Commentationes Mathematicae Universitatis Carolinae

Sabina Schmidt

Representation of the Hausdorff measure of noncompactness in special Banach spaces

Commentationes Mathematicae Universitatis Carolinae, Vol. 30 (1989), No. 4, 733--735

Persistent URL: http://dml.cz/dmlcz/106794

Comment.Math.Univ.Carolinae 30,4(1989)733–735

Representation of the Hausdorff measure of noncompactness in special Banach spaces

SABINA SCHMIDT

Abstract. In this paper we give a representation for the Hausdorff measure of noncompactness in separable Banach spaces.

Keywords: Hausdorff measure of noncompactness, Gelfand-Phillips property

Classification: 46B20

It is known (Bourgain-Diestel [1]) that separable and (more generally) wcg spaces have the Gelfand-Phillips property, i.e., any limited set in E is relatively compact. (We recall that a bounded set A in a Banach space E is said to be limited if, for any $(x_n^*)_{n\in\mathbb{N}} \subseteq E^*$ converging weak* to zero, we have $\lim\limits_{n\to\infty} \sup\limits_{a\in A} |x_n^*(a)| = 0$.)

We will deduce this result from a representation for the Hausdorff measure of noncompactness β [3] in separable Banach spaces; cf. Theorem 1.

For the proof of our representation theorem we use the following result.

Proposition 1. *Let E be a (separable) Banach space and $(E_n)_{n\in\mathbb{N}}$ an increasing sequence of finite-dimensional subspaces dense in E. Then for any bounded set $A \subseteq E$*

$$\beta(A) = \lim_{n\to\infty} \sup_{a\in A} \operatorname{dist}(a, E_n).$$

Theorem 1. *Let E be a separable Banach space. Then for every bounded set $A \subseteq E$*

$$(\star) \qquad \beta(A) = \max\{\overline{\lim_{n\to\infty}} \sup_{a\in A} |x_n^*(a)| : (x_n^*)_{n\in\mathbb{N}} \subseteq S(0,1) \subseteq E^* \text{ converges weak* to zero}\}.$$

PROOF : Let $\varepsilon > 0$ be fixed. Then, by definition of $\beta(A)$, there is some finite set of centers $\{y_i : 1 \le i \le r\}$ with

$$A \subseteq \bigcup_{i=1}^{r} S(y_i, \beta(A) + \varepsilon).$$

For $(x_n^*)_{n\in\mathbb{N}} \subseteq S(\theta, 1) \subseteq E^*$ converging weak* to zero we obtain

$$\begin{aligned}\sup |x_n^*(a)| &\le \max_{i=1}^{r} \sup\{|x_n^*(a - y_i)| + |x_n^*(y_i)| : a \in S(y_i, \beta(A) + \varepsilon)\}\\ &\le \beta(A) + \varepsilon + \max_{i=1}^{r} |x_n^*(y_i)|.\end{aligned}$$

Since the limit of the right-hand side is $\beta(A)+\varepsilon$ and ε is chosen arbitrarily, we arrive at

$$\overline{\lim_{n\to\infty}} \sup_{a\in A} |x_n^*(a)| \le \beta(A).$$

To prove equality in $(\star)$, choose an increasing sequence of finite-dimensional subspaces $(E_n)_{n\in\mathbb{N}}$ with $\bigcup_{n\in\mathbb{N}} E_n$ dense in E. Then by Proposition 1 $\beta(A) = \lim_{n\to\infty} \sup_{a\in A} \operatorname{dist}(a, E_n)$.

Defining $\beta_n = \sup_{a\in A} \operatorname{dist}(a, E_n)$ for each $n\in\mathbb{N}$ we can find an $a_n\in A$ such that

$$\beta_n - \frac{1}{n} \le \operatorname{dist}(a_n, E_n).$$

The theorem of Hahn-Banach gives a sequence $(x_n^*)_{n\in\mathbb{N}} \subseteq E^*$ with the properties $\|x_n^*\| = 1, x_n^*(x) = 0$ for $x\in E_n$ and $x_n^*(a_n) = \operatorname{dist}(a_n, E_n)$.

Therefore

$$\beta(A) = \lim_{n\to\infty}\left(\beta_n - \frac{1}{n}\right) \le \overline{\lim_{n\to\infty}} \sup_{a\in A} |x_n^*(a)|.$$

To prove that $(x_n^*)_{n\in\mathbb{N}}$ converges weak* to zero, fix $x\in E$ and let $\varepsilon > 0$. The density of $\bigcup_{n\in\mathbb{N}} E_n$ in E implies the existence of $N\in\mathbb{N}$ and $y\in E_N$ such that $\|x-y\|\le\varepsilon$. From the properties of $(x_n^*)_{n\in\mathbb{N}}$ we obtain

$$|x_n^*(x)| = |x_n^*(x-y)| \le \varepsilon \quad \text{for} \quad n\ge N.$$

The first part of the proof gives now

$$\beta(A) = \overline{\lim_{n\to\infty}} \sup_{a\in A} |x_n^*(a)|. \qquad \blacksquare$$

As an application, formula $(\star)$ immediately implies Darbo's theorem for separable spaces, i.e, $\beta(A) = \beta(\overline{\operatorname{conv}} A)$.

A Banach space E is called a wcg (weakly compactly generated) space, if there exist some weakly compact subsets K whose linear hull is dense in E. The following property of wcg spaces leads for countable bounded subsets to the same result as in Theorem 1.

Proposition 2 ([2]). *Let X be a separable subspace of some wcg space E. Then there exist a closed separable subspace Y with $X\subseteq Y$ and a continuous linear projection $P: E\to Y$ with $\|P\| = 1$.*

For a subspace Y of E with $A\subseteq Y$ bounded, $\beta_Y(A)$ denotes the Hausdorff measure of noncompactness of A in Y, i.e.,

$$\beta_Y(A) = \inf\{\varepsilon > 0 : A \subseteq \bigcup_{i=1}^{n(\varepsilon)} S(y_i^\varepsilon, \varepsilon), y_i^\varepsilon \in Y\}.$$

Theorem 2. *Let E be a wcg space. Then for any bounded separable set $A \subseteq E$*

$$\beta(A) = \max\{\overline{\lim_{n\to\infty}} \sup_{a\in A} |x_n^*(a)| \subseteq S(\theta,1) \subseteq E^* \textit{converges weak}^* \textit{ to zero}\}.$$

PROOF : The first part of the proof of Theorem 1 shows that it suffices to find a sequence $(x_n^*)_{n\in\mathbb{N}} \subseteq S(\theta,1) \subseteq E^*$ converging weak* to zero such that

$$\beta(A) = \overline{\lim_{n\to\infty}} \sup_{a\in A} |x_n^*(a)|.$$

For a separable set A, Proposition 2 gives a separable subspace Y with $A \subseteq Y$ and a linear projection $P: E \to Y$ with $\|P\| = 1$. From the definition of β and β_Y it is easily seen that $\beta(A) = \beta_Y(A)$. Therefore, by Theorem 1, there exists a sequence $(y_n^*)_{n\in\mathbb{N}} \subseteq Y^*$ converging weak* to zero and $\|y_n^*\| \leq 1$ such that

$$\beta(A) = \overline{\lim_{n\to\infty}} \sup_{a\in A} |y_n^*(a)|.$$

Defining $x_n^* = y_n^* \circ P$ for $n \in \mathbb{N}, (x_n^*)_{n\in\mathbb{N}} \subseteq E^*$ is the desired sequence. ■

Now let A be a limited subset of a wcg space E. Then from Theorem 2 we obtain $\beta(B) = 0$ for every separable subset B of A, i.e., every separable subset of A is relatively compact. This is equivalent to the relative compactness of A, and so E has the Gelfand-Phillips property.

REFERENCES

[1] Bourgain J., Diestel J., *Limited operators and strict cosingularity*, Math. Nachr. **119** (1984), 55–58.

[2] Diestel J., *Geometry of Banach Spaces–Selected topics*, Lecture Notes in Math. 485, Springer-Verlag, Berlin - Heidelberg - New York, 1975.

[3] Sadovskii B.N., *Limit-compact and condensing operators*, Russ. math. Survs. **27** (1972), 85–155.

Mathematisches Institut I, Universität Karlsruhe, Kaiserstr. 12, D-7500 Karlsruhe 1

(*Received* October 7,1988, *revised* August 17,1989)