On pushing out frames

B. BANASCHEWSKI

Dedicated to the memory of Zdeněk Frolík

Abstract. This paper deals with the preservation of monomorphisms by pushouts in the category of frames. It shows that, for a frame L, pushout along every $u: L \longrightarrow M$ preserves monomorphisms iff the congruence frame of L is Boolean, and derives several further consequences. Among the lemmas needed, it is proved that, for regular L, the coequalizer of any $f,g: L \longrightarrow M$ is the map $M \longrightarrow \uparrow s$, $x \rightsquigarrow x \vee s$, for $s = \bigvee f(x) \wedge g(y)$ $(x \wedge y = 0)$

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Among the question discussed at one of the problem sessions during the Conference on Locales and Topological Groups in Curaçao in August 1989 was the following:

For pushouts

$$(*) \qquad \begin{array}{ccc} L & \xrightarrow{v} & N \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle \overline{u}} \\ M & \xrightarrow{\overline{v}} & P \end{array}$$

in the category of frames, what conditions will ensure that $\overline{v}$ is monic whenever v is monic?

As it stands, this question permits various specific interpretations, depending on the nature of the conditions envisaged. Thus, one might have in mind the possibility of conditions involving both, u and v, as in the fairly obvious observation, based on Stone Duality for finite distributive lattices, that the desired conclusion holds whenever L, M, and N are finite. On the other hand, one may consider the case that focusses on L and ask:

For which frames L does pushout along every homomorphism $u: L \longrightarrow M$ preserve monomorphisms?

This is the question which is settled in this note.

We recall a number of basic notions. A <u>frame</u> is a complete lattice L satisfying the distribution law

$$a \wedge \bigvee S = \bigvee a \wedge t (t \in S) \qquad (a \in L,\ S \subseteq L),$$

a frame homomorphism is a map between frames preserving all finitary meets and arbitrary joins, and Frm is the resulting category. The monomorphisms of Frm are exactly the one–one homomorphisms.

A frame L is called regular if, for each $a \in L$,

$$a = \bigvee x \ (x \prec a)$$

where $x \prec a$ means that $x \wedge z = 0$ and $a \vee z = e$ for some $z \in L$, 0 being the zero (= bottom) and e the unit (= top) of L.

Each element a of a frame L has a pseudocomplement $a^* = \bigvee x \ (x \wedge a = 0)$, and $L_{**} = \{x \in L \mid x = x^{**}\}$ is a frame homomorphism. Also, for any frame L, the frame congruences on L form a frame $\mathfrak{C}L$, the congruence frame of L, and the map

$$a \rightsquigarrow \nabla_a = \{(x, y) \mid x \vee a = y \vee a\}$$

is an embedding $L \longrightarrow \mathfrak{C}L$. Here, each ∇_a is complemented in $\mathfrak{C}L$ with complement

$$\Delta_a = \{(x, y) \mid x \wedge a = y \wedge a\},$$

and $\mathfrak{C}L$ is generated, as a frame, by the ∇_a and Δ_b $(a, b \in L)$; it follows, in particular, that $L \longrightarrow \mathfrak{C}L$ is an epimorphic embedding. The correspondence $L \rightsquigarrow \mathfrak{C}L$ is functorial such that, for any $h \colon L \longrightarrow M$, $\mathfrak{C}h \colon \mathfrak{C}L \longrightarrow \mathfrak{C}M$ takes a congruence on L to the congruence generated by its image by $h \times h$.

For further results concerning frames see Johnstone [**3**].

We begin with some lemmas, the first of which presents an explicit description of certain coequalizers which may be of independent interest. In the following, for any elements of a frame, $\uparrow s = \{x \mid x \geq s\}$.

Lemma 1. *For a regular frame L, the coequalizer of any frame homomorphisms $f, g \colon L \longrightarrow M$ is $M \to \uparrow s$, $x \rightsquigarrow x \vee s$, where $s = \bigvee f(x) \wedge g(y)$ $(x, y \in L$, $x \wedge y = 0)$.*

PROOF : Consider any $h \colon M \longrightarrow N$ such that $hf = hg$. Then,

$$h(s) = \bigvee hf(x) \wedge hg(y)(x \wedge y = 0) = \bigvee hf(x) \wedge hf(y)(x \wedge y = 0) = 0,$$

hence $h(a) = h(a \vee s)$ for each $a \in M$, and therefore h factors through $\uparrow s$. It remains to show that $f(a) \vee s = g(a) \vee s$ for each $a \in L$. Take any $x \prec a$ in L with $x \wedge z = 0$ and $a \vee z = e$. Then $f(x) \wedge g(z) \leq s$, so that

$$g(a) \vee s \geq g(a) \vee (f(x) \wedge g(z)) = (g(a) \vee f(x)) \wedge (g(a) \vee g(z)) = g(a) \vee f(x) \geq f(x).$$

Taking join over all $x \prec a$ and regularity then implies that $f(a) \leq g(a) \vee s$, and by symmetry one obtains $f(a) \vee s = g(a) \vee s$. ∎

Next, we have a fairly obvious observation on the preservation of arbitrary meets, based on the familiar fact that any frame homomorphism preserves the complements of complemented elements, by the uniqueness of complements in bounded distributive lattices, and that complete Boolean algebras satisfy the general deMorgan law $\sim \bigwedge S = \bigvee \sim t (t \in S)$ where $\sim$ stands for complementation.

Lemma 2. *For Boolean L, any frame homomorphism $h\colon L \longrightarrow M$ preserves arbitrary meets.*

Finally, our main step towards the desired result is

Lemma 3. *For any regular frame L, if*

$$\begin{array}{ccc} L & \xrightarrow{v} & N \\ \big\downarrow{\scriptstyle u} & & \big\downarrow{\scriptstyle \overline{u}} \\ M & \xrightarrow{\overline{v}} & P \end{array}$$

is a pushout in Frm *where u and v preserve arbitrary meets then $\overline{v}$ is dense whenever v is monic.*

Proof : It has to be shown, for monic v, that $\overline{v}$ only maps zero to zero. We use the standard construction of pushouts (*) as the coequalizer of a suitable pair of maps from L into the coproduct $M \oplus N$ of M and N, employing the description of the latter given in Banaschewski [1]. For this, recall that $M \oplus N$ can be obtained by first taking the frame $\mathfrak{S}$ of all Scott closed subsets of the product $M \times N$, and then the closure system $\mathfrak{L}$ in $\mathfrak{S}$ determined by the following condition on $X \in \mathfrak{S}$:
(J) For any $(a,b) \in M \times N$ and any finite $Z \subseteq M$, if $(a \frown t, b) \in X$ for all $t \in Z$ hen $(a \frown \bigvee Z, b) \in X$, and analogously for any finite $Z \subseteq N$.

Here, a Scott closed subset $X \subseteq M \times N$ is a downset (if $(a,b) \leq (s,t)$ and $(s,t) \in X$ then $(a,b) \in X$) which is closed under updirected joins, where updirected sets are by definition non–void.

The closure operator λ or $\mathfrak{S}$ associated with $\mathfrak{L}$ is then a nucleus, so that $\mathfrak{L}$ is a frame; it is the coproduct of M and N with coproduct maps

$$\begin{array}{ll} k\colon M \longrightarrow M \times N & x \rightsquigarrow (x,e) \rightsquigarrow {\downarrow}(x,e) \rightsquigarrow \lambda{\downarrow}(x,e) \\ \qquad\qquad \mathfrak{S} \xrightarrow{\ \lambda\ } \mathfrak{L} & \\ \ell\colon N \longrightarrow M \times N & y \rightsquigarrow (e,y) \rightsquigarrow {\downarrow}(e,y) \rightsquigarrow \lambda{\downarrow}(e,y) \end{array}$$

where ${\downarrow}(a,b) = \{(s,t) \mid s \leq a,\ t \leq b\}$.

Note that the condition (J) can be reduced to the two particular cases $Z = \emptyset$ and $Z = \{s,t\}$, which is the way we shall deal with it.

Now, the pushout (*) is obtained as the coequalizer of

$$L \underset{\ell v}{\overset{ku}{\rightrightarrows}} \mathfrak{L}$$

and since L is regular Lemma 1 says this is the map

$$\mathfrak{L} \longrightarrow {\uparrow}S,\ X \rightsquigarrow X \vee S$$

for

$$S = \bigvee_{a,b \in L,\ a \frown b = 0} ku(a) \frown \ell v(b).$$

To describe S more explicitly, we first observe that, for any $x \in M$ and $y \in N$,

$$k(x) = \downarrow(x, e) \cup \downarrow(e, 0) \text{ and } \ell(y) = \downarrow(e, y) \cup \downarrow(0, e)$$

which results by direct checking: the indicated sets are Scott closed, as finite unions of such sets, (J) for $Z = \emptyset$ holds in virtue of the appearance of $(e, 0)$ and $(0, e)$, respectively, and (J) for $Z = \{s, t\}$ is verified by going through all possible cases for the occurrence of the elements $(a \frown s, b)$ and $(a \frown t, b)$, respectively $(a, b \frown s)$ and $(a, b \frown t)$.

Next, since binary meet in $\mathfrak{L}$ is just intersection,

$$k(x) \frown \ell(y) = \downarrow(x, y) \cup \downarrow(e, 0) \cup \downarrow(0, e)$$

and therefore

$$S = \bigvee_{a,b \in L,\, a \frown b = 0} \downarrow(u(a), v(b)) \cup \downarrow(e, 0) \cup \downarrow(0, e).$$

Finally, we wish to establish that S is actually just the <u>union</u> W of the indicated sets, that is, W is Scott closed and satisfies (J). Here, it suffices to consider only the sets $\downarrow(u(a), v(b))$ for $a \frown b = 0$ in L, the apparently missing $\downarrow(e, 0)$ and $\downarrow(0, e)$ being covered by the cases $a = e$, $b = 0$, and $a = 0$, $b = e$.

The condition (J) only has to be checked for the case $Z = \{s, t\}$. If $Z \subseteq M$ and $(c \frown s, d)$, $(c \frown t, d) \in W$ for some $(c, d) \in M \times N$, then there are a, b, a', $b' \in L$ such that

$$a \frown b = 0 = a' \frown b',$$

$$(c \frown s, d) \leq (u(a), v(b)) \text{ and } (c \frown t, d) \leq (u(a'), v(b')),$$

and it follows that

$$(c \frown (s \vee t), d) \leq (u(a \vee a'), v(b \frown b'))$$

where $(a \vee a') \frown (b \frown b') = 0$, and hence $(c \frown (s \vee t), d) \in W$. The case $Z \subseteq N$ being entirely analogous, this established (J).

For Scott closedness, we first note that the hypothesis on u and v implies the following: For each $x \in M$ and $y \in N$, if

$$\overline{x} = \bigwedge_{a \in L,\, x \leq u(a)} a, \qquad \overline{y} = \bigwedge_{b \in L,\, y \leq v(b)} b$$

then $x \leq u(\overline{x})$, $y \leq v(\overline{y})$ and the maps $x \rightsquigarrow \overline{x}$, $y \rightsquigarrow \overline{y}$ are order preserving. Of course, these maps are precisely the left adjoints of the maps u and v. Their usefulness here lies in the fact that, for any $(x, y) \in M \times N$, $(x, y) \in W$ iff $\overline{x} \frown \overline{y} = 0$.

Now, consider any updirected $T \subseteq W$. Then, for

$$a = \bigvee\{\overline{x} \mid (x, y) \in T\} \text{ and } b = \bigvee\{\overline{y} \mid (x, y) \in T\}$$

we have

$$a \frown b = \bigvee\{x_1 \frown y_2 \mid (x_1, y_1), (x_2, y_2) \in T\} = 0$$

since T is updirected, and hence

$$\bigvee T = \Big(\bigvee_{(x,y)\in T} x, \bigvee_{(x,y)\in T} y \Big) \leq \Big(\bigvee_{(x,y)\in T} u(\overline{x}), \bigvee_{(x,y)\in T} v(\overline{y}) \Big) = (u(a), v(b))$$

shows $\bigvee T \in W$. In all, this establishes that $S = W$.

Finally, for any $x \in M$, $k(x) \vee S = S$ iff $k(x) \subseteq S$ iff $\downarrow(x, e) \leq (u(a), v(b))$ for some $a, b \in L$ such that $a \frown b = 0$. Here, $e = v(b)$ implies $b = e$ for monic v, and then $a = 0$ so that $x = 0$. This proves $\overline{v}$ is dense, as claimed. ∎

Now we have

Proposition. *Pushout along every $u\colon L \longrightarrow M$ preserves monomorphisms iff the congruence frame $\mathfrak{C}L$ of L is Boolean.*

PROOF : $(\Rightarrow)$ For any L, the diagram

$$\begin{tikzcd}
L \arrow[r, "su"] \arrow[d, "u"] & (\mathfrak{C}L)_{**} \arrow[d, "id"] \\
\mathfrak{C}L \arrow[r, "s"] & (\mathfrak{C}L)_{**}
\end{tikzcd}$$

where u is the usual embedding and $s = (.)^{**}$ is a pushout since u is epic. Moreover, su is monic since

$$su(x) = u(x)^{**} = \nabla_x^{**} = \nabla_x,$$

but s is not monic unless $\mathfrak{C}L$ is Boolean.

$(\Leftarrow)$ Assuming $\mathfrak{C}L$ is Boolean, consider the following enlargement of the pushout diagram (*).

$$\begin{tikzcd}
\mathfrak{C}L \arrow[rrr, "\mathfrak{C}v"] \arrow[dd, "\mathfrak{C}u"'] & & & \mathfrak{C}N \arrow[ddd, "t"] \\
& L \arrow[r, "v"] \arrow[d, "u"] & N \arrow[d, "\overline{u}"] & \\
\mathfrak{C}M \arrow[d] & M \arrow[l] \arrow[r, "\overline{v}", "w"'] & P & \\
(\mathfrak{C}M)_{**} \arrow[rrr, "s"] & & & Q
\end{tikzcd}$$

where the outer square is also a pushout, the maps $L \longrightarrow \mathfrak{C}L, \ldots$ are the standard embeddings, $\mathfrak{C}M \longrightarrow (\mathfrak{C}M)_{**}$ is the usual map $(.)^{**}$, and w is the unique map resulting from the pushout property of the inner square and the various commuting parts of the diagram.

Now, for monic v, $\mathfrak{C}v$ is dense and hence monic because $\mathfrak{C}L$ is Boolean. Moreover, the maps of the outer pushout originating at $\mathfrak{C}L$ preserve arbitrary meets by Lemma 2, and since any Boolean frame is regular Lemma 3 applies. This makes s dense and hence monic because $(\mathfrak{C}M)_{**}$ is Boolean. Finally, since the map $M \longrightarrow (\mathfrak{C}M)_{**}$ is also monic, as already noted in the first part of this proof, it follows that $w\overline{v}$ and therefore $\overline{v}$ is monic. ■

Corollary 1. *Pushout along $u\colon L \longrightarrow M$ preserves monomorphisms whenever L is Boolean or finite.*

PROOF : For Boolean L, the usual $L \longrightarrow \mathfrak{C}L$ is an isomorphism so that $\mathfrak{C}L$ is also Boolean. If L is finite, $\mathfrak{C}L$ is finite, and since any $\mathfrak{C}L$ is generated, as a frame, by the complemented elements ∇_a and Δ_a, $a \in L$, this makes every element of $\mathfrak{C}L$ complemented. ■

In analogy with a familiar model theoretic notion, a frame L is said to have the <u>amalgamation property</u> if, in any pushout (*), $\overline{u}$ and $\overline{v}$ are monic whenever u and v are. Since the ($\Rightarrow$) proof of the Proposition involves the monomorphism $u\colon L \longrightarrow \mathfrak{C}L$, we then also have

Corollary 2. *A frame L has the amalgamation property iff $\mathfrak{C}L$ is Boolean.*

Finally, we have the following partial strengthening of the Proposition, obtained as a consequence of the first part of the proof:

Corollary 3. *For any frame L, $\mathfrak{C}L$ is Boolean whenever pushout along any quotient map $L \longrightarrow L/\Theta$ preserves monomorphisms.*

PROOF : We observe first that, for any congruence Θ on L and any monomorphism $v\colon L \longrightarrow N$, the pushout of v along the quotient map $u\colon L \longrightarrow L/\Theta$ is given by

$$\begin{tikzcd} L \arrow[r, "v"] \arrow[d, "u"] & N \arrow[d, "\overline{u}"] \\ L/\Theta \arrow[r, "\overline{v}"] & N/\mathfrak{C}v(\Theta) \end{tikzcd}$$

where $\overline{u}$ is the indicated quotient map and $\overline{v}$ the unique homomorphism such that $\overline{v}u = \overline{u}v$, resulting from the fact that the $\Theta = \mathrm{Ker}(u) \subseteq \mathrm{Ker}(\overline{u}v)$. Hence, if $\overline{v}$ is monic then

$$\overline{u}v(x) = \overline{u}v(y) \text{ iff } u(x) = u(y)$$

for all x, $y \in L$, that is, $\Theta = (v \times v)^{-1}[\mathfrak{C}v(x)]$. This makes the homomorphism $\mathfrak{C}v\colon \mathfrak{C}L \longrightarrow \mathfrak{C}N$ one–one. It follows that, for any pushout

$$\begin{tikzcd} L \arrow[r, "v"] \arrow[d, "u"] & N \arrow[d, "\overline{u}"] \\ \mathfrak{C}L \arrow[r, "\overline{v}"] & P \end{tikzcd}$$

where u is now the usual embedding $L \longrightarrow \mathfrak{C}L$, $\bar{v}$ is monic in view of the commuting square

$$\begin{array}{ccc} L & \xrightarrow{v} & N \\ \downarrow & & \downarrow \\ \mathfrak{C}L & \xrightarrow{\mathfrak{C}\bar{v}} & \mathfrak{C}N \end{array}$$

Applying this to the pushout already considered in the first part of the proof of the Proposition we conclude as before that $\mathfrak{C}L$ is Boolean. ■

Remark 1. The frames that occur in the Proposition have been considered elsewhere. Thus, Beuzer and Macnab [2] characterize them as those frames L for which each $\uparrow a$, $a \in L$, has a smallest dense element. Also, Simmons [6] shows, for the frame $\mathfrak{D}X$ of open sets of a T_0 space X, that $\mathfrak{C}(\mathfrak{D}X)$ is Boolean iff X is scattered.

Remark 2. There is an alternative proof of the Proposition, based on an auxiliary result different from Lemma 3. Joyal and Tierney [4] show that pushout along any $v: L \longrightarrow M$ takes open monomorphisms to open monomorphisms, where in general a frame homomorphism is called open if it preserves all meets and the relative pseudocomplement

$$a \rightarrow b = \bigvee x(a \wedge x \leq b)$$

(see also Pitts [5]). This can be applied in the ($\Leftarrow$) part of the proof of the Proposition, using the natural extension of Lemma 2 that any frame homomorphism with Boolean domain is open. The part of the latter concerning $a \rightarrow b$ results from the observation that $a \rightarrow b = (\sim a) \vee b$ in any frame if the element a has a complement $\sim a$.

It might be noted that the Proposition says more than what is immediately implied by the Joyal–Tierney result since there obviously exist non–open monomorphisms $v: L \longrightarrow N$ for which $\mathfrak{C}L$ is Boolean, as is the case, for instance, for any finite non–Boolean L.

Remark 3. There are certain subcategories of K of Frm in which the monomorphisms are still exactly the one–one homomorphisms but pushout along each $u: L \longrightarrow M$ preserves the monomorphisms in K, even though these pushouts are pushouts in Frm and there are $L \in K$ for which $\mathfrak{C}L$ is not Boolean. One of these is the category CohFrm of coherent frames and coherent homomorphisms (Johnstone [3]). CohFrm is equivalent to the category **D** of (bounded) distributive lattices, and the result for it follows from the corresponding one for **D**. The latter can be obtained by using that **D** has enough injectives, which comes from the corresponding fact for Boolean algebras where it amounts to the familiar Sikorski Theorem that a Boolean algebra is injective iff it is complete. Alternatively, Stone Duality for **D** will also give the desired result. However, there also is a choice–free, and indeed constructive, argument which uses a reduction to finite lattices followed by a direct proof. Another subcategory of Frm of the same kind is the full subcategory of Frm given by the compact regular frames. In this case, the result in question follows from the duality between these frames and compact Hausdorff spaces, which in turn follows from the Boolean Ultrafilter Theorem. We do not know whether there is a constructive proof or, at least, a choice–free one.

Remark 4. By the result of Joyal and Tierney [4] quoted in Remark 1, pushout along every $u: L \longrightarrow M$ preserves monomorphisms whenever L has the property that any monomorphism $v: L \longrightarrow N$ is open. This observation, however, is also an easy consequence of the Proposition, specifically of Corollary 1, since L has this property exactly if it is Boolean. Indeed, for any L, if the familiar embedding $L \longrightarrow \mathfrak{C}L$ is open then, for each $a \in L$,

$$\nabla_{a^*} = (\nabla_a)^* = \Delta_a$$

so that

$$\nabla_{a^* \vee a} = \nabla_{a^*} \vee \nabla_a = \Delta_a \vee \nabla_a = \nabla_e,$$

therefore $a^* \vee a = e$, and thus a is complemented. The converse had already been noted.

Remark 5. The question settled by the Proposition has two variants focussing on u and v respectively, namely:

For which $u: L \longrightarrow M$ does pushout along u preserve monomorphisms?

For which monomorphisms $v: L \longrightarrow N$ does pushout along arbitrary $u: L \longrightarrow M$ produce monomorphisms?

Concerning the first, one might note that any L rather trivially appears as the domain of some u asked for by that question: take $M = L$ and $u = id_L$ or M = the terminal (=one–element) frame. Further, an argument based on Lemma shows that all $u: L \longrightarrow 2$ with regular L have the property involved. This, in turn, leads to the result that, *for regular L, pushout along $u: L \longrightarrow M$ preserves monomorphisms whenever M is spatial.* An immediate corollary of this, obtained from the special pushout used in the first part of the proof of the Proposition, in that, for regular L, $\mathfrak{C}L$ is Boolean whenever it is spatial. This could also be derived, in a very different manner, by means of the criterion of Beuzer and Macnab [2] quoted in Remark 1, or from the results of Simmons [6]. Concerning pushout along arbitrary $u: L \longrightarrow 2$, we note that the frame of open sets of any infinite space X with the cofinite topology provides an example of such a u pushout along which does not always preserve monomorphisms: u takes each non–void open set to 1, and v is the identical embedding into the power set of X.

As to the second question, the monomorphisms involved include, by what has been proved and said above, the v with Boolean $\mathfrak{C}L$ and the open v. In addition there are, rather more obviously, the left invertible v. Finally, there is the equally obvious point that both, the u in the first and the v in the second question, are closed under composition. Complete answers, in either case, seem to be rather more difficult than what has been dealt with here.

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also managed to disprove the claim that the result given in Remark 5 holds without the restriction to spatial M, which I had put forward at the problem session mentioned earlier.

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Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario L8S 4K1 Canada

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