Commentationes Mathematicae Universitatis Carolinae

Aleksander Błaszczyk; Szymon Plewik; Sławomir Turek

Topological multidimensional van der Waerden theorem

Commentationes Mathematicae Universitatis Carolinae, Vol. 30 (1989), No. 4, 783--787

Persistent URL: http://dml.cz/dmlcz/106802

Topological multidimensional van der Waerden theorem

ALEKSANDER BŁASZCZYK, SZYMON PLEWIK, SŁAWOMIR TUREK

Abstract. We give a topological proof of the multidimensional van der Waeren theorem.
Keywords: multidimensional van der Waerden theorem, minimal dynamical system,
Classification: 54H20, 05A17

Responding to Furstenberg [4] we describe a direct proof of the topological version of multidimensional van der Waerden theorem. This theorem says that if X is a compact space and (X, G) is a minimal dynamical system, where G is a commutative group of homeomorphisms, then for each non-empty open set $U \subset X$ and $T_1, \dots T_k \in G$ there exists a natural number $n \geq 1$ such that $T_1^n(U) \cap \dots \cap T_k^n(U) \neq \emptyset$.

Furstenberg and Weiss [6] gave a direct proof of this theorem in the metric case (cf. also [5] and [7]) and derived the multidimensional van der Waerden theorem from it. We give a proof valid for all compact spaces and describe another way for obtaining the multidimensional van der Waerden theorem from its topological version.

If X is a topological space and G a group of its homeomorphisms, then the pair (X, G) is called a minimal dymamical system if there is no proper closed subset $F \subset X$ such that $T(F) = F$ for each $T \in G$. If X is compact, then (X, G) is minimal iff for each non-empty open set $U \subset X$ there exist $S_1, \dots, S_n \in G$ such that $X = S_1(U) \cup \dots \cup S_n(U)$.

Theorem 1. *(Topological Multidimensional van der Waerden Theorem)*
Let X be a compact topological space and G a commutative group of its homeomorphisms such that the dynamical system (X, G) is minimal. Then for each non-empty open set $V \subset X$ and each finite set $\{T_1, \dots T_k\} \subset G$ there exists a natural number $n \geq 1$ such that $V \cap T_1^n(V) \cdots \cap T_k^n(V) \neq \emptyset$

PROOF : We proceed by the induction on k.

1. Assume $k = 1$. Fix $T \in G$ and let $V \subset X$ be a non-empty open set. Since (X, G) is minimal, there exists $S_1, \dots, S_p \in G$ such that $S_1(V) \cup \dots \cup S_p(V) = X$. We construct a sequence $W_0, W_1, \dots$ of non-empty open sets such that:

(a) $W_0 = V$,
(b) $T^{-1}(W_n) \subset W_{n-1}$ for $n \geq 1$,
(c) for every n there exists t, $1 \leq t \leq p$, such that $W_n \subset S_t(V)$.

For the definition of W_{n+1} we choose a natural number t such that $1 \leq t \leq p$ and $W_{n+1} = T(W_n) \cap S_t(V) \neq \emptyset$.

If the sequence $W_0, W_1, \dots$ is defined, then we choose natural numbers i, j and t such that $i < j$ and $W_i \cup W_j \subset S_t(V)$. We set $U = S_t^{-1}(W_j)$ and $n = j - i$. By (b) we get

$T^{-n}(U) = T^{-n}(S_t^{-1}(W_j)) = S_t^{-1}(T^{-n}(W_j)) \subset S_t^{-1}(T^{-n+1}(W_{j-1})) \subset \cdots \subset S_t^{-1}(W_i) \subset V$. Therefore $U \subset T^n(V)$, $U \subset V$ and $V \cap T^n(V) \neq \emptyset$.

2. Assume the theorem is true for every collection of k elements of G. Fix a non-empty open set $V \subset X$ and $T_1, \ldots, T_{k+1} \in G$. Since the choice of transformations is free (we can set T_i^{-1} instead T_i) it suffices to show that there exists an open non-empty set $W \subset X$ such that $W \cup T_1^n(W) \cup \cdots \cup T_{k+1}^n(W) \subset V$ holds for some $n \geq 1$

By the minimality of (X, G) there exists $S_1, \ldots, S_p \in G$ such that $S_1(V) \cup \cdots \cup S_p(V) = X$.

Inductively we construct a sequence $W_0, W_1, \ldots$ of non-empty open sets and a sequence $p_0, p_1, \ldots$ of natural numbers such that :

(A) $W_0 = V$ and $p_0 = 0$,
(B) $T_1^{p_n}(W_n) \cup \cdots \cup T_{k+1}^{p_n}(W_n) \subset (W_{n-1})$ for every n,
(C) for every n there exists t, $1 \leq t \leq p$, such that $W_n \subset S_t(V)$.

If W_{n-1} and p_{n-1} are defined, then we apply induction assumption for W_{n-1} and homeomorphisms $T_{k+1} \circ T_1^{-1}, \ldots, T_{k+1} \circ T_k^{-1}$. There exists a natural number p_n such that

$$W_{n-1} \cap (T_{k+1} \cdot T_1^{-1})^{p^n}(W_{n-1}) \cap \cdots \cap (T_{k+1} \cdot T_k^{-1})^{p_n}(W_{n-1}) \neq \emptyset$$

For some t, $1 \leq t \leq p$, we get

$$W_n = T_{k+1}^{-p_n}(W_{n-1}) \cap T_1^{-p_0}(W_{n-1}) \cap \cdots \cap T_k^{-p_0}(W_{n-1}) \cap S_t(V) \neq \emptyset.$$

It is easy to see that conditions (B) and (C) hold for W_n and p_n.

If the sequence $W_0, W_1, \ldots$ is defined, then we choose natural numbers i, j and t such that $i < j$, $1 \leq t \leq p$ and $W_i \cup W_j \subset S_t(V)$. We set $n = p_{i+1} + \cdots + p_j$. For $1 \leq r \leq k+1$ we get $T_r^n(W_j) \subset W_i$. Indeed, $T_r^n(W_j) = T_r^{p_{i+1}+\cdots+p_j}(W_j) \subset T_r^{p_{i+1}+\cdots+p_{j-1}}(W_{j-1}) \subset T_r^{p_{i+1}}(W_{i+1}) \subset W_i$.

Let $W = S_t^{-1}(W_j)$. We have $W_j \subset S_t(V)$ and $W \subset V$. For $1 \leq r \leq k+1$, by the commutativity of G, we get

$$T_r^n(W) = T_r^n(S_t^{-1}(W_j)) = S_t^{-1}(T_r^n(W_j)) \subset S_t^{-1}(W_i) \subset V,$$

which finishes the proof. ∎

Corollary 1. *Let $T_1, \ldots, T_k$ be a commuting family of one-to-one continuous functions of a compact space X into itself and P be an open cover of X. Then there exists a natural number $n \geq 1$ such that $T_1^{-n}(U) \cap \cdots \cap T_k^{-n}(U) \neq \emptyset$ for some $U \in P$.*

PROOF : Consider minimal closed set $Y \subset X$ such that $T_i(Z) \subset Z$ for every $i \leq k$; one has to use Zorn Lemma to obtain such a set. The minimality of Z follows that $T_i(Z) = Z$ for any $I \leq k$. Indeed, suppose $T_i(Z) = Y \subsetneq Z$ for some $i \leq k$. Then for every $j \leq k$,

$$T_j(Y) = T_j(T_i(Z)) = T_i(T_j(Z)) \subset T_i(Z) = Y$$

Since $Y \neq Z$, we get a contradiction.

Set $G_i = T_i/Z$ for all $i \leq k$. Then the family $\{G_1, \ldots, G_k\}$ is a commuting family of homeomorphisms of Z into itself. The choice of the set Z follows that the system (Z, H), where H is the group of homeomorphisms of Z induced by $\{G_1, \ldots, G_k\}$ is a minimal dynamical system. Now it suffices to choose $U \in P$ such that $U \cap Z \neq 0$ and apply the Theorem 1.

Let βS denotes the Čech-Stone compactification of a (Tychonoff) space S. If S is a discreet space, βS is just the set of all ultrafilters over the set S; see [2] for details. In this case the topology on βS is generated by the family $\{U* : U \subset S\}$, where $U^* = \{v \in \beta S : U \in v\}$. Clearly $(U \cap V)^* = U^* \cap V^*$ and if $\{U_1, \ldots, U_n\}$ is a partition of S, then $\{U_1^*, \ldots, U_n^*\}$ is an open partition of βS. For every mapping f from S into S, the formula $\bar{f}(v) = \{U \subset S : f^{-1}(U) \in v\}$ defines the unique continuous extension of f over βS. One can easily check that $\bar{f}^{-1}(U^*) = (f^{-1}(U))^*$ for every $U \subset S$. Also, if $g : S \to S$ is another function, then $\bar{f} \circ \bar{g} = \overline{f \circ g}$. In particular, if f and g commutes, then $\bar{f}$ and $\bar{g}$ commutes as well. Clearly, $\bar{f}$ is one-to-one whenever f is one-to-one.

Let N denotes the set of natural numbers and $N^r = \{(k_1, \ldots, k_r)\}$: $k_i \in N$ for $1 \leq i \leq r$. If $a = (a_1, \ldots, a_r)$ and $b = (b_1, \ldots, b_r)$ and $n \in N$, then $b + na = (b_1 + na_1, \ldots, b_r + na_r)$. ■

Theorem 2. *(Multidimensional van der Waerden Theorem) If $\{U_1, \ldots, U_p\}$ is a partition of N^r then one the sets U_i has the property, that for every finite set $F \subset N^r$ there exists $n \in N$ and $b \in N^r$ such that $b + na \in U_i$ for all $a \in F$.*

PROOF : Clearly, every finite set $F \subset N^r$ is contained in a cube $\{1, \ldots, k\}^r = \{a_1, \ldots, a_t\}$, $t = k^r$, for some $k \in N$. Since the partition is finite, it suffices to show that there exists $b \in N^r$ and $q \leq p$ such that for some $n \in N$, $\{b+na_1, \ldots, b+na_t\} \subset U_q$. To do this let us consider functions $f_j : N^r \to N^r$ defined by $f_j(x) = x + a_j$ for $j \leq t$. Clearly, $\{\bar{f}_1, \ldots, \bar{f}_t\}$ is a commuting family of one-to-one continuous functions of βN^r into itself. By the Corollary there exist $q \leq p$ and a natural number $n \geq 1$ such that

$$\bar{f}_1^{-n}(U_q^*) \cap \cdots \cap \bar{f}_t^{-n}(U_q^*) \neq \emptyset$$

Then, by the remarks preceding the theorem, we get

$$f_1^{-n}(U_q) \cap \cdots \cap f_t^{-n}(U_q) \neq \emptyset$$

Take a point b belonging to this set. Then for every $j \leq t$ we have

$$b + na_j = f_j^n(b) \in U_q;$$

which completes the proof. ■

Theorem 1 and Theorem 2 are in fact equivalent. The lacking implication can be obtained by use of the trick from Balcar, Kalašek and Williams [1].

The next Proposition unable us to formulate Theorem 1 in a slight stronger from.

Proposition. *If X is a compact space and G is a commutative semigroup of continuous mappings of X onto itself, then there exist a compact space $\widetilde{X}$ and a continuous mapping $\pi : \widetilde{X} \to X$ such that $w(\widetilde{X}) \leq w(X) + |G| + w$ and for every $g \in G$ there exists a unique homeomorphism $\widetilde{g} : \widetilde{X} \to \widetilde{X}$ satisfying condition $\pi \circ \widetilde{g} = g \circ \pi$. Moreover, if (X, G) is a minimal dynamical system, then $(\widetilde{X}, \{\widetilde{g} : g \in G\})$ is a minimal system as well.*

PROOF : We define a partial ordering on G : we say that $f_1 \leq f_2$ whenever there exists $h \in G$ such that $f_1 \circ h = f_2$, By commutativity of G, h is unique. Indeed,if $f_1 \circ h = f_2$ and $f_1 \circ g = f_2$, then $g \circ f_1 = h \circ f_1$. Hence $h = g$, because f_1 is "onto". Observe that the ordering is directed, e.i. for any f, $g \in G$ there exists $h \in G$ satisfying $f \leq h$ and $g \leq h$. To do this it suffices to set $h = f \circ g$ and use the commutativity of G.

Now consider the inversive system $\zeta = \{X_f, \pi^f_g, G\}$, where $X_f = X$ for every $f \in G$ and $\pi^f_g = h$, where h is the unique element of G such that $g \circ h = f$; see Engelking [3] for the notions not explained here. Let $X = \lim_{\leftarrow} \zeta$, e.i. $\widetilde{X} = \{x \in \prod\{X_f : f \in G\}$: for every $f, g \in G$, $g \leq f$ implies $x_g = \pi^f_g(x_f)\}$. For every $f \in G$, $\pi_f : \widetilde{X} \to X_f$ is the canonical projection, e.i. $\pi_f(x) = x_f$. Clearly, the weight of $\widetilde{X}$ is not greater than the weight of the product $\prod\{X_f : f \in G\}$ and so it is not than the greatest cardinal among $w(X)$, $|G|$ and ω.

Every mapping $g \in G$ appoints a morphism of the system ζ into itself. This morphism is the identity in the set of indexes and for every $f \in G$ the mapping of X_f onto X_f equals g. This is indeed a morphism of inverse system since $g \circ \pi^f_h = \pi^f_h \circ g$ whenever $h \leq f$ and $f, g \in G$. Thus we get a unique continuous mapping $\widetilde{g} : \widetilde{X} \to \widetilde{X}$ such that $\pi_f \circ g = g \circ \pi_f$ holds true for every $f \in G$. We set $\pi = \pi_f$, where f is an arbitrary element of G. Since X is compact and g is "onto" (all bonding mappings are "onto"), it suffices to show that g is one-to-one. To do this fix different elements x, $y \in \widetilde{X}$. There exists $f \in G$ such that $x_f \neq y_f$. We set $h = g \circ f$. Since $f \leq h$, $\pi_f = \pi^f_h \circ \pi_h$. But $\pi^h_f = g$. Thus $g(\pi_h(x)) \neq g(\pi_h(y))$, which means that $\widetilde{g}(x) \neq \widetilde{g}(y)$.

It remains to show that the minimality of the systems (X, G) implies the minimality of $(X, \{\widetilde{g} : g \in G\})$. Fix $x \in \widetilde{X}$ and open non-empty set $U \subset \widetilde{X}$. Since $\widetilde{X}$ is an inverse limit over a directed set, there exist $f \in G$ and a non-empty open set $W \subset X_f$ such that $\pi_f^{-1}(W) \subset U$. By the minimality of (X, G), there exists $g \in G$ such that $g(\pi_f(x)) \in W$. Thus $\pi_f(\widetilde{g}(x)) \in W$ and therefore $g(x) \in \pi_f^{-1}(W)$, which completes the proof. ■

Corollary 2. *If (X, G) is a minimal dynamical system, where X is a compact space and G is a commutative semigroup of continuous functions mapping X into itself, then for every non-empty open set $U \subseteq X$ and every $T_1, \ldots, T_k \in G$, there exists $n \in N$ such that $U \cap T_1^{-n}(U) \cap \cdots \cap T_k^{-n}(U) \neq \emptyset$.*

PROOF : First observe that, by the minimality of (X, G), all mappings from G have to be "onto". Then we use the Proposition. The family $\{\widetilde{T} : T \in G\}$ generate a group of homeomorphisms of $\widetilde{X}$ into itself. By Theorem 1, there exist $n \in N$ such that

$$\pi^{-1}(U) \cap \widetilde{T}_1^{-1}(\pi^{-1}(U)) \cap \cdots \cap \widetilde{T}_k^{-n}(\pi^{-1}(U)) \neq \emptyset.$$

Since for every $T \in G$ there is $\widetilde{T}^{-n}(\pi^{-1}(U)) = \pi^{-1}(U)) = \pi(T^{-n}(U))$, we get

$$\pi^{-1}(U \cap T_1^{-n}(U) \cap \cdots \cap T_k^{-n}(U)) \neq \emptyset,$$

which completes the proof. ∎

REFERENCES

[1] B. Balcar, P. Kalášek and S.W. Williams, *On the multiple Birkhoff recurrence theorem in dynamics*, Comment. Math. Univ. Carolinae 28 (1987), 607-612.
[2] W.W. Comfort, S. Negrepontis, *The theory of ultrafilters*, Springer-Verlag, Heidelberg, 1974.
[3] R. Engelking, *General Topology*, Warszawa PWN, 1977.
[4] H. Furstenberg, *Ergodic behaviour of diagonal measures and a theorem of Szemeredi on arithmetic progressions*, J. d'Analyse Math. 31 (1977), 204-256.
[5] H. Furstenberg, *Recurrence in ergodic theory and combinatorial number theory*, Princeton University Press, New Jersey, 1981.
[6] H. Furstenberg, B. Weiss, *Topological dynamics and combinatorial number theory*, J. d'Analyse Math. 34 (1978), 61-85.
[7] R.L. Graham, B.L. Ritschild and J.H. Spencer, *Ramsey theory*, J. Wiley and Sons, New York, 1980.

Instytut Matematyki, Uniwersytetu Śląskiego, ul. Bankowa 14, 40 007 Katowice, Poland

(*Received* May 5,1989)