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Abstract. It is shown that "uniformly descending sequences" of degrees below $\underset{\sim}{O}'$ bound 1-generic degrees. Limits on a "uniform structure" of initial segments of degrees below $\underset{\sim}{O}'$ are concluded.

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It is the aim of this note to present a result concerning limits of a "uniform structure" of initial segments of degrees below $\underset{\sim}{O}'$.

The original motivation was to study the behaviour of PA degrees below $\underset{\sim}{O}'$. C. Jockusch asked whether every PA degree below $\underset{\sim}{O}'$ bounds a 1-generic degree. At present there are several ways how to show the positive answer to this question. The easiest one is to combine the fact that every PA degree $\leq \underset{\sim}{O}'$ bounds a nonzero r.e. degree ([5]) with the well known fact that every nonzero r.e. degree bounds a 1-generic degree (see [1],or [8] VI. 3.9). Nevertheless, the first attempt to solve this problem followed a slightly different way and yielded a result which has some other interesting consequences. We present it here.

Our notation and terminology is fairly standard. Let us recall that PA degrees are degrees of complete extensions of Peano arithmetic or, equivalently, degrees of 0-1 valued diagonally nonrecursive functions.

Definition. A sequence of degrees $\{\underset{\sim}{a}_i\}_i$ is called uniformly descending if for all i $\underset{\sim}{a}_{i+1} < \underset{\sim}{a}_i$ and for some choice of representatives $A_i \in \underset{\sim}{a}_i$ $A_{i+1} \leq_T A_i$ holds uniformly in i.

Theorem. *Let $\{\underset{\sim}{a}_i\}_i$ be a uniformly descending sequence of degrees and $\underset{\sim}{a}_0 \leq \underset{\sim}{O}'$. Then there is a 1-generic degree $\underset{\sim}{b}$ such that $\underset{\sim}{b} \leq \underset{\sim}{a}_i$ for all i.*

PROOF : Let $\{\underset{\sim}{a}_i\}_i$ be a sequence of sets such that $A_i \in \underset{\sim}{a}_i$ and $A_{i+1} <_T A_i$ for all i, $A_0 \leq_T \emptyset'$ and $A_{i+1} \leq_T A_i$ holds uniformly in i.

It follows easily from Upward Domination Lemma ([7], or [6] III. 5.4) that there is a sequence of functions $\{g_i\}_i$ such that $g_{i+1} \leq_T g_i$ holds uniformly in i, $g_i \in \underset{\sim}{a}_i$ and g_i is dominated by no function of degree $\leq \underset{\sim}{a}_{i+1}$ (in fact, by no function of degree $< \underset{\sim}{a}_i$) for all i.

We will modify the standard construction of a 1-generic set recursive in $\emptyset'$ replacing the $\emptyset'$ -construction by a construction where, roughly speaking, we allow to act for the sake of the eth requirement only when it is permitted by the function g_e.

The construction.

Step 0. Let $\sigma_0 = \emptyset$.

Step $s+1$. Ask whether there is a number $e \leq s$ such that there is a string τ such that

$$\sigma_s \subseteq \tau, \qquad lh(\tau) \leq g_e(s),$$
$$\phi_{e,g_e(s)}(\sigma_s)(e) \uparrow \text{ and } \phi_{e,g_e(s)}(\tau)(e) \downarrow .$$

If no such e exists, let $\sigma_{s+1} = \sigma_s * 0$. If such e exists, let e_s be the least such e, let τ be the least corresponding string and let $\sigma_{s+1} = \tau * 0$.

Let $B = \bigcup_s \sigma_s$.

Since $g_{i+1} \leq_T g_i$ holds uniformly in i it is easy to see that B is recursive in g_0 and, thus, in A_0. The same argument also shows that B is recursive in A_i for all i (although not uniformly in i).

We claim that the set B is 1-generic. Suppose for a contradiction that B is not 1-generic and let e be the least number such that $\phi_e(B)(e) \uparrow$ and for every s there is a string τ such that $\sigma_s \subseteq \tau$ and $\phi_e(\tau)(e) \downarrow$. Let us fix s_0 such that for no $s > s_0$ we have $e_s \leq e$ (i.e. if e_s exists then $e_s > e$). Since $g_{i+1} \leq_T g_i$ holds uniformly in i it is easy to verify that the whole further construction after the step s_0 is recursive in g_{e+1}. Thus, by the standard argument there is a function of degree $\leq \underset{\sim}{a}_{e+1}$ which dominates the function g_e. More precisely, if $t(s)$ is the least number such that there is a string τ for which $\sigma_s \subseteq \tau$, $lh(\tau) \leq t(s)$ and $\phi_{e,t(s)}(\tau)(e) \downarrow$ then for all $s > s_0$ we have $t(s) > g_e(s)$. Moreover, the function $t(s)$ can be shown to be A_{e+1} -recursive. A contradiction. Our claim is proved and B is 1-generic. ∎

It is well-known that there are PA degrees which fail to bound a 1-generic degree (e.g. any hyperimmune free PA degree). On the other hand, our Theorem shows that it does not apply to PA degrees below $\underset{\sim}{0}'$.

Corollary 1. *Every PA degree below $\underset{\sim}{0}'$ bounds a 1-generic degree.*

PROOF : It is well-known that every PA degree bounds a uniformly descending sequence of degrees (see, e.g., [4]). ∎

As we already mentioned there is another way how to prove this Corollary.

We mention some other applications of our Theorem. One of them concerns limits of a "uniform structure" of initial segments of degrees below $\underset{\sim}{0}'$. Let us consider in this connection recursively presentable countable distributive lattices with a least element. It is well-known that such lattices are isomorphic both to initial segments of degrees below $\underset{\sim}{0}''$ and to initial segments of degrees below $\underset{\sim}{0}'$ (see [6], [2]). But there is a great difference concerning uniformity in these two cases.

Given a recursively presentable countable distributive lattice L with a least element we can find recursively in $\underset{\sim}{0}''$ a sequence of sets $\{A_i\}_i$ such that the structure of degrees $\{dg(A_i)\}_i$ forms an initial segment of degrees isomorphic to L and, moreover, $A_j \leq_T A_i$ holds uniformly in i and those j for which $dg(A_j) \leq dg(A_i)$. On the other hand, methods for embedding such lattices below $\underset{\sim}{0}'$ are much more complicated (see [6]) and do not give us such uniformity. Our Theorem shows immediately that such uniformity is impossible in the latter case. Indeed, let us take the countable linear ordering $L = \{1 > u_1 > u_2 \cdots > 0\}$ and observe that there is no initial segment of degrees below $\underset{\sim}{0}'$ isomorphic to L having the described uniform structure.

Corollary 2. *No uniformly descending sequence of degrees below $\underset{\sim}{O}'$ can form an initial segment of degrees.*

Since every 1-generic degree bounds a uniformly descending sequence of degrees ([**3**]) we can easily conclude the following.

Corollary 3. *Uniformly descending sequences of degrees below $\underset{\sim}{O}'$ are downward dense, i.e., if $\{\underset{\sim}{a}_i\}_i$ is a uniformly descending sequence of degrees below $\underset{\sim}{O}'$ then there is a uniformly descending sequence of degrees $\{\underset{\sim}{b}_i\}_i$ such that $\underset{\sim}{b}_0 \leq \underset{\sim}{a}_i$ for all i.*

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