

# Commentationes Mathematicae Universitatis Carolinae

---

Marco Baronti; Pier Luigi Papini

Bicontractive projections in sequence spaces and a few related  
kinds of maps

*Commentationes Mathematicae Universitatis Carolinae*, Vol. 30 (1989), No. 4,  
665--673

Persistent URL: <http://dml.cz/dmlcz/106787>

## Terms of use:

© Charles University in Prague, Faculty of Mathematics and Physics, 1989

Institute of Mathematics of the Academy of Sciences of the Czech Republic  
provides access to digitized documents strictly for personal use. Each copy  
of any part of this document must contain these *Terms of use*.



This paper has been digitized, optimized for electronic  
delivery and stamped with digital signature within the  
project *DML-CZ: The Czech Digital Mathematics Library*  
<http://project.dml.cz>

## Bicontractive projections in sequence spaces and a few related kinds of maps

MARCO BARONTI, PIER LUIGI PAPINI

**Abstract.** Norm-one projections onto subspaces of a Banach space play an important role in approximation theory. In some classical sequence spaces, subspaces for which such a projection exists have been characterized recently. Bicontractive projections are not common: here we characterize the subspaces of  $c_0$  and  $l_p$  for which a projection of this type exists. Golomb defined central proximity maps: they are useful to construct a best approximation by the alternating method. Here we show that this class of maps often coincide with the class of bicontractive projections.

**Keywords:** Projections, Sequence spaces, Central proximity map

**Classification:** Primary 47A30, Secondary 47A05, 46A45, 41A45

### 1. Introduction.

Let  $X$  be a Banach space over the real field  $R$ ; we denote by  $Y$  a closed subspace of  $X$ . A map  $P : X \rightarrow Y$  is *idempotent* if  $P^2 = P$ ; a linear idempotent map  $P : X \rightarrow Y$  is called a *projection* onto  $Y$ . A projection  $P$  is said to be *bicontractive* if  $\|P\| = \|I - P\| = 1$ .

In general, Banach spaces are not rich of contractive projections, as Kakutani theorem and its generalizations say (see e.g. [1], §12). In some sequence spaces a few characterizations are known (see [2] and [3]). Of course bicontractive projections in these spaces are still less: in the present paper we try to produce results on this matter for some classes of subspaces in a few classical sequence spaces.

The interest in the class of bicontractive projections sprung for us when we studied central proximity maps (see [4]): in fact there are strong connections between the latter and the former.

We recall that an idempotent map  $P : X \rightarrow Y$  is called a *central proximity map* when the following condition holds:

$$(1) \quad \|x - Px - y\| = \|x - Px + y\| \quad \text{for all } x \text{ in } X \text{ and } y \text{ in } Y.$$

These maps were introduced by M. Golomb three decades ago: for general results on them we send to [10] (see also [4] for more references).

A number of conditions are sufficient that a central proximity map be linear: for example (see [4]) the condition that  $Y$  is hyperplane. Other conditions will be indicated in Section 3 below. In Section 4 we shall prove a few characterizations concerning bicontractive projections in sequence spaces. Finally, relations concerning central proximity maps and bicontractive projections will be considered in Section 5.

## 2. General known results and definitions.

The following results are known:

**Theorem 1.** (see [4]). *If  $P : X \rightarrow Y$  is a projection, then  $P$  satisfies (1) if and only if  $\|2P - I\| = 1$ .*

**Theorem 2.** (see [5]). *Let  $X = L_p(A, \Sigma, \mu)$ ,  $1 \leq p < +\infty$ , or  $X$  a pre-dual  $L_1$ -space. Then a projection  $P$  is bicontractive if and only if  $2P - I$  is an isometry.*

Note that the if part in Theorem 2 above trivially holds for any  $X$ ; moreover, if  $P$  a projection then  $2P - I$  is involutive (i.e.  $(2P - I)^2 = I$ ).

To state some of our results, the following definitions are necessary.

If  $x, y$  are in  $X$ , then we shall write  $x \perp y$  ( $x$  is orthogonal to  $y$ ) when  $\|x + ty\| \geq \|x\|$  for all  $t$  in  $R$ . We shall also write  $A \perp B$  when  $a \perp b$  for all  $a$  in  $A$  and  $b$  in  $B$ . The following characterization is possible. Set:

$$(2) \quad \tau(x, y) = \lim_{t \rightarrow 0^+} \frac{\|x + ty\| - \|x\|}{t}.$$

Then  $x \perp y$  if and only if  $-\tau(x, -y) \leq 0 \leq \tau(x, y)$ . In particular if we set:

$$(3) \quad \tau'(x, y) = \lim_{t \rightarrow 0^+} \frac{(\|x + ty\| - \|x - ty\|)}{2t} = (\tau(x, y) - \tau(x, -y))/2$$

then  $\tau'(x, y) = 0$  implies  $x \perp y$  (note that  $-\tau(x, -y) \leq \tau'(x, y) \leq \tau(x, y)$  for any pair  $x, y$ ). The condition  $x \perp y$  can also be expressed in this way: there exists a norm-one functional  $J$  assuming the norm at  $x$  and such that  $J(y) = 0$ . Recall that for a projection  $P : X \rightarrow Y$ ,  $\|P\| = 1$  is equivalent to:

$$(4) \quad Px \perp x - Px \quad \text{for all } x \text{ in } X.$$

If  $Y = \bigcap_{k=1}^n f_k^{-1}(0)$  is a subspace of  $X$  of finite codimension  $n$  ( $f_1, \dots, f_n$  in  $X^*$ ), then a projection

$P : X \rightarrow Y$  has a specific form, namely:

$$(5) \quad Px = x - \sum_{i=1}^n f_i(x)z_i \quad \text{where } f_i(z_j) = \delta_{i,j} \quad ((I - P)X = \text{span} \{z_1, \dots, z_n\}).$$

In this case  $P$  is bicontractive if and only if:

$$(6) \quad Y \perp \text{span} \{z_1, \dots, z_n\} \quad \text{and} \quad \text{span} \{z_1, \dots, z_n\} \perp Y.$$

We shall write, for  $\alpha$  in  $R$ ,  $\text{sgn} \alpha = \frac{|\alpha|}{\alpha}$  (eventually  $\text{sgn} 0$  will mean 0). For a result related to existence of bicontractive projection, see Proposition 2 in [12].

### 3. Central proximity maps and linearity.

As proved in [4], if  $P : X \rightarrow Y$  is a central proximity map, then we have:

$$(7) \quad \tau'(x - Px, y) = 0 \quad \text{for all } x \text{ in } X, \quad y \text{ in } Y.$$

**Theorem 3.** *Let  $P : X \rightarrow Y$  be a central proximity map. Then we have*

$$(8) \quad \tau'(y, x - Px) = 0 \quad \text{for all } x \text{ in } X, \quad y \text{ in } Y.$$

PROOF : Recall (see [4]) that any central proximity map satisfies:

$$(9) \quad P(\alpha x + y) = \alpha Px + y \quad \text{for } x \text{ in } X, \quad y \text{ in } Y \text{ and } \alpha \text{ in } R.$$

Therefore (by using (1)) we obtain:

$$\begin{aligned} \tau'(y, x - Px) &= \lim_{\alpha \rightarrow 0^+} \frac{\|y + \alpha(x - Px)\| - \|y - \alpha(x - Px)\|}{2\alpha} = \\ &= \lim_{\alpha \rightarrow 0^+} \frac{\|\alpha x + y - P(\alpha x + y) + y\| - \|\alpha x + y - P(\alpha x + y) - y\|}{2\alpha} = 0. \end{aligned}$$

■

By using Theorem 3 we obtain:

**Theorem 4.** *Let  $X$  be such that  $\tau'(x, y)$  is additive in its second argument; then every central proximity map in  $X$  is linear.*

PROOF : Let  $P : X \rightarrow Y$  be an idempotent map satisfying (1), thus, by Theorem 3, also (8). Take  $x, z$  in  $X$  and  $y$  in  $Y$ ; by using the linearity of  $\tau'(\cdot, \cdot)$  in the second argument we obtain:  $\tau'(y, Px + Pz - P(x + z)) = \tau'(y, Px - x + Pz - z - (P(x + z) - x - z)) = 0$ . If we set  $y = Px + Pz - P(x + z)$ , by recalling that  $\tau'(x, x) = \|x\|$  for any  $x$  in  $X$ , we obtain  $Px + Pz = P(x + z)$ . ■

**Corollary 5.** *Let  $X$  be a smooth space or  $X = L_1(X, \Sigma, \mu)$  and  $P : X \rightarrow Y$  a central proximity map. Then  $P$  is linear.*

PROOF : If  $X$  is smooth, then  $-\tau(x, -y) = \tau(x, y) = \tau'(x, y)$  for every pair  $x, y$  and  $\tau(x, y)$  is linear in  $y$ . If  $X = L_1$  then  $\tau'$  is linear in its second argument (see e.g. [11]), ■

For examples of central proximity maps which are not linear we send to [10].

### 4. Bicontractive projections in sequence spaces.

In this section we want to discuss characterizations of bicontractive projections in some classical sequence spaces. We shall denote by  $(e_i)_{i \in \mathbb{N}}$  or by  $(e_i^*)_{i \in \mathbb{N}}$  the elements of the natural basis in  $X$  or  $X^*$  respectively,  $X$  being one of these spaces  $((e_i)_j = (e_i^*)_j = \delta_{i,j})$ .

Let  $X = C_0(K)$  be the space of all continuous functions vanishing at infinity on a locally compact Hausdorff space  $K$  with values in  $R$  (endowed with the sup norm). Then bicontractive projections onto subspaces of  $Y$  have been characterized in [7] (Proposition 1.19 and Lemma 1.17). For the particular case of  $X = c_0$  we have:

**Theorem 6.** *The projection  $P : c_0 \rightarrow Y$  is bicontractive if and only if there exist a map  $\alpha : R \rightarrow (-1, 1)$  and an isomorphism  $h$  of  $R$  such that:*

$$(10) \quad Pf = 1/2(f + \alpha f \circ h), f \in c_0$$

and

$$(11) \quad \alpha \circ h = \alpha, \quad h^2 = I.$$

*In particular, if  $Y = f^{-1}(0)$  (with  $f$  in  $l^1$ ) is the range of a bicontractive projection, then either  $f$  has only one non null component or  $f$  has at most two non null components  $f_i, f_j$  with  $|f_i| = |f_j|$ .*

**PROOF :** The proof of the above theorem was based on Theorem 2. We shall give a direct proof of the last part by using some results from [6].

Let  $P : X \rightarrow Y = f^{-1}(0)$ ,  $\|P\| = \|f\| = 1$ . Let  $Px = x - f(x)z$  ( $f(z) = 1$ ); we know (see [6]) that  $|f_i| \geq \frac{1}{2}$  for at least one index  $i$ . Also: if this is true for a single index  $i$ , then we must have (see again [6])  $z = (1/f_i)e_i$ . Now let  $\|I - P\| = 1$ , therefore:

$$\|(I - P)(x)\| = \|f(x)z\| = \left| \frac{f(x)}{f_i} \right| \leq \|x\| \quad \text{for all } x \in X.$$

If we set  $x_k = \sum_{j=1}^k e_j \operatorname{sgn} f_j$  then we have ( $\|x_k\| = 1$ ):  $|f(x_k)| = \sum_{n=1}^k |f_n| \leq |f_i|$ . Letting  $k \rightarrow \infty$  we obtain  $\|f\| \leq |f_i|$  which implies  $f_n = 0$  for any  $n \neq i$ . If  $f$  has two components  $f_h, f_k$  such that  $|f_h| = |f_k| = 1/2$ , then we obtain  $f_n = 0$  for  $n \neq h, k$ .

Conversely, if  $f$  has only one non null component,  $f_i$ , then if we define  $z = (1/f_i)e_i$  it is easy to prove that  $Px = x - f(x)z$  is bicontractive. If  $f$  has only two non null components,  $f_h$  and  $f_k$ , such that  $|f_h| = |f_k|$ , then if we define  $z = (\operatorname{sgn} f_h)e_h + (\operatorname{sgn} f_k)e_k$  it is easy to prove that the projection  $Px = x - f(x)z$  is bicontractive. ■

To prove our characterizations for the spaces  $l^p$ ,  $1 \leq p < +\infty$ , we need the following result.

**Theorem 7.** (see [2], [3]). *Let  $X = l^p$ ;  $1 \leq p < +\infty$ ,  $p \neq 2$ , and let  $Y \subset X$  be a hyperplane of finite codimension  $n$ . Then a norm one projection  $P : X \rightarrow Y$  exists if and only if  $Y$  can be expressed as the intersection of  $n$  functionals having at most two non null components.*

Now we can prove our characterizations.

**Theorem 8.** *Let  $X = l^p$ ,  $1 < p < +\infty$ ,  $p \neq 2$ , and  $Y \subset X$  be a hyperplane of finite codimension  $n$ . Then  $Y$  is the range of a bicontractive projection if and only if there exist  $n$  functionals  $f_1, \dots, f_n$  in  $l^q$  ( $1/p + 1/q = 1$ ) and  $n$  different indices  $t_1, \dots, t_n$  such that:*

$$i) \quad Y = \bigcap_{i=1}^n (f_i^{-1}(0))$$

$$ii) \quad (f_i)_{t_j} = \delta_{i,j} \quad (1 \leq i, j \leq n)$$

iii) *for any  $i$ , there exists at most one index  $s_i \notin \{t_1, \dots, t_n\}$  such that  $(f_i)_{s_i} \neq 0$ ; moreover if  $s_i$  is such, then  $|(f_i)_{s_i}| = 1$  and  $(f_j)_{s_i} = 0$  for any  $j \neq i$ .*

PROOF: "If" part. Let  $f_1, \dots, f_n$  be  $n$  functionals satisfying i), ii), iii). Assume, for simplicity of notations:  $t_j = j$  ( $j = 1, \dots, n$ ); moreover, let  $s_1, \dots, s_m$  ( $0 \leq m \leq n$ ) be  $m$  different indexes  $\in \{n+1, n+2, \dots\}$  such that  $|(f_i)s_j| = \delta_{i,j}$  for  $0 \leq i \leq m$  (for  $m = 0$  no such index exists).

Consider the following elements  $z_1, \dots, z_n$ :

$$(12) \quad z_i = \frac{1}{2}e_i + \frac{1}{2}\operatorname{sgn}(f_i)s_i e_{s_i} \quad \text{if } 1 \leq i \leq m; \quad z_i = e_i \quad \text{if } m < i \leq n$$

It is routine to show that conditions (6) hold for  $z_1, \dots, z_n$  as above: thus the projection  $P$  defined by (5) is bicontractive, which proves the "if" part of the theorem.

"Only if" part. Let  $Y$  be as indicated and let  $P: X \rightarrow Y$ ,  $\|P\| = \|I - P\| = 1$ . By Theorem 7 there exist  $n$  functionals  $f_1, \dots, f_n$  satisfying i) and ii) and such that each of them has at most two non null components; for simplicity of notations, we shall assume  $t_i = i$  ( $i=1, 2, \dots, n$ ), thus  $(f_i)_j = \delta_{i,j}$  ( $1 \leq i, j \leq n$ ).

Also let  $Px = x - \sum_{i=1}^n f_i(x)z_i$ : by our assumption condition (6) must be satisfied.

Now suppose that for some  $s \notin \{1, 2, \dots, n\}$  we have  $(f_i)_s \neq 0$  for more than one index  $i$ : again for simplicity of notations we shall assume

$(f_i)_s \neq 0$  for  $i = 1, 2, \dots, m$  ( $2 \leq m \leq n$ ) and (eventually, if  $m < n$ )  $(f_i)_s = 0$  for  $m < i \leq n$ .

From  $f_i(z_1) = \delta_{i,1}$  ( $1 \leq i \leq n$ ) we obtain:

$$(13) \quad f_1(z_1) = (z_1)_1 + (f_1)_s(z_1)_s = 1 \quad \text{and so} \quad (z_1)_s = \frac{1 - (z_1)_1}{(f_1)_s}$$

$$(14) \quad f_j(z_1) = 0 \quad \text{and so} \quad (z_1)_j = (f_j)_s(z_1)_s = \frac{-(f_j)_s(1 - (z_1)_1)}{(f_1)_s}$$

for  $2 \leq j \leq m$ .

Now set:

$$(15) \quad x = \sum_{j=1}^m (f_j)_s e_j - e_s = ((f_1)_s, \dots, (f_m)_s, 0, \dots, -1, 0, \dots)$$

Recall that for any point in  $X$  there exists exactly one norm-one functional attaining its norm at such point. Since  $x \in Y$  we have  $x \perp z_1$  and this implies:

$$(16) \quad x = \sum_{j=1}^m |(f_j)_s|^{p-2} (f_j)_s (z_1)_j - (z_1)_s = 0$$

If we substitute (13) and (14) into (16), we obtain:

$$0 = |(f_1)_s|^{p-2} (f_1)_s (z_1)_1 + \sum_{j=2}^m |(f_j)_s|^{p-2} (f_j)_s \frac{-(f_j)_s(1 - (z_1)_1)}{(f_1)_s} - \frac{1 - (z_1)_1}{(f_1)_s} =$$

$$\frac{1}{(f_1)_s} \left( \sum_{j=1}^m |(f_j)_s|^p (z_1)_1 - \sum_{j=2}^m |(f_j)_s|^p - 1 + (z_1)_1 \right).$$

Now we set:

$$(17) \quad b = \frac{1}{1 + \sum_{j=1}^m |(f_j)_s|^p}$$

Then we obtain:

$$(18) \quad (z_1)_1 = b(1 + \sum_{j=2}^m |(f_j)_s|^p) = 1 - b|(f_1)_s|^p > 0$$

and then, by using (14):

$$(19) \quad (z_1)_j = -b(f_j)_s |(f_1)_s|^{p-2} (f_1)_s \quad \text{for } 2 \leq j \leq m.$$

Moreover by (13):

$$(20) \quad (z_1)_s = b|(f_1)_s|^{p-2} (f_1)_s.$$

Now we shall use the condition  $z_1 \perp x$ : this means that  $J(x) = 0$  where  $J$  is the unique norm-one functional attaining its norm at  $z_1$ . Thus we have (we intend eventually  $0^{p-2} = 0$ ):

$$J(x) = \sum_{j=1}^m |(z_1)_j|^{p-2} (z_1)_j (f_j)_s - |(z_1)_s|^{p-2} (z_1)_s = 0.$$

By using (18) and (19) we obtain:

$$[b(1 + \sum_{j=2}^m |(f_j)_s|^p)]^{p-1} (f_1)_s - \sum_{j=2}^m b^{p-1} |(f_1)_s|^{(p-2)p} |(f_j)_s|^p (f_1)_s - |(z_1)_s|^{p-2} (z_1)_s = 0.$$

But (20) implies  $|(z_1)_s|^{p-2} (z_1)_s = b^{p-1} |(f_1)_s|^{p^2-2p} (f_1)_s$ ; if we substitute in the previous equation, then we divide by  $b^{p-1} (f_1)_s \neq 0$  we obtain:

$$(1 + \sum_{j=2}^m |(f_j)_s|^p)^{p-1} - \sum_{j=2}^m |(f_1)_s|^{(p-2)p} |(f_j)_s|^p - |(f_1)_s|^{(p-2)p} = 0$$

and so:

$$(1 + \sum_{j=2}^m |(f_j)_s|^p)^{p-2} = |(f_1)_s|^{(p-2)p}, \text{ which implies } (p \neq 2):$$

$$(21) \quad |(f_1)_s|^p = 1 + \sum_{j=2}^m |(f_j)_s|^p.$$

By reasoning in a similar way, but starting from  $z_2$  (instead of from  $z_1$ ) we obtain:

$$(22) \quad |(f_2)_s|^p = 1 + \sum_{j=1, j \neq 2}^m |(f_j)_s|^p.$$

and by subtraction:

$$\begin{aligned} |(f_1)_s|^p - |(f_2)_s|^p &= |(f_2)_s|^p - |(f_1)_s|^p \text{ and so:} \\ |(f_1)_s| &= |(f_2)_s| \text{ (we denote by } a \text{ this quantity).} \end{aligned}$$

In a similar way we obtain:

$$|(f_1)_s| = \dots = |(f_m)_s| = a.$$

By using (21) we have:

$a^p = 1 + (m - 1)a^p$  which is absurd if  $m \geq 2$ . This shows that if  $s_i > n$  is an index such that  $(f_i)_{s_i} \neq 0$ , then the other  $n - 1$  functionals among  $f_1, \dots, f_n$  have the  $s_i - th$  component equal to zero. This implies  $x' = (f_i)_{s_i} e_i - e_{s_i}$  is in  $Y$ . Now we reason as above but on  $z_i$ ; we shall obtain from here:

$$(23) \quad |(f_i)_{s_i}|^{p(p-2)} = 1,$$

which proves the "Only if" part of the theorem since  $p \neq 2$ . ■

**Remark 1.** Theorem 8, for the case of hyperplanes, had already been proved in [9].

Now we shall prove a similar characterization for  $l^1$ .

**Theorem 9.** Let  $X = l^1$  and let  $Y$  be a subspace of  $X$  of finite codimension  $n$ . Then  $Y$  is the range of a bicontractive projection  $P : X \rightarrow Y$  if and only if there exist  $n$  functionals  $f_1, \dots, f_n$  in  $l^\infty$  satisfying the conditions i), ii), iii) of Theorem 8.

**PROOF :** We assume again, for simplicity of notations:  $t_j = j$  ( $j = 1, \dots, n$ ).

"If" part. Let  $f_i = e_i^* + (f_i)_{k_i} e_{k_i}^*$ ,  $|(f_i)_{k_i}| = 1$  for  $i = 1, \dots, m$  ( $1 \leq m \leq n$ ;  $k_i > n$ );  $f_i = e_i^*$  for  $m < i \leq n$ . Choose now (in  $l^1$ )  $n$  elements  $z_1, \dots, z_n$  according to (12), then define  $P$  according to (5). It is easy to see that the conditions (6) are satisfied, thus proving the "If" part of the Theorem.

"Only if" part. Reason as in the proof of the "Only if" part of Theorem 8 (where we set  $p = 1$ ), but some remarks are due, mainly since  $l^1$  is not smooth. Concerning the condition  $z_i \perp x$  the following remarks apply.

From [3] we obtain  $(z_i)_s \neq 0$  and  $(z_i)_k = 0$  for  $k > n$ ,  $k \neq s$  (if  $(f_i)_s \neq 0$ ). Also, consider (if  $m < n$ )  $f_k$  with  $m + 1 \leq k \leq n$ ; because of Theorem 7, we certainly can write:

$$f_k = e_k^* + (f_k)_{h_k} e_{h_k}^* \text{ with } h_k > n; h_k \neq s \text{ and eventually } (f_k)_{h_k} = 0$$

Thus  $f_k(z_i) = 0$  implies  $(z_i)_k = 0$  and so:  $z_i = \sum_{j=1}^m (z_i)_j e_j + (z_i)_s e_s$ .

Therefore the condition  $x \perp z_i$  implies (16) since any norm-one functional attaining its norm at  $x$  takes the same value at  $z_i$ . Again we should eventually intend  $t/|t| = 0$  if  $t = 0$ . For the same reason the condition  $z_i \perp x$  implies (21) with  $p = 1$ . ■



**Remark 2.** To prove Theorems 8 and 9 we could use Theorem 2: to obtain the "only if" part by using the condition  $\|2Px - x\| = \|x\|$  for all  $x \in X$  we should indicate all relations between the conditions  $f_k \neq 0$  and  $z_k \neq 0$ .

**Remark 3.** Theorems 8 and 9 show that in  $l^p$  ( $1 \leq p < \infty, p \neq 2$ ), if the codimension of  $Y$  is  $n$  and  $Y$  is the range of a bicontractive projection, then  $Y$  can be expressed as the intersection of  $n$  hyperplanes with the same property. The converse is not true: for example, if  $Y = f^{-1}(0) \cap g^{-1}(0)$  where  $f = e_1^* + e_3^*$  and  $g = e_1^* + e_2^*$ , then we cannot express  $Y$  in that way, and so it is not the range of a bicontractive projection: this can be proved following the lines of the proof of Theorem 8.

### 5. Central proximity maps and bicontractive projections.

Some of the results indicated in Section 3 and 4 can be summarized as follows. We shall say that  $X$  is B-L space if it is one of the spaces considered in Theorem 2.

**Theorem 10.** *Let  $P : X \rightarrow Y$  be an idempotent map. Consider the following properties:*

- (a)  $\|2Px - x\| = \|x\|$  for all  $x$  in  $X$
  - (b)  $P$  is a central proximity map
- Also, if  $P$  is linear, let:*
- (c)  $\|P\| = \|I - P\| = 1$
  - (d)  $\|2P - I\| = 1$
  - (e)  $2P - I$  is an isometry.

*Then we always have: (e)  $\Rightarrow$  (d)  $\Rightarrow$  (c); (d)  $\Rightarrow$  (b)  $\Rightarrow$  (a). If  $X$  is a B-L space, then (c)  $\Leftrightarrow$  (d)  $\Leftrightarrow$  (e). If  $X$  is smooth, or  $X = L_1(X, \Sigma, \mu)$ , then (b)  $\Rightarrow$  (e). In particular, if  $X = L_p(X, \Sigma, \mu)$  ( $1 \leq p < +\infty$ ), then all properties (b) through (e) are equivalent.*

**PROOF :** The implications (e)  $\Rightarrow$  (d)  $\Rightarrow$  (c) and (b)  $\Rightarrow$  (a) are trivial (note that  $2P = 2P - I + I$  and  $2(I - P) = I - (2P - I)$ ). The implication (d)  $\Rightarrow$  (b) was proved in [10], Lemma 4.5.

In a B-L space we have (see Theorem 2) (c)  $\Rightarrow$  (e), thus the equivalence between (c), (d) and (e). If  $X$  is smooth or  $X = L_1(X, \Sigma, \mu)$ , then (b) implies  $P$  linear (see Corollary 5), thus (b)  $\Rightarrow$  (e) (so (b)  $\Leftrightarrow$  (d)  $\Leftrightarrow$  (e)). The last sentence of the theorem is a consequence of the previous ones. ■

The existence of some relation between (e) and (b) had already been observed in [5], where it was indicated that a projection  $P$  satisfies (e) if and only if

$$(b') \quad \|y + z\| = \|y - z\| \text{ for } y \text{ in } Y \text{ and } z = (I - P)z.$$

This means that (b) is satisfied when  $Pz = 0$ : but in fact the assumption (d) (which is weaker than (e)) already implies (b).

**Remark 4.** For examples showing that (b) does not imply  $P$  linear e.g. in  $C[0, 1]$ , and (c) does not imply (a), we send to [4]. Probably (c) does not imply (d) also if  $X$  is assumed to be smooth.

By using Theorem 10 and the results of [2] and [3] it is possible to characterize central proximity maps onto subspaces of finite dimension or codimension in  $l_p$ ,  $1 \leq p < +\infty, p \neq 2$ . Also, by recalling some facts indicated e.g. in [13], p. 779 - 788, we have:

**Theorem 11.** *Let  $X = L_p(A, \Sigma, \mu)$ ,  $1 \leq p < +\infty$ ,  $p \neq 2$ , and let  $Y \subset X$  a proper subspace of finite dimension or codimension in  $X$ . Then no central proximity map onto  $Y$  exists if the measure is  $\sigma$ -finite and contains no atom. Also, if  $X = C[0, 1]$  and  $Y$  is as above, then no central proximity map onto  $Y$  (when it exists) can be linear.*

Finally, we recall that a necessary condition for the existence a bicontractive projection onto a subspace  $Y$  was indicated in [8].

## REFERENCES

- [1] D.Amir, *Characterizations of inner product spaces*, Birkhäuser, Basel, 1986.
- [2] M.Baronti - P.L.Papini, *Norm-one projections onto subspaces of  $l^p$* , *Annali Mat. Pura Appl. (IV)*, **152** (1988), 53-61.
- [3] M.Baronti - P.L.Papini, *Norm-one projections onto subspaces of finite condimension in  $l^1$  and  $c_0$* , preprint.
- [4] M.Baronti - P.L.Papini, *Proximity maps are seldom central*, in: *Approximation Theory VI* (C.K.Chui, L.L.Schumaker eds.) Academic Press, New York, 1989.
- [5] S.J.Bernau - H.E.Lacey, *Bicontractive projections and reordering of  $L^p$ -spaces*, *Pacific J. Math.* **69** (1977), 291-302.
- [6] J.Blatter - E.W.Cheney, *Minimal projections on hyperplanes in sequence spaces*, *Annali Mat. Pura Appl. (IV)*, **101** (1974), 215 - 227.
- [7] Y.Friedman - B.Russo, *Conctractive projections on  $C_0(K)$* , *Trans. Amer. Math. Soc.* **273** (1982), 57 - 73.
- [8] L.V.Gladun, *The relation between the numerical ranges of the characteristics of a Banach space and its subspaces*, (Russian), *Teor. Funktsij., Funktsional. Anal. i Prilozhen.* **46** (1986), 21-26.
- [9] S.Kinnunen, *On projections and Birkhoff-James orthogonality in Banach spaces*, *Nieuw Arch.Wisk.* (4), **2** (1984), 235 - 255.
- [10] L.W.Light - E.W.Cheney, *Approximation theory in tensor product spaces*, Springer Berlin, 1985.
- [11] P.M.Miličić, *Sur les espaces semi-lisses*, *Mat. Vesnik* (21), **36** (1984), 222 - 226.
- [12] R.Nussbaum, *The ball intersection property for Banach spaces*, *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astron. Phys.* **19** (1971), 931 - 936.
- [13] R.Sine, *Rigidity properties of nonexpansive mappings*, *Nonlin. Anal.* **11** (1987), 777 - 794.

Dipartimento di matematica, Facoltà di Ingegneria Via delle Brecce Bianche, I-60100 ANCONA,  
Dipartimento di matematica, Piazza Porta S. Donato 5, I-40127 BOLOGNA

(Received June 19,1989)