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Asymptotic equivalence and homeomorphism of the families of endomorphisms and surjections of the metrizable vector fibering

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Abstract. By means of the notion of abstract vector fibering of V.M.Millionshikov, the generalization of some theorems - classifiers playing an important role in the theory of asymptotic integration, is carried out.

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The perturbed differential equations may be classified on the basis of the notion of asymptotic equivalence [1 - 3]. Many theorems - classifiers playing an important role in the theory of asymptotic integration have appeared in mathematical literature lately. What kind of interpretation ensures such level of community that unites some of the results and liquidates numerous repetitions?

Evidently such interpretation is possible by means of the notion of abstract vector fibering of V.M.Millionshikov [4].

1. Let (E,p,B) be an abstract metrizable vector fibering, [4]. The surjection (E,p,B) is to be called the pair of mappings (Y,X^1), $Y : E \to E$, $X^1 : B \to B$ where $pY = X^1p$, and for any $b \in B$ the restriction $Y[b] \overset{\text{def}}{=} Y|p^{-1}(b)$ of the mapping Y on the stratum over the point b is the surjection $p^{-1}(b) \to p^{-1}(X^1b)$.

Let $M \subset \mathcal{R}$ and $+\infty$ be the accumulation point of the set M. The mapping F_0 of the set M into the set $\{(Y,X^1)\}$ of all surjections (E,p,B) is called the family of the surjections of abstract vector fibering.

Let us suppose that there are defined families of endomorphisms, [4], $F_1 : M \to \{(X,X)\}$ and surjections $F_2 : M \to \{(Y,X^1)\}$ of the vector fibering (E,p,B), where for some $b \in B$ $p^{-1}(X_tb) \subseteq p^{-1}(X_t^{-1}b)$ and $\|X_tz\| \leqslant K(t_0,z)Q(t)$ on these strata. X_ty and $Y(t : t_0,z)$ are accordingly the family of endomorphisms F_1 and the family of surjections F_2 in the point $(t,z) \in M \times p^{-1}(b)$, $K(t_0,z)$ is a non-negative number, $t \geqslant t_0$, $Q : M \to (0,+\infty)$, $X_{t_0}z = z$, $Y(t : t_0,z) = y$, $z \in p^{-1}(b)$. Henceforth $X.z$ and $Y(\cdot : t_0,z)$ will be called the values of these families in the point $z \in p^{-1}(b)$. Suppose that $\Phi_1 = \{X.z : z \in p^{-1}(b)\}$, $\Phi_2 = \{Y(\cdot : t_0,z),$ $z \in p^{-1}(b)\}$, Φ_3 is the set of all values of all families of surjections F_3 such that $F_3 : M \to \{(Z,X^2)\}$, $p^{-1}(X_t^2b) = p^{-1}(X_t^{-1}b)$, and if $Z(\cdot : t_0,z) \in \Phi_3$, then $z \in p^{-1}(b)$, $\|Z(t : t_0,z)\| \leqslant K_3Q(t)$, where the positive number K_3 depends on the family of surjections and on the point $(t_0,z) \in M \times p^{-1}(b)$. Obviously, $\Phi_1 \subset \Phi_2$.

Let us assume that the linear operations are defined into Φ_3 in natural way and

$$\|Z\|_{\Phi_3} = \sup_{t \geqslant t_0} \frac{\|Z(t : t_0,z)\|}{Q(t)}, \quad Z \in \Phi_3.$$

Then Φ_3 is the known Banach space.

Definition 1. The families F_1 and F_2 are called asymptotically equivalent according to Levinson on the stratum $p^1(b)$ with respect to the function Q, if there exist the bijection $P : p^{-1}(b) \to p^{-1}(b)$ such that $X_t z = Y(t : t_0, Pz) + o(Q(t))$ for $t \to +\infty$ and for any $z \in p^{-1}(b)$. The families F_1 and F_2 are called homeomorphic on the stratum $p^{-1}(b)$, if Φ_1 and Φ_2 are homeomorphic in the topology of the space Φ_3.

Definition 2. The families F_1 and F_2 are called asymptotically equivalent according to Nemitsky on the stratum $p^{-1}(b)$ with respect to the function Q, if they are asymptotically equivalent according to Levinson on this stratum and P is a homeomorphism.

Theorem 1. *Let us assume that*

$$Y(t : t_0, z) = X_t z + X_t T_t Y(\cdot : t_0, z) \tag{1}$$

and, on the contrary, if the surjection $\varphi(t)$ satisfies the equation (1), then $\varphi(t) \equiv Y(t : t_0, z)$ for $t \geqslant t_0$ and $\forall z \in p^{-1}(b)$, where:

a) $T_t : \Phi_3 \to p^{-1}(b)$ *for* $t \geqslant t_0$,

b) $T_t Z(\cdot : t_0, z) = \alpha + H_t Z(\cdot : t_0, z)$, $\alpha \in p^{-1}(b)$ *and depends on* $Z(\cdot : t_0, z) \in \Phi_3$, $H_t 0 = 0$,

c) $\|H_t Z_1(\cdot : t_0, z_1) - H_t Z_2(\cdot : t_0, z_2)\| \leqslant \|g_t\| \|Z_1(\cdot : t_0, z_1) - Z_2(\cdot : t_0, z_2)\|$ *for* $\forall z_1, z_2 \in p^{-1}(b)$, $g_t \in p^{-1}(b)$ *for any* $t \geqslant t_0$, $\|g_t\| \to 0$ *for* $t \to +\infty$

Then the families F_1 and F_2 are asymptotically equivalent according to Levinson on the stratum $p^{-1}(b)$ with respect to the function Q.

PROOF : We consider the operator

$$L : \Phi_3 \to \Phi_3 \tag{2}$$

where

$$LZ_2(\cdot : t_0, z_2) = Z_1(\cdot : t_0, z_1)$$
$$Z_1(t : t_0, z_1) = X_t z + X_t H_t Z_2(\cdot : t_0, z_2).$$

Then L is the contracting operator in Φ_3 and, consequently, for any $y \in p^{-1}(b)$ there exists a fixed point of L in Φ_3, that is

$$Z(t : t_0, z_0) = X_t z + X_t H_t Z(\cdot : t_0, z_0), \quad Z(\cdot : t_0, z_0) \in \Phi_3.$$

Then on the basis of b) we have

$$Z(t : t_0, z_0) = X z_0 + X_t T_t Z(\cdot : t_0, z_0), \quad z_0 \in p^{-1}(b).$$

Taking into consideration (1), hence, it follows $Z(\cdot : t_0, z0) \in \Phi_2$. Here $Pz = z_0$, where P is the bijection $p^{-1}(b)$ on $p^{-1}(b)$, $T_t Z(\cdot : t_0, z + \alpha) = \alpha + H_t Z(\cdot : t_0, z + \alpha)$ and $Z(t : t_0, Pz) = X_t z + o(Q(t))$ for $t \to +\infty$. ■

Lemma 2. *Let D be a Banach space and $f : D \to D$ be a contracting operator, U and V non-empty subsets of D such that $(I - f)V \subset U$ (I is an identity operator). If $S : U \to V$ satisfies the correlation*

$$Sy = y + fSy$$

then S is the homeomorphism of U on V.

PROOF : Since f is a contracting operator, then for any y Sy is a unique solution of the equation

$$Sy = y + fSy, \quad y \in V \tag{3}$$

Suppose that $y_1 \neq y_2$. Let us show that $Sy_1 \neq Sy_2$. If we assume the contrary, then

$$Sy_1 = y_1 + fSy_1, \quad Sy_2 = y_2 + fSy_2$$

and $y_1 - y_2 = 0$. Hence, $Sy_2 \neq Sy_1$. Let x be a fixed element. Let us consider the equation

$$x = y + fx \tag{4}$$

In this case $y = x - fx$, that is the equation (4) has the unique solution. Hence, there exists S^{-1}. Let us prove the continuity of S and S^{-1}. Let $y_n \to y_0$ for $n \to +\infty$. Then

$$Sy_n - Sy_0 = y_n - y_0 + fSy_n - fSy_0$$

and

$$\|Sy_n - Sy_0\| \leqslant \|y_n - y_0\| + q\|Sy_n - Sy_0\|, \quad 0 < q < 1.$$

Hence, it follows

$$\|Sy_n - Sy_0\| \leqslant \frac{1}{1-q}\|y_n - y_0\|$$

that is

$$Sy_n \to Sy_0 \quad \text{for} \quad y_n \to y_0.$$

Let us prove the continuity of S^{-1}. It is easy to notice that $S^{-1}y$ satisfies the correlation

$$y = S^{-1}y + fy, \quad y \in V.$$

Then, if $y_n \to y_0$ for $n \to +\infty$, then

$$\|S^{-1}y_n - S^{-1}y_0\| \leqslant \|y_n - y_0\| + q\|y_n - y_0\|$$

that is $S^{-1}y_n \to S^{-1}y_0$. The lemma is proved.

Theorem 2. *Let us assume that the conditions a) and b) of Theorem 1 are fulfilled. If*

$$\|H_tZ_1(\cdot : t_0, z_1) - H_tZ_2(\cdot : t_0, z_2)\| \leqslant q\|Z_1(\cdot : t_0, z_1) - Z_2(\cdot : t_0, z_2)\|_{\Phi_3}, \quad 0 < q < 1$$

then the sets Φ_1 *and* Φ_2 *are homeomorphic in the topology of the space* Φ_3.

PROOF : On the basis of (2) we consider the operator

$$(5) \qquad \begin{aligned} L_XZ = X + FZ, \quad X = X.z \in \Phi_1, \quad Z \in \Phi_3, \quad FZ = Z_1 \\ Z_1(t : t_0, z_1) = X_tH_tZ(\cdot : t_0, z) \end{aligned}$$

F is a contracting operator here. We define the operator $S : \Phi_1 \to \Phi_2$ as follows: for any $X \in \Phi_1$ SX is the fixed point of the contracting operator L_X. Then

$$SX = L_XSX$$

If we assume that $U = \Phi_1$, $V = \Phi_2$, then, on the basis of the Lemma, S is a homeomorphism of Φ_1 on Φ_2. ∎

Corollary. *From Theorem 2 it follows that the families* F_1 *and* F_2 *are homeomorphic on the stratum* $p^{-1}(b)$. *Moreover, they are asymptotically equivalent according to Nemitsky on this stratum.*

3. Now we consider the case, when on each stratum $p^{-1}(X_t, b)$

$$(6) \qquad \|X_tz\| \leqslant K_i(t_0, z)Q_i(t)$$

where

$$z \in \Xi_i, \bigcup_i \Xi = p^{-1}(b), K_i : M \times \Xi_i \to (0, +\infty), Q_i : M \to (0 + \infty), i = \overline{1, n}.$$

Suppose that $\Phi_1^{(i)} = \{X_0z : z \in \Xi_i\}$, $\Phi_3^{(i)}$ is the set of all values of all families of surjections $F_3^{(i)}$ such that $F_3^{(i)} : M \to \{(Z, X^2)\}$, $p^{-1}(X_t^2b) = p^{-1}(X_t^1b)$ and if $Z(\cdot : t_0, z) \in \Phi_3^{(i)}$, then $z \in p^{-1}(b)$, $\|Z(t : t_0, z)\| \leqslant K_3^{(i)}Q_i(t)$, where the positive number $K_3^{(i)}$ depends on the family of surjections and on the point $(t_0, z) \in M \times p^{-1}(b)$, $\Phi_1^{(i)} \in \Phi_3^{(i)}$, $i = \overline{1, n}$.

Theorem 3. *Assume that*

$$(7) \qquad Y(t : t_0, z) = X_tz + X_tT_t^{(i)}y(\cdot : t_0, z), \quad i = \overline{1, n}$$

and, on the contrary, if $\phi(t)$ *satisfies the equation (7), then* $\phi(t) \equiv Y(t : t_0, z)$ *for* $t \geqslant t_0$ *and* $\forall z \in \Xi_i$, *where*

a) $T_t^{(i)} : \Phi_3^{(i)} \to p^{-1}(b)$ *for* $t \to t_0$,

b) $T_t^{(i)}Z(\cdot : t_0, z) = \alpha + H_t^{(i)}Z(\cdot : t_0, z)$, $\alpha \in p^{-1}(b)$ *and depends on* $Z(\cdot : t_0, z) \in \Phi_3^{(i)}, H_t^{(i)}0 = 0$

c) $\|H_t^{(i)}Z_1(\cdot : t_0, z_1) - H_t^{(i)}Z_2(\cdot : t_0, z_2)\| \leqslant \|gt\|\|Z_1(\cdot : t_0, z_1) - Z_2(\cdot : t_0, z_2)\|_{\Phi_3^{(i)}}$ *for* $z_1, z_2 \in p^{-1}(b)$, $g_i \in p^{-1}(b)$ *for any* $t \geqslant t_0$, $\|g_t\| \to 0$ *for* $t \to +\infty$.

Then there exist the sets $\overline{\Xi}_i \subset p^{-1}(b)$, $\bigcup_i \overline{\Xi}_i = p^{-1}(b)$ *and the bijections* $P_i : \Xi_i \to \overline{\Xi}_i$ *such that* $X_t z = Y(t : t_0, P_i z) + o(Q_i(t))$ *for* $t \to +\infty$ *and for any* $z \in \Xi_i$, $i = \overline{1,n}$.

The proof of Theorem 3 is analogous to the proof of Theorem 1.

Theorem 4. *Let us assume that* $\Phi_2^{(i)} = \{Y(\cdot : t_0, z) : z \in \overline{\Xi}_i\}$ *and the conditions a) and b) of Theorem 3 are fulfilled. Then, if*

$$\|H_t^{(i)} Z_1(\cdot : t_0, z_1) - H_t^{(i)} Z_2(\cdot : t_0, z_2)\| \leqslant$$
$$\leqslant q_i \|Z_1(\cdot : t_0, z_1) - Z_2(\cdot : t_0, z_2)\|_{\Phi_3^{(i)}}, \quad 0 < q_i < 1, \quad i = \overline{1,n}$$

then the sets $\Phi_1^{(i)}$ *and* $\Phi_2^{(i)}$ *are homeomorphic in the topology of the space* $\Phi_3^{(i)}$ *for any* $1 \leqslant i \leqslant n$.

The proof of Theorem 4 does not differ from the proof of Theorem 2.

Theorem 1 is the transference of the notion of asymptotic equivalence of the differential equations according to Levinson [3],[5], when one of them is a linear homogeneous equation, onto metrizable abstract vector fiberings. Specifically the known Levinson theorem follows from Theorem 1. Theorem 2 is a new theorem. From the Lemma it follows that in the conditions of the Levinson theorem for the considered differential equations to be fulfilled the asymptotical equivalence to Nemitsky take place.

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