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A note on Baire isomorphism

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Dedicated to the memory of Zdeněk Frolík

Abstract. We give an example of an absolute Baire space which is not Baire isomorphic to any compact Hausdorff space. This answers a question asked by Z.Frolík (see also [1] Problem N 54)

Keywords: Baire set, Baire isomorphism

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Introduction. A set A of a topological space X is called Baire set if A belongs to the smallest σ -algebra of subsets of X which contains all zero-sets of X . If X is metrizable space, then σ -algebras of Baire and Borel sets of X coincide. A mapping $f : X \rightarrow Y$ is called a Baire mapping if an inverse image of zero-set of Y is a Baire set of X . A bijection $f : X \rightarrow Y$ is a Baire isomorphism, if f and f^{-1} are Baire mappings. A Tychonoff space X is called absolute Baire space if X is a Baire subset of $\beta(X)$. Any Čech-complete Lindelöf space X is absolute Baire space as X is the intersection of sequence of cozero-sets of $\beta(X)$.

In this paper we give a negative answer to following question: whether every absolute Baire space is Baire isomorphic to some compact Hausdorff space? We think that this was originally formulated by Z.Frolík (see also [1] Problem N 54).

We construct a Čech-complete Lindelöf space M , which is not Baire isomorphic to any compact Hausdorff space.

We use the following notations: $|X|$ is cardinality of X , $w(X)$ is weight of X , $c = |I = [0, 1]|$, P is the space of irrational numbers, $\exp_{\aleph_0} X = \{A : A \subseteq X, |A| \leq \aleph_0\}$, if γ is a family of subsets of X , $A \subseteq X$, then $\gamma(A)$ is a star of A , i.e. $\gamma(A) = \cup\{U \in \gamma : U \cap A \neq \emptyset\}$.

We shall use the following results

I) [2] If X and Y are metrizable compact spaces and $f : X \rightarrow Y$ is a continuous mapping, then $\mathcal{Z}(f) = \{x : |f^{-1}f(x)| > 1\}$ is F_σ -set of X .

II) (that is special case of the result from [3]). Let A be a Borel set in P . If A is not a subset of any σ -compact set in P , then A contains a subset that is closed in P and homeomorphic to P .

III) Let $f : X \rightarrow P$ be a mapping onto P and X be a σ -compact separable metric space. Then P contains a closed F , homeomorphic to P , so that $f^{-1}(F)$ is not F_σ in X .

This result easily follows from [4].

We recall that a spectrum $(X_\alpha, P_\beta^\alpha, \mathcal{U})$ is sigma-spectrum [5] if $w(X_\alpha) \leq \aleph_0, \alpha \in \mathcal{U}, \mathcal{U}$ is \aleph_0 -complete i.e. every countable chain $\beta \subseteq \mathcal{U}$ has the least upper bound $\gamma = \gamma(\beta)$, and X_γ is naturally homeomorphic to $\lim(X_\alpha, P_\beta^\alpha, \beta)$.

IV) Let $X = \lim S_1, Y = \lim S_2, S_1 = (X_\alpha, \pi_\beta^\alpha, \mathcal{U}), S_2 = (Y_\alpha, \mu_\beta^\alpha, \mathcal{U})$, where $S_i, i = 1, 2$ are sigma-spectrum, and all $\pi_\beta^\alpha, \mu_\beta^\alpha$ are perfect mappings. If X and Y are Baire isomorphic spaces then there are closed and cofinal subset $\mathcal{U}' \subseteq \mathcal{U}$, and Baire isomorphisms $f_\alpha : X_\alpha \rightarrow Y_\alpha, \alpha \in \mathcal{U}'$, such that $\mu_\beta^\alpha \circ f_\alpha = f_\beta \circ \pi_\beta^\alpha$ for every $\alpha, \beta \in \mathcal{U}'$, $\alpha \geq \beta$. In fact, that is proved in [6].

Construction of M . We shall consider a generalization of the Alexandrov duplicate construction, which is similar to the construction from [7].

If $\gamma = \{F_\alpha : \alpha \in A\}$ is a pair-wise disjoint collection of closed sets of P and $B \subseteq A$, we may define a topology on $P(B) = P \times \{0\} \cup \{F_\alpha \times \{1\} : \alpha \in B\}$ as follows: $\cup\{F_\alpha \times \{1\} : \alpha \in B\}$ is open and has the topology of direct sum $\oplus\{F_\alpha : \alpha \in B\}$, and neighborhoods of a point $(x, 0)$ in $P \times \{0\}$ are have the form $U(\alpha) = (U \times \{0\}) \cup ((U \setminus F_\alpha) \times \{1\})$, where $x \in U, U \subseteq P$ is open and $\alpha \in B$. Then $P(B)$ is a regular Lindelöf space and $P \times \{0\}$ has original topology. If $B_1 \supseteq B_2$, then $P(B_2)$ is a closed subset of $P(B_1)$. For $B_1 \supseteq B_2$ we may define a mapping $\mu_{B_2}^{B_1} : P(B_1) \rightarrow P(B_2)$ as follows: $\mu_{B_2}^{B_1}(x) = x$ if $x \in P(B_2)$ and $\mu_{B_2}^{B_1}(x) = \pi_p(x)$ if $x \in P(B_1) \setminus P(B_2)$. A mapping $\mu_{B_2}^{B_1}$ is a retraction and, consequently, a quotient mapping. Let us show that $\mu_{B_2}^{B_1}$ is closed (and hence perfect). First of all, $(\mu_{B_2}^{B_1})^{-1} \mu_{B_2}^{B_1}(U(\alpha)) = U(\alpha)$ and $(\mu_{B_2}^{B_1})^{-1} \mu_{B_2}^{B_1}(V \times \{1\}) = V \times \{1\}$, where $V \subseteq F_\alpha$ and V is open in $F_\alpha, \alpha \in B_2$ and the sets $U(\alpha), V \times \{1\}$ form a base for $P(B_2)$. Let $p \in F_\alpha, \alpha \in B_1 \setminus B_2$. Then any neighborhood of a set $p \times \{0, 1\}$ contains a neighborhood of the form $U(\alpha) \cup (V \times \{1\})$, where $p \in U, U \subseteq F_\alpha$ and a set V is open in F_α . Let W be an open set of P and $W \cap F_\alpha = V$. If $U' = U \cap W$, then $U'(\alpha) \cup (V \times \{1\})$ is an open set of $P(B_1)$ and $(\mu_{B_2}^{B_1})^{-1} \mu_{B_2}^{B_1}(U'(\alpha) \cup (V \times \{1\})) = U'(\alpha) \cup (V \times \{1\})$. Consequently, $\mu_{B_2}^{B_1}$ is a perfect mapping.

So we have constructed the spectrum $S_1 = \{P(B_1) : \mu_{B_2}^{B_1}, B_1 \supseteq B_2, \exp_{\aleph_0} A\}$. Since all the mappings $\mu_{B_2}^{B_1}$ are perfect, $A \supseteq B_1 \supseteq B_2$, it easy follows that $\lim S_1 = P(A)$ and S_1 is sigma-spectrum.

The result space M we shall construct like $P(A)$, choosing a suitable $\gamma = \{F_\alpha : \alpha \in A\}$.

Let $\mathcal{F} = \{\varphi : \varphi : P \rightarrow X \text{ is a Baire isomorphism and } X \subseteq \mathbb{R}^{\aleph_0}\}$ and let $\Gamma = \{\gamma : \gamma \text{ is pair-wise disjoint collection of closed subsets of } P \text{ such that } 1) \text{ for every } F \in \gamma, F \text{ is homeomorphic to } P \text{ and } 2) |\gamma| = c\}$. Then $|\mathcal{F}| = c$ [2]. For every $\gamma \in \Gamma$, let $\mathcal{F}(\gamma) = \{\varphi \in \mathcal{F} : \text{there exists a } \gamma(\varphi) \in \Gamma \text{ such that } 1) \gamma(\varphi) \text{ is a refinement of } \gamma, 2) \text{ every } F \in \gamma \text{ contains at most own set of } \gamma(\varphi), 3) \gamma(\varphi) \text{ can be represented as a disjoint union of countable families } \gamma_\alpha(\varphi), \alpha < c, \text{ such that } \varphi(\cup \gamma_\alpha(\varphi)) \text{ is a } \sigma\text{-compact set for every } \alpha < c\}$.

V) There exists $\underline{\gamma} \in \Gamma$, such that for every $\varphi \in \mathcal{F}(\underline{\gamma}) \{ \{F : F \in \underline{\gamma}\}, \varphi(F) \text{ is not } \sigma\text{-compact set and } \varphi(F) \cap \varphi(F') = \emptyset \text{ for every } F' \in \underline{\gamma} \setminus \{F\} \} = c$.

PROOF : Take an arbitrary $\delta \in \Gamma$. If $\mathcal{F}(\delta) = \emptyset$ then let $\underline{\gamma} = \delta$. Assume that $\mathcal{F}(\delta) \neq \emptyset$. Due to $|\mathcal{F}(\delta)| \leq c$ we construct by transfinite induction the families

μ_α , $\alpha < c$, such that 1) for every $\alpha < c$ there exist $\varphi \in \mathcal{F}(\delta)$ and $\beta < c$ such that $\mu_\alpha = \delta_\beta(\varphi)$ 2) $\delta(\cup \mu_\alpha) \cap \delta(\cup \mu_\beta) = \phi$ if $\alpha \neq \beta$, 3) for every $\varphi \in \mathcal{F}(\delta) | T_\varphi = \{\alpha : \mu_\alpha = \delta_\beta(\varphi) \text{ for some } \beta < c\} = c$. Let $\alpha \in T_\varphi$. Then the set $\varphi(\cup \mu_\alpha)$ is σ -compact, i.e. $\varphi(\cup \mu_\alpha) = \bigcup_{j=1}^{\infty} B_j$, where B_j is a compact set, $j \in N$. Let $\Phi_\alpha \in \mu_\alpha$.

Since Φ_α is not σ -compact set and $\Phi_\alpha = \bigcup_{j=1}^{\infty} (\Phi_\alpha \cap \varphi^{-1}(B_j))$, then for some $\gamma_0 \in N$ there is not a σ -compact set T such that $\Phi_\alpha \cap \varphi^{-1}(B_{j_0}) \subseteq T \subseteq \Phi_\alpha$. The set $\Phi_\alpha \cap \varphi^{-1}(B_{j_0})$ is Borel, consequently, according to II) there exists a closed set Φ'_α which is homeomorphic to P and $\Phi'_\alpha \subseteq \Phi_\alpha \cap \varphi^{-1}(B_{j_0})$. If a set $\varphi(\Phi'_\alpha)$ is not σ -compact, then let $\Phi''_\alpha = \Phi'_\alpha$. Suppose $\varphi(\Phi'_\alpha)$ is σ -compact set, then, according to III) there is a closed set $\Phi \subseteq \Phi'_\alpha$, homeomorphic to P and such that $\varphi(\Phi)$ is not F_σ in $\varphi(\Phi'_\alpha)$, hence, it is not σ -compact. Let $\Phi''_\alpha = \Phi$. The family $\gamma = \{\Phi''_\alpha : \alpha < c\}$ is found. So the V) is proved. ■

Let $M = P(A)$, where $\gamma = \{F_\alpha : \alpha \in A\}$ has all the properties if V) The space M is Čech-complete and Lindelöf, since $\mu_\phi^A : M \rightarrow P \times \{0\}$ is a perfect mapping from M onto P .

The space M is not Baire isomorphic to any compact Hausdorff space. Suppose on the contrary, that M is Baire isomorphic to some compact Hausdorff space Y . Then, according to [8], $w(M) = w(Y)$. Hence, we can assume that $Y \subseteq I^A$. Let $Y_B = \pi_B(Y) \subseteq I^B$, where $B \subseteq A$, and if $A \supseteq B_1 \supseteq B_2$ then $\pi_{B_2}^{B_1} = \pi_{B_2} | Y_{B_1} : Y_{B_1} \rightarrow Y_{B_2}$. Clearly Y is the limit space of the sigma-spectrum $S_2 = \{Y_{B_1}, \pi_{B_2}^{B_1}, B_1 \supseteq B_2, \exp_{N_0} A\}$. According to IV) there is $U' \subseteq \exp_{N_0} A$, closed and cofinal subset of $\exp_{N_0} A$ and Baire isomorphisms $f_B : P(B) \rightarrow Y_B$, $B \in U'$, so that $f_{B_2} \circ \mu_{B_2}^{B_1} = \pi_{B_2}^{B_1} \circ f_{B_1}$ for all $B_1 \supseteq B_2, B_1, B_2 \in U'$. Without loss of generality we can assume that $U \ni B_0$, where B_0 is minimal.

VI) There is a family of pairs $\{(B_\alpha, B'_\alpha) : 1 \leq \alpha < c\}$ so that $B_\alpha \supseteq B'_\alpha$, $B_\alpha, B'_\alpha \in U'$ and $(B_\alpha \setminus B'_\alpha) \cap (B_\beta \setminus B'_\beta) = \phi$ if $\alpha \neq \beta$.

We shall construct the family by induction. Let (B_1, B'_1) be an arbitrary pair, where $B_1 \supseteq B'_1$, $B_1, B'_1 \in U'$ (this pair does exist, because of cofinality of U'). Suppose there are pairs $(B_\alpha, B'_\alpha) : 1 \leq \alpha < \beta < c$. Let us set $B_\beta = \{B_\alpha : \alpha < \beta\} \cup \{B'_\alpha : \alpha < \beta\}$. Then $|B_\beta| < c$ and since U' is cofinal there is $B'_\beta \supseteq B_\beta$, such that $B'_\beta \subseteq U'$, $|B'_\beta| = |B_\beta|$ and B'_β is directed by inclusion. Let $x_\beta \in A \setminus \cup B'_\beta$ and $T_0 \in U'$, $x_\beta \in T_0$. Let us construct by induction the sets $T_i, T_i \subseteq T_{i+1}$, $T_i \in U'$, $i \geq 0$, $T_i \cap (\cup B'_\beta) = \{y_{im}\}_{m=1}^{\infty}$ and $S_i, S_i \subseteq S_{i+1}$, $S_i \in B_\beta$, $i \geq 1$, $i \in N$ and so that $T_{i+1} \supseteq S_i$, $S_{i+1} \supseteq \{y_{km} : k, m \leq i\}$. Then both $T = \bigcup_{i=0}^{\infty} T_i$ and $S = \bigcup_{i=1}^{\infty} S_i$ belong to U' , since U' is closed, $T \cap (\cup B'_\beta) = S$, $x_\beta \in T \setminus S \neq \phi$, $T \supseteq S$. In addition $(T \setminus S) \cap B_\alpha = \phi$ for every $\alpha < \beta$. So a pair $B_\beta = T$, $B'_\beta = S$ is found. The VI) is proved.

We note, that $f_{B_0} | P \times \{0\} : P \times \{0\} \rightarrow f_{B_0}(P \times \{0\})$ is Baire isomorphism and $f_{B_0} | P \times \{0\} \in \mathcal{F}$. Let $f_{B_0} | P \times \{0\} = \varphi_0$. We shall show that $\varphi_0 \in \mathcal{F}(\gamma)$.

The following diagram is commutative

$$\begin{array}{ccc}
 P(B_\alpha) & \xrightarrow{f_{B_\alpha}} & Y_{B_\alpha} \\
 \mu_{B'_\alpha}^{B_\alpha} \downarrow & & \downarrow \pi_{B'_\alpha}^{B_\alpha} \\
 P(B'_\alpha) & \xrightarrow{f_{B'_\alpha}} & Y_{B'_\alpha} \\
 \mu_{B_0}^{B'_\alpha} \downarrow & & \downarrow \pi_{B_0}^{B'_\alpha} \\
 P(B_0) & \xrightarrow{f_{B_0}} & Y_{B_0}
 \end{array}$$

Since $B_\alpha \supseteq B'_\alpha \supseteq B_0$ and diagram is commutative, then $f_{B_\alpha}(\mathcal{Z}(\mu_{B'_\alpha}^{B_\alpha})) = \mathcal{Z}(\pi_{B'_\alpha}^{B_\alpha})$ and $f_{B_0}(\mu_{B_0}^{B'_\alpha}(\mathcal{Z}(\mu_{B'_\alpha}^{B_\alpha}))) = \pi_{B_0}^{B'_\alpha}(\mathcal{Z}(\pi_{B'_\alpha}^{B_\alpha}))$. According to 1) $\pi_{B_0}^{B'_\alpha}(\mathcal{Z}(\pi_{B'_\alpha}^{B_\alpha}))$ is σ -compact and $\mu_{B_0}^{B'_\alpha}(\mathcal{Z}(\mu_{B'_\alpha}^{B_\alpha})) = \cup\{F_\alpha \times 0 : \alpha \in B_\alpha \setminus B'_\alpha\}$. Hence $\varphi_0(\cup\{F_\alpha \times 0 : \alpha \in B_\alpha \setminus B'_\alpha\})$ is σ -compact for every $1 \leq \alpha < c$. Since $|B_\alpha| \leq \aleph_0$ and the family $\{B_\alpha \setminus B'_\alpha\}_{1 \leq \alpha < c}$ is pair-wise disjoint, we have $\varphi_0 \in \mathcal{F}(\gamma)$. Since $\varphi_0 \in \mathcal{F}(\gamma)$, then $|\mathcal{A}(\varphi_0) = \{\alpha : \varphi_0(F_\alpha) \text{ is not } \sigma\text{-compact and } \varphi_0(F_\alpha) \cap \varphi_0(F_\beta) = \emptyset \text{ for every } \alpha \neq \beta, F_\beta \in \gamma\}| = c$. Let us choose $\alpha_0 \in \mathcal{A}(\varphi_0) \setminus B_0$. According to cofinality of \mathcal{U}' there is $\tilde{B} \in \mathcal{U}'$, such that $\tilde{B} \supseteq B_0 \cup \{\alpha_0\}$. Then $\varphi_0(\cup\{F_\alpha : \alpha \in \tilde{B} \setminus B_0\})$ is σ -compact. But since $\varphi_0(F_{\alpha_0}) \cap \varphi_0(\cup\{F_\alpha : \alpha \in \tilde{B} \setminus B_0\}) = \varphi_0(F_{\alpha_0})$, then $\varphi_0(F_{\alpha_0})$ is closed in σ -compact set $\varphi_0(\cup\{F_\alpha : \alpha \in \tilde{B} \setminus B_0\})$. Thus $\varphi_0(F_{\alpha_0})$ is σ -compact. But this contradicts to $\alpha_0 \in \mathcal{A}(\varphi_0)$ and so we complete the proof. ■

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