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Abstract. We deal with an application of an abstract theorem concerning critical points of functionals with symmetries (see [2]) to the problem

$$-\Delta u = \lambda \cdot u + g(u), \qquad u \in H_0^1(\Omega), \qquad \Omega \subset R^n.$$

For $\lambda \in R$, we prove the existence of infinitely many solutions under assumptions which generalize the previous results of de Candia/Fortunato [4] et al.

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This note deals with the semilinear boundary value problem

$$-\Delta u = \lambda \cdot u + g(u), \qquad u \in H_0^1(\Omega), \tag{1}$$

where $\Omega \subset R^n$ $(n \geq 2)$ is a bounded smooth domain, under the following general assumptions on the function g:

(i) $g : R \to R$ is odd and continuous.

(ii) There are positive constants c_1, c_2 such that for some $p \in (2, 2n/(n-2))$ we have

$$|g(t)| \leq c_1 + c_2 \cdot |t|^{p-1}, \qquad t \in R.$$

(iii) For any $c_3 > 0$, there is a constant c_4 such that

$$|g(t)| \geq c_3 \cdot |t| - c_4, \qquad t \in R.$$

The solutions of (1) are the critical points of the C^1-functional on $H_0^1(\Omega)$ defined by

$$F(u) = \frac{1}{2}\int |\nabla u|^2 - \frac{1}{2}\lambda \cdot \int u^2 - \int G(u), \tag{2}$$

where $G(t) = \int_0^t g(s)\,ds$, $t \in R$.

If G is "superquadratic", i.e. satisfies the additional condition

$$g(t) \cdot t - 2 \cdot G(t) \geq c \cdot G(t) - c', \qquad t \in R \tag{iv$_1$}$$

for some positive constants c, c', then it is known that (1) possesses an infinite sequence of distinct solutions for any $\lambda \in R$ (see [1], [3], [5]). Easy arguments show that $(\text{iv})_1$ does nor apply to functions $G(t)$ growing nearly as $|t|^2$, for example, to $G(t) = t^2 \cdot (ln(1+t^2))^\beta$, $\beta > 0$. From [4] we learned that under the assumption

$$(\text{iv})_2 \qquad t \cdot g(t) - 2 \cdot G(t) \geq c \cdot |t|^\alpha - c' \qquad t \in R,$$
$$(c, c' > 0 \text{ and } \alpha \geq \max(2, \frac{2n}{n+2}(p-1))$$

which covers this case, there also exist infinitely many solutions of (1). Nevertheless, it is easy to see that $(\text{iv})_1$ does not follow from $(\text{iv})_2$, too.

The aim of this note is to improve the range of α allowed in the de Candia/Fortunato condition $(\text{iv})_2$ and to discuss certain more implicit assumptions generalizing both $(\text{iv})_1$ and $(\text{iv})_2$. Our considerations are based on a result of Bartolo/Benci/Fortunato [2, Theorem 2.4]:

Proposition. *Suppose that the functional $f \in C^1(H, R)$, where H is a Hilbert space, with dual H', satisfies the following properties:*

- (f_1) *Every bounded sequence $\{u_k\} \subset f^{-1}((0,\infty))$, for which $\{f(u_k)\}$ is bounded and $f'(u_k) \to 0$, possesses a convergent subsequence.*
- (f_2) *For any $c \in (0,\infty)$ there exist positive reals $d < c, R$, and α such that*
$$\|f'(u)\|_{H'} \cdot \|u\|_H \geq \alpha$$
for all $u \in f^{-1}([c-d, c+d])$ satisfying $\|u\|_H \geq R$.
- (f_3) *There exist two closed subspaces H^+, H^- of H with codim $H^+ < +\infty$, and two constants $c_\infty > c_0 > f(0) \geq 0$ such that*

$$(3) \qquad f(u) \geq c_0, \qquad u \in H^+ \text{ with } \|u\|_H = \delta \text{ for some } \delta > 0$$

$$(4) \qquad f(u) < c_\infty, \qquad u \in H^-.$$

- (f_4) *f is even.*

Then, if $\dim H^- \geq \operatorname{codim} H^+$, f possesses at least $m := \dim H^- - \operatorname{codim} H^+$ distinct pairs of critical points whose corresponding critical values belong to $[0,\infty]$.

In the following we prove that our variational functional (2) satisfies the assumptions (f_1) – (f_4) where $H := H_0^1(\Omega)$ and $H' := H^{-1}(\Omega)$. Let us mention that (f_1), (f_3), (f_4) follow from the general conditions (i), (ii), (iii) on $g(t)$ in a standard way. Only for showing (f_2) we need some additional assumption. It turns out that (f_2) with $f = F$ immediately follows form the following property:

$(\text{f}_2)'$ For arbitrary $M > 0$, if $|F(u)| \leq M$ and $\|u\|_H \to \infty$, then $\int g(u) \cdot u - 2G(u) \to \infty$.

Indeed, since $\|F'(u)\|_{H'} \|u\|_H \geq |\langle F'(u), u\rangle|$, for (f_2) it is sufficient to show that

$$(5) \qquad \gamma(u) := \langle F'(u), u\rangle = \int |\nabla u|^2 - \lambda \int u^2 - \int g(u) \cdot u < -a < 0$$

under the restriction $F(u) \in (0, 2c)$ and $\|u\|_H \geq R$ for large R. But due to

$$(6) \qquad \gamma(u) = 2F(u) - \int (g(u) \cdot u - 2G(u))$$

$(\text{f}_2)'$ yields (5) in an obvious way.

Thus, we have to look for more explicit conditions (analogous to $(\text{iv})_1$ or $(\text{iv})_2$) which imply, together with (i) – (iii), the property $(\text{f}_2)'$.

Lemma. *Under the assumptions* (ii), (iii), *for the problem* (1) *and the corresponding functional* (2) *the following implications hold:*

$$\left.\begin{array}{l}(iv)_1 \Rightarrow (iv)_3 \Rightarrow \\ (iv)_2 \Rightarrow (iv)_2' \Rightarrow\end{array}\right\} \quad (f_2)' \Rightarrow (f_2)$$

where $(\mathrm{iv})_2'$ *and* $(\mathrm{iv})_3$ *are given by:*

$(\mathrm{iv})_2'$ *There are positive constants* c_5, c_6 *and* $\alpha > \max(1, \frac{n}{2}(p-2))$ *such that*

$$t \cdot g(t) - 2G(t) \geq c_5 \cdot |t|^\alpha - c_6.$$

$(\mathrm{iv})_3$ *There are positive constants* c_7, c_8 *and* q *such that for* $u \in H (:= H_0^1(\Omega))$

$$\int (g(u) \cdot u - 2G(u)) \geq c_7 \cdot \left(\int G(u)\right)^q - c_8.$$

PROOF : $(f_2)' \Rightarrow (f_2)$ has already been proved. $(\mathrm{iv})_1 \Rightarrow (\mathrm{iv})_3$ is obvious with $q = 1$. $(\mathrm{iv})_2 \Rightarrow (\mathrm{iv})_2'$ is clear because $\max(1, \frac{n}{2}(p-2)) < \max(2, \frac{2n}{n+2}(p-1))$ for arbitrary $p \in (2, \frac{2n}{n-2})$. For proving $(\mathrm{iv})_3 \Rightarrow (f_2)'$ we observe that (iii) and $|F(u)| \leq M$ yield the inequality

$$\|u\|_H^2 = \int |\nabla u|^2 = (2F(u) + \lambda \cdot \int u^2 + 2\int G(u)) \leq \leq c + c' \cdot \int G(u) \qquad (c, c' > 0). \tag{7}$$

Thus, if $\|u\|_H \to \infty$, then $\int G(u) \to \infty$ and by $(\mathrm{iv})_3$ also $\int g(u) \cdot u - 2G(u) \to \infty$.

It remains to establish $(\mathrm{iv})_2' \Rightarrow (f_2)'$. As above we can rely on (7). But by (ii) we actually have

$$\|u\|_H^2 \leq c' + c'' \|u\|_{L_p}^p. \tag{8}$$

Next we apply a Gagliardo–Nirenberg-type inequality

$$\|u\|_{L_p} \leq c \cdot \|u\|_{L_\alpha}^{1-\Theta} \cdot \|u\|_H^\Theta, \qquad u \in H \tag{9}$$

which is valid with $\Theta = n(\frac{1}{\alpha} - \frac{1}{p})/(1 + n(\frac{1}{\alpha} - \frac{1}{2})) \in (0,1)$ for any $1 < \alpha < p < \frac{2n}{n-2}$ (for details, see [6, ch.2.4.2]). From (8) and (9) it follows that $\|u\|_H \to \infty$ implies $\|u\|_{L_\alpha} \to \infty$ whenever $\Theta p < 2$ which is equivalent to $\alpha > \frac{n}{2}(p-2)$. Thus, by $(\mathrm{iv})_2'$ we finally get $\int g(u) \cdot u - 2G(u) \to \infty$ which proves $(f_2)'$. The lemma is established. ∎

We are now able to formulate the main result.

Theorem. *The problem* (1) *has infinitely many distinct solutions if the assumptions* (i), (ii), (iii), *and any one of the conditions stated in the Lemma is fulfilled.*

PROOF : As indicated above, we have to examine (f_1) – (f_4). (f_2) is proved in the lemma, (f_4) is obvious from (i).

Now, consider a sequence $\{u_k\}$ as assumed in (f_1). Observe that

$$-\Delta u_k - \lambda u_k - g(u_k) \to 0 \quad \text{in } H'.$$

Since $(-\Delta)^{-1} : H' \to H$ is continuous, we have

$$u_k - \lambda \cdot (-\Delta)^{-1} u_k - (-\Delta)^{-1} g(u_k) \to 0 \quad \text{in } H. \tag{10}$$

$\{u_k\}$ is bounded in H, hence possesses a weak convergent subsequence still denoted by $\{u_k\}$, i.e.

$$u_k \rightharpoonup u_0 \qquad \text{(weak convergence in } H). \tag{11}$$

Let $W_q^r := W_q^r(\Omega)$ denote the usual Sobolev spaces, $1 < q < \infty$. Next we show that

$$u \to (-\Delta)^{-1} g(u) : H \to H \qquad \text{is compact.} \tag{12}$$

Let $u \in H \subset W_2^1$. The well-known Sobolev embedding theorem leads to $u \in L_{\frac{2n}{n-2}}$, and by (ii) we obtain $g(u) \in L_q$,

$$q = \frac{2n}{(n-2)(p-1)} \qquad \text{(for } n = 2 \text{ arbitrary } q < \infty \text{ is allowed).}$$

Since $p < \frac{2n}{n-2}$, we have $q > \frac{2n}{n+2} \geq 1$. Elliptic regularity yields the boundedness of $(-\Delta)^{-1} \circ g : H \to W_q^2$, and according to the compact embedding $W_q^2 \hookrightarrow\hookrightarrow W_2^1$ we get (12). It remains to observe that

$$u \to (-\Delta)^{-1} u : H \to H \qquad \text{is compact.} \tag{13}$$

Properties (10) – (13) imply (f_1).

In order to verify (f_3) we choose the following subspaces:

$$H_l^- := \bigoplus_i M_i, \qquad i = 1, 2, \dots, l$$
$$H_{l'}^+ := \overline{\bigoplus_j M_j}, \qquad j = l', l'+1, \dots$$

where M_s denotes the eigenspaces corresponding to the eigenvalues $0 < \lambda_1 < \lambda_2 < \lambda_3 < \dots$ of the laplacian. Let l, l' be such that $m := \dim H_l^- - \operatorname{codim} H_{l'}^+ > 0$ and $\lambda < \lambda_l, \lambda_{l'}$. By the definition of $F(u)$ and (ii), (iii) we have

$$\begin{aligned} 2F(u) &= \int |\nabla u|^2 - \lambda \int u^2 - 2 \int G(u) \\ &\geq \|u\|_H^2 - c' \|u\|_{L_p}^p - c'', \qquad c', c'' > 0. \end{aligned} \tag{14}$$

On the other hand, by the definition of $H_{l'}^+$ we have $\|u\|_{L_2}^2/\|u\|_H^2 \leq 1/\lambda_{l'}$ for $u \in H_{l'}^+$, which, substituted into (9) for $\alpha = 2$, gives

$$\|u\|_{L_p} \leq c \cdot \|u\|_H \cdot (\lambda_{l'})^{-(1-\Theta)/2}, \qquad u \in H_{l'}^+$$

where $\Theta = n(\frac{1}{2} - \frac{1}{p}) < 1$. Together with (14) this yields

$$F(u) \geq \frac{1}{2}(\delta^2 - c' \cdot c \cdot \delta^p \cdot \lambda_{l'}^{-(1-\Theta)p/2} - c'')$$

for all $u \in H_{l'}^+$, $\|u\|_H = \delta$. This, in turn, implies (3) with an arbitrary large c_0 if first δ and then l' are chosen large enough.

Finally, to show (4), we take c_3 in (iii) such that $c_3/2 > \lambda_l - \lambda$. According to (iii) we have

$$\int G(u) \geq \int \frac{c_3}{2}u^2 - c_4|u| \geq \frac{c_3}{4}\|u\|_{L_2}^2 - c_4'.$$

Furthermore,

$$\|u\|_H^2 \leq \lambda_l \cdot \|u\|_{L_2}^2, \qquad u \in H_l^-.$$

Thus, we obtain

$$\begin{aligned} 2F(u) &\leq \|u\|_H^2 - \lambda\|u\|_{L_2}^2 - \frac{c_3}{2}\|u\|_{L_2}^2 + 2c_4' \\ &\leq (\lambda_l - \lambda - \frac{c_3}{2}) \cdot \|u\|_{L_2}^2 + 2c_4' \leq 2c_4', \qquad u \in H_l^- \end{aligned}$$

and (f_3) is completely established.

Now, the theorem follows from the proposition. ∎

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