) - x_2(r))_+ \, dr, \quad t \in T,
\]

or equivalently

\[
\|x_1(t) - x_2(t)\| \leq \|x_0 - x_0^2\| + \int_0^t \|f_1(r) - f_2(r)\| \, dr.
\]

If \( A \) is densely defined, linear, \( m \)-accretive, then the notion of integral solution coincides with that of mild solution.

Recall that a "strong solution" of (*) is a continuous function \( x : T \to X \) (i.e. \( x(\cdot) \in C(T, X) \)), for which we have that \( x(t) \in D(A) \), is differentiable a.e. on \((0, b)\) and satisfies (*) a.e. with \( x(0) = x_0 \in \overline{D(A)} \).

Every strong solution is an integral solution. The converse is true only if we impose additional hypotheses on \( X, A \) and \( f \). We are not going to go into the details of that problem. We only mention that if \( X = \mathbb{R}^n \) and \( A \) is maximal monotone or if \( X \) is a Hilbert space and \( A = \partial \phi \), with \( \phi \) being a proper, closed, convex function on \( X \), then every integral solution is also strong for any initial condition \( x_0 \in \overline{D(A)} \). For further details we refer to Barbu [2], Brezis [4] and Schechter [13].

3. The Theorem.

In this section we will establish the existence of an integral solution for the following multivalued evolution equation:

\[
(*) \quad \begin{cases} 
\dot{x}(t) \in Ax(t) + F(t, x(t)) \\
x(0) = x_0 
\end{cases}
\]

By an integral solution of (**), we mean a function \( x \in C(T, X) \), which is an integral solution (as defined in Section 2) of \( \dot{x}(t) \in Ax(t) + f(t), x(0) = x_0 \) for some \( f \in S^1_{F,(x(\cdot))} \).

Let \( T = [0, b] \) and let \( X \) be a separable Banach space. We will need the following hypotheses:

- **H(A):** \( A : X \to 2^X \) is an \( m \)-dissipative operator, which generates a semigroup of compact nonlinear contractions (i.e. \( S(t) : \overline{D(A)} \to \overline{D(A)} \) is compact for \( t > 0 \)),

- **H(F):** \( F : T \times X \to P_f(X) \) is a multifunction s.t.

  1. \( (t, x) \to F(t, x) \) is graph measurable,

  2. for every \( t \in T \), \( x \to F(t, x) \) is l.s.c.,

  3. \( |F(t, x)| = \sup \{|y| : y \in F(t, x)\} \leq a(t) + b(t)\|x\| \) a.e.

- **H_0:** \( x_0 \in \overline{D(A)} \).

We have the following existence result concerning (**).
Theorem. If hypotheses \( H(A), H(F) \) and \( H_0 \) hold, then (**) admits an integral solution.

Proof: We will start by determining an a priori bound for the integral solutions of (**). Suppose \( x(t) \in C(T,X) \) is such a solution of (**). Recalling that \( S(t) \) is the integral solution of \( \dot{y}(t) = Ay(t), y(0) = x_0 \) and using the inequalities of Section 2, we have:

\[
\|x(t) - S(t)x_0\| \leq \int_0^t \|f(s)\| ds
\]

for all \( t \in T \) and some \( f \in S(t,x(t)) \). Since \( t \rightarrow S(t)x_0 \) is continuous on \( T \) and using hypothesis \( H(F) \) (3), we have

\[
\|x(t)\| \leq M_1 + \int_0^t [a(s) + b(s)]\|x(s)\| ds
\]

for some \( M_1 > 0 \). Applying Gronwall's inequality, we get that

\[
\|x(t)\| \leq K \exp\|b\|_1 = M_2
\]

where \( K = M_1 + \|a\|_1 \). Then define a new multifunction \( \hat{F} : T \times X \rightarrow P_f(X) \) as follows:

\[
\hat{F}(t,x) = \begin{cases} F(t,x) & \text{if } \|x\| \leq M_2 \\ F(t, \frac{M_2 x}{\|x\|}) & \text{if } \|x\| > M_2. \end{cases}
\]

Observe that \( \hat{F}(t,x) = F(t,p_{M_2}(x)) \), where \( p_{M_2}(\cdot) \) is the \( M_2 \)-radial retraction. We have \( \text{Gr}\hat{F} = \{(t,x,y) \in T \times X \times X : (t,p_{M_2}(x),y) \in \text{Gr}\hat{F}\} \). Let \( r : T \times X \times X \rightarrow T \times X \times X \) be defined by \( r(t,x,y) = (t,p_{M_2}(x),y) \). Recalling that \( p_{M_2}(\cdot) \) is 2-Lipschitz, we have that \( r(\cdot,\cdot,\cdot) \) is continuous, hence measurable. So, since \( \text{Gr}\hat{F} \in \Sigma \times B(X) \times B(X) \), we have \( \text{Gr}r^{-1}(\text{Gr}\hat{F}) = \text{Gr}\hat{F} \in \Sigma \times B(X) \times B(X) \) i.e. \( \hat{F}(\cdot,\cdot,\cdot) \) is graph measurable. Also since \( \hat{F}(t,\cdot) \) is the composition of the Lipschitz function \( p_{M_2}(\cdot) \) with the l.s.c. multifunction \( F(t,\cdot) \), we have that \( \hat{F}(t,\cdot) \) is l.s.c. Finally note that \( |F(t,x)| \leq a(t) + M_2 b(t) = \gamma(t) \) a.e. with \( \gamma(\cdot) \in L^1_+ \).

In the sequel we will consider the following multivalued Cauchy problem:

\[
(**') \quad \begin{cases}
\dot{x}(t) \in Ax(t) + \hat{F}(t,x(t)) \\
x(0) = x_0
\end{cases}
\]

Let \( h \in L^1(X) \) and consider the Cauchy problem

\[
(**) \quad \begin{cases}
\dot{x}(t) \in Ax(t) + h(t) \\
x(0) = x_0
\end{cases}
\]

We know (see Section 2), that (**) has a unique integral solution. Let \( r : L^1(X) \rightarrow C(T,X) \) be the map that to each \( L^1(X) \)-perturbation \( h(\cdot) \) assigns the corresponding unique integral solution \( r(h)(\cdot) \in C(T,X) \) of (**). Let \( B(\gamma) = \)
\[ h \in L^1(X) : \|h(t)\| \leq \gamma(t) \) a.e.\}. Our claim is that \( K = r(B(\gamma)) \) is relatively compact in \( C(T,X) \).

To this end, first we will show that for every \( t \in T, K(t) = r(B(\gamma))(t) = \{x(t) : x(\cdot) = r(h)(\cdot), h \in B(\gamma)\} \) is compact in \( X \). For \( t = 0 \), we have \( K(0) = \{x_0\} \) and so the claim is automatically verified. Hence let \( t > 0, t \in T \). Note that \( B(\gamma) \) is a uniformly integrable subset of \( L^1(X) \). So given \( t \in (0,\infty) \) and \( \varepsilon > 0 \), we can find \( \delta(\varepsilon) \in (0,t) \) s.t. for \( B \subseteq T \) Lebesgue measurable with \( \lambda(B) < \delta \), we have:

\[
\int_B \|h(s)\| \, ds < \varepsilon
\]

for all \( h \in B(\gamma) \). Now consider the following Cauchy problem on \([t - \delta, t] \):

\[
\begin{cases}
\dot{x}(\delta)(s) \in Ax(\delta)(s) \\
x(\delta)(t - \delta) = r(h)(t - \delta)
\end{cases}
\]

where \( h \in B(\gamma) \). From the inequalities of Section 2, we have:

\[
\|x(\delta)(t) - r(h)(t)\| \leq \int_{t-\delta}^t \|h(s)\| \, ds < \varepsilon
\]

for all \( h \in B(\gamma) \). Also recall that

\[
x(\delta)(t) = \dot{S}(\delta)r(h)(t - \delta) \subseteq S(\delta)K(t - \delta)
\]

and the latter is relatively compact in \( X \), since \( K(t - \delta) = \{y(t - \delta) : y(\cdot) \in K\} \) is bounded and \( S(\delta) \) is a compact contraction (see hypothesis \( H(A) \)). Therefore \( S(\delta)K(t - \delta) \) is compact. So for every \( t \in T \), every \( \varepsilon > 0 \) and every \( x \in K(t) \), there exists an element \( x_\varepsilon \) in the compact set \( S(\delta)K(t - \delta) \) s.t. \( \|x - x_\varepsilon\| < \varepsilon \implies K(t) \) is compact.

Next, recall that since the semigroup \( S(t) \) is compact, for \( B \subseteq X \) nonempty, bounded, we have that \( t \to \{S(t)x : x \in B\} \) is equicontinuous on \( T \). Hence given \( \varepsilon > 0 \), we can find \( \delta_1(\varepsilon) > 0 \) s.t. for \( |t' - t| < \delta_1 \) and for all \( x \in K(t - \delta) \) we have:

\[
\|S(t' - t + \delta)x - S(\delta)x\| < \varepsilon
\]

\[
\implies \|S(t' - t + \delta)r(h)(t - \delta) - S(\delta)r(h)(t - \delta)\| < \varepsilon
\]

\[
\implies \|x(\delta)(t') - x(\delta)(t)\| < \varepsilon.
\]

So finally for \( \delta_2 = \min(\delta, \delta_1) \) and for \( |t' - t| < \delta_2 \), we have

\[
\|r(h)(t') - r(h)(t)\|
\]

\[
\leq \|r(h)(t') - x(\delta)(t')\| + \|x(\delta)(t') - x(\delta)(t)\| + \|x(\delta)(t) - r(h)(t)\| < \varepsilon + \varepsilon + \varepsilon = 3\varepsilon
\]

\[
\implies K = r(B(\gamma)) \text{ is equicontinuous.}
\]

Invoking the Arzela-Ascoli theorem, we conclude that \( \overline{K} \) is compact in \( C(T,X) \). Thus by Mazur's theorem \( \overline{K} = \overline{\text{conv } K} \) is compact.
Next let $R : \mathcal{K}_c \to 2^{L^1(X)}$ be defined by

$$R(x) = S^1_{\hat{F}(\cdot, x(\cdot))}.$$  

Since $\hat{F}(\cdot, \cdot)$ is graph measurable, it is easy to check as before, that $t \to \hat{F}(t, x(t))$ is graph measurable and integrably bounded by $\gamma(\cdot)$ and so $R(\cdot)$ has nonempty values, in fact $R(\cdot) \gamma L^1(X)$-valued. Also since $\hat{F}(\cdot, \cdot)$ is l.s.c. and using Theorem 4.1 of [11], we have that if $x_n \to \bar{x}$ in $\mathcal{K}_c$, then $R(x) \subseteq s - \lim R(x_n)$, where $s$ indicates the strong topology on $L^1(X)$. So $R(\cdot)$ is l.s.c. (see section 2). Hence we can apply Fryszkowski's selection theorem [7], to get $v : \mathcal{K}_c \to L^1(X)$ continuous s.t. $v(x) \in R(x)$ for all $x \in \mathcal{K}$. Set $p = r \circ v$. Clearly $p : \mathcal{K}_c \to \mathcal{K}_c$ is continuous. Apply Schauder's fixed point theorem to get $x \in \mathcal{K}_c$ s.t. $x = p(x) = r(v(x))$. Hence we have that $\hat{x}(\cdot)$ is an integral solution of

$$\begin{cases}
\dot{x}(t) \in A\dot{x}(t) + v(\dot{x})(t) \\
\dot{x}(0) = x_0
\end{cases}$$

with $v(\dot{x})(\cdot) \in S^1_{\hat{F}(\cdot, \dot{x}(\cdot))}$. So $\dot{x}(\cdot) \in C(T, X)$ is an integral solution of $(**)'$. From the definition of $\hat{F}(t, x)$ and hypothesis $H(F)(3)$, we see easily that $|\hat{F}(t, x)| \leq a(t) + b(t)||x||$ a.e.. So as before, through Gronwall's inequality, we get $||\dot{x}(t)|| \leq M_2$, $t \in T \implies \hat{F}(t, \dot{x}(t)) = F(t, x(t))$, $t \in T \implies \dot{x}(\cdot)$ is the desired integral solution of $(**)$.  

As we mentioned in Section 2, when $X = \mathbb{R}^n$, then every integral solution is a strong solution. So we can state as a corollary to our theorem, an extension of the existence result of Cellina-Marchi [6].

So let $T = [0, \delta]$, $X = \mathbb{R}^n$ and make the following hypothesis about $A$:

$$H(A)' : A : D(A) \subseteq \mathbb{R}^n \to 2^{\mathbb{R}^n}$$

is a maximal monotone operator.

Then we get as a corollary to our theorem, the following extension of the work of Cellina-Marchi [6].

Corollary. If hypotheses $H(A)'$, $H(F)$ and $H_0$ hold, then $(**)$ admits a strong solution.

Remarks. (1) In Cellina-Marchi [6], the multivalued perturbation $F(t, x)$ was assumed to be jointly Hausdorff continuous.

(2) Hypotheses $H(F)$ (1) and (2), cover the case where $t \to F(t, x)$ is graph measurable and $x \to F(t, x)$ is Hausdorff continuous (see Theorem 3.3 of [10]).

4. An example.

Let $\Omega$ be a bounded open domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega = \Gamma$.

Let $r > (n - 2)/n$ and consider the following multivalued, nonlinear, parabolic partial differential equation on $T \times \Omega$:
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\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{\partial x(t,z)}{\partial t} - \Delta x(t,z)|x(t,z)|^{r-1} \in F(t,z,x(t,z)) \\
x(t,z) = 0 \quad \text{on} \quad T \times \Gamma \\
x(0,z) = x_0(z) \quad \text{on} \quad \{0\} \times \Omega
\end{array} \right.
\end{align*}
\]

(****)

Here \( F : T \times \Omega \times \mathbb{R} \to P_f(\mathbb{R}) \) is a multifunction which is l.s.c. in the third variable and \( (t,y) \to S^t_{F(t,y)} \) is graph measurable on \( T \times L^1(\Omega) \). It is easy to check that this is the case if \( (t,z) \to F(t,z,r) \) is measurable and \( r \to F(t,z,r) \) is Hausdorff continuous. Also assume that \( |F(t,z,r)| = \sup\{|v| : v \in F(t,z,r)\} \leq a(t,z) + b(t,z)r \) a.e. with \( a(\cdot,\cdot) \in L^1_+(T \times \Omega) \) and \( b(t,\cdot) \in L^\infty(\Omega) \) while \( t \to \|b(t,\cdot)\|_\infty \) belongs in \( L^1(\Omega) \). Furthermore let \( \hat{x}_0 = x_0(\cdot) \in L^1(\Omega) \).

Take \( X = L^1(\Omega) \). This is a separable Banach space. Consider the nonlinear operator \( A : D(A) \subseteq X \to X \) defined by \( Ax = \Delta x|x|^{r-1} \) with \( D(A) = \{ x \in X : x, x^{r-1} \in W_0^{1,1}(\Omega), \quad \Delta x|x|^{r-1} \in L^1(\Omega) \} \). From Brezis [5] we know that the operator \( A \) defined above is \( m \)-dissipative and the nonlinear semigroup it generates is compact for \( t \in (0,b) \). Also let \( \hat{F} : T \times X \to P_f(L^1(X)) \) be defined by \( \hat{F}(t,x) = S^t_{F(t,x(\cdot))} \). Then \( \hat{F}(\cdot,\cdot) \) is graph measurable, \( \hat{F}(t,\cdot) \) is l.s.c. (see Theorem 4.1 of [11]) and

\[
|\hat{F}(t,x) \leq \hat{a}(t) + \hat{b}(t)||x||_1 \quad \text{a.e}
\]

with

\[
\hat{a}(t) = \|a(t,\cdot)\|_1 \quad \text{and} \quad \hat{b}(t) = \|b(t,\cdot)\|_\infty.
\]

Rewrite the initial-boundary value problem (****) as the following abstract multivalued evolution equation:

(****)'

\[
\begin{align*}
\left\{ \begin{array}{l}
\hat{x}(t) \in Ax(t) + \hat{F}(t,x(t)) \\
x(0) = \hat{x}_0
\end{array} \right.
\end{align*}
\]

We see all hypotheses of our theorem are satisfied and so we know that (****)' has an integral solution \( \hat{x}(\cdot) \in C(T,L^1(\Omega)) \). Set \( x(t,z) = \hat{x}(t)(z)z \in \Omega \). This is a generalized solution of (****).

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