
Salvador Romaguera; Sergio Salbany

Quasi-metrization and hereditary normality of compact bitopological spaces

Commentationes Mathematicae Universitatis Carolinae, Vol. 31 (1990), No. 1, 113--122

Persistent URL: `http://dml.cz/dmlcz/106826`

Quasi–metrization and hereditary normality of compact bitopological spaces

SALVADOR ROMAGUERA AND SERGIO SALBANY

Dedicated to the memory of Zdeněk Frolík

Abstract. We characterize two classes of bitopological spaces which admit a compatible quasi–pseudometric in terms of a G_δ condition and hereditary pairwise normality. In particular, well–known theorems of V. E. Šneĭder and M. Katětov are extended. As applications we give a characterization of regular compact pseudometrizable spaces (not necessarily Hausdorff) and a metrization theorem for compact ordered spaces.

Keywords: pairwise compact space, 2compact space, quasi–metric, quasi–pseudometric, hereditarily pairwise normal spaces, compact ordered space, G_δ-diagonal

Classification: 54E55, 54E35, 54F05, 54D30, 54E45

1. Introduction. In [14] V. E. Šneĭder showed that every Hausdorff compact space with a G_δ-diagonal is necessarily metrizable. This G_δ-diagonal type of theorem has been extended to other classes of spaces and a unification of these results is achieved through the notion of an M-space due to K. Morita: A Hausdorff space is metrizable if and only if it is an M-space with a G_δ-diagonal (see [**5**, corollary 3.8]).

Using Šneĭder's theorem, M. Katětov [**6**] proved that a Hausdorff compact space (X,T) is metrizable if and only if the space $(X \times X \times X, T \times T \times T)$ is hereditarily normal.

Our interest is the study of spaces which admit a compatible distance function which is not necessarily symmetric. In this paper we prove generalizations of the theorems of Šneĭder and Katětov and characterize two classes of bitopological spaces which admit a compatible quasi–pseudometric in terms of a G_δ condition and hereditary pairwise normality. As applications we give a characterization of regular compact pseudometrizable spaces (not necessarily Hausdorff) and also of compact metrizable ordered spaces in terms of the hereditary normality of $X \times X \times X$.

2. Preliminaries. In the following the letter $\mathbb{N}$ will denote the set of all positive integers. If P and Q are topologies for a set X, and if $A \subset X$, we write $P \operatorname{cl} A$ for the closure of A taken in P and $(P \times Q) \operatorname{cl} A$ for the closure of A taken in $P \times Q$ when $A \subset X \times X$.

A quasi–pseudometric on a set X is a non–negative real–valued function d on $X \times X$ such that, for all x, y, $z \in X$

(i) $d(x,x) = 0$;
(ii) $d(x,y) \leq d(x,z) + d(z,y)$.

If d separates points in the sense that $d(x,y) > 0$ whenever $x \neq y$, then d is called a quasi–metric on X.

A quasi–pseudometric d on X generates a topology $T(d)$ on X, where a basic $T(d)$-open neighbourhood of x is the d-ball of radius $r > 0$, $B_d(x,r) = \{y \in X : d(x,y) < r\}$. Since the conjugate of d, d^{-1}, given by $d^{-1}(x,y) = d(y,x)$, is also quasi–pseudometric, there is another topology $T(d^{-1})$ associated with d. A quasi–pseudometric d is called separating if $d(x,y) + d^{-1}(x,y) > 0$ whenever $x \neq y$.

We shall consider the bitopological space $(X, T(d), T(d^{-1}))$, naturally associated with a quasi–pseudometric d. Observe that d is a quasi–metric if and only if $T(d)$ is a T_1 topology.

A bitopological space (X, P, Q) is said to be quasi–(pseudo)metrizable if there is a quasi–(pseudo)metric d on X such that $T(d) = P$ and $T(d^{-1}) = Q$.

The following concepts from the theory of bitopological spaces will be recalled, specially because there are different notions of bitopological compactness and bitopological T_2 separation (see e.g. [2] and [**13**]).

A bitopological space (X, P, Q) is called:

(i) pairwise regular [7] if, for all $x \in X$, the Q-closed P-neighbourhoods of x form a base for the P-neighbourhoods of x and P-closed Q-neighbourhoods of x form a base for the Q-neighbourhoods of x.

(ii) pairwise normal [7] if for every pair of disjoint sets A and B, where A is P-closed and B is Q-closed, there is a P-open set U and a disjoint Q-open set V such that $B \subset U$ and $A \subset V$.

(iii) pairwise (countably) compact ([**13**]) [3] if every proper P-closed set is Q-(countably) compact and every proper Q-closed set is P-(countably) compact.

(iv) 2compact [**11**] if $P \vee Q$ is compact.

(v) pairwise Hausdorff [7] if, for $x \neq y$, there is a P-neighbourhood of x and disjoint Q-neighbourhood of y.

(vi) 2separated [**11**] if, for $x \neq y$, there is a P-open set V and a disjoint Q-open set W such that $x \in V$, $y \in W$ or $y \in V$, $x \in W$; equivalently, if $P \vee Q$ is Hausdorff.

Remarks.

(i) It is clear that every 2compact space is pairwise compact.

(ii) From [**3**, theorem 10] it follows that if (X, P, Q) is a pairwise Hausdorff 2compact space, then $P = Q$.

(iii) When (X, P, Q) is pairwise regular, then it is 2compact if and only if every P-closed set is Q-compact and every Q-closed set is P-compact.

The following examples illustrate fundamental differences between pairwise compact and 2compact spaces; pairwise Hausdorff and 2separated spaces.

Example 1. Let $X = \{1/n : n \in \mathbb{N}\}$ and let $d(1/n, 1/m) = 1/m$ if $n \neq m$ and $d(1/n, 1/n) = 0$ for all $n \in \mathbb{N}$. Then d is a quasi–metric on X with $T(d)$ the Zariski (cofinite) topology on X and $T(d^{-1})$ the discrete topology on X. The space $(X, T(d), T(d^{-1}))$ is pairwise Hausdorff and pairwise compact but, clearly, is not 2compact.

Example 2. Let $X = [0,1]$ and let $u(x,y) = (y-x) \vee 0$. Then, u is a quasi-pseudometric on X and basic $T(u)$-open sets are of the form $[0,a)$, $0 < a < 1$; basic $T(u^{-1})$-open sets are of the form $(a,1]$, $0 < a < 1$. The space $(X, T(u), T(u^{-1}))$ is 2compact (with $T(u \vee u^{-1})$ being the usual topology on X) and 2separated, but not pairwise Hausdorff.

3. The specialization order. With the bitopological space $(X, T(u), T(u^{-1}))$ in mind, where u is the quasi–pseudometric of example 2, the following partial order relations are natural ones:

(i) $x \le y \iff y \in T(u)\,\mathrm{cl}\{x\}$;
(ii) $x \le y \iff x \in T(u^{-1})\,\mathrm{cl}\{y\}$ and
(iii) $x \le y \iff y \in T(u)\,\mathrm{cl}\{x\}$ and $x \in T(u^{-1})\,\mathrm{cl}\{y\}$.

In our discussion, we shall be interested in pairwise regular spaces, in which case all three partial orders defined above are the same.

We shall denote the graph of the partial order $\le$ by $[\le]$, so that, $[\le] = \{(x,y) : x \le y\}$. For the discrete order ($x \le y \iff x = y$) the graph $[\le]$ is the diagonal $\Delta = \{(x,x) : x \in X\}$.

The following proposition is easily verified and we omit the proof.

Proposition 1. *Suppose (X,P,Q) be pairwise regular and let $x \le y$ be given by $y \in P\,\mathrm{cl}\{x\}$. Then $[\le] = (Q \times P)\,\mathrm{cl}\,\Delta$.*

4. Hereditary pairwise normality. A bitopological space(X, P, Q) will be called hereditarily pairwise normal if for every subset A, the bitopological space $(A, P \mid A, Q \mid A)$ is pairwise normal, where $T \mid A$ denotes the relative topology on A induced by a topology T on X.

Proposition 2. *A bitopological space (X,P,Q) is hereditarily pairwise normal if and only if whenever A and B are subsets of X such that $A \cap Q\,\mathrm{cl}\,B$ and $B \cap P\,\mathrm{cl}\,A$ are disjoint, then there is a P-open set U and a disjoint Q-open set V such that $B \subset U$ and $A \subset V$.*

PROOF : Let (X,P,Q) be hereditarily pairwise normal. Suppose $A \cap Q\,\mathrm{cl}\,B = B \cap P\,\mathrm{cl}\,A = \emptyset$. Put $M_1 = X - P\,\mathrm{cl}\,A$, $M_2 = X - Q\,\mathrm{cl}\,B$ and $M = M_1 \cup M_2$. Clearly, $M \cap P\,\mathrm{cl}\,A \cap Q\,\mathrm{cl}\,B = \emptyset$. By assumption, there is a P-open set U_1 and a Q-open set V_1 such that $M \cap U_1 \cap V_1 = \emptyset$, $M \cap P\,\mathrm{cl}\,A \subset M \cap V_1$ and $M \cap Q\,\mathrm{cl}\,B \subset M \cap U_1$. Putting $U = U_1 \cap M_1$ and $V = V_1 \cap M_2$ we obtain $B \subset U$, $A \subset V$ and $U \cap V = \emptyset$, as required. Conversely, let $Y \subset X$ and suppose that A is a $P \mid Y$ closed subset of Y and B is a disjoint $Q \mid Y$ closed subset of Y. Then, $A \cap Q\,\mathrm{cl}\,B = B \cap P\,\mathrm{cl}\,A = \emptyset$. By assumption, there is a P-open set U and a disjoint Q-open set V such that $A \subset V \cap Y$ and $B \subset U \cap Y$. Hence, $(Y, P \mid Y, Q \mid Y)$ is pairwise normal. ∎

The following proposition is an analogue of M. Katětov's theorem on the special properties of spaces with a hereditarily normal product. Because of the two topologies that are involved and the weak separation of the spaces we want to consider, we require the following concepts:

Definition. Let (X, P, Q) be a bitopological space and $A \subset X$. We say that $x \in \overline{\lim}A$ if $A \cap P\,\mathrm{cl}\{x\} = \emptyset$ and $x \in Q\,\mathrm{cl}\,A$. Similarly, $x \in \underline{\lim}A$ if $A \cap Q\,\mathrm{cl}\{x\} = \emptyset$ and $x \in P\,\mathrm{cl}\,A$.

Observe that if (X, P, Q) is pairwise Hausdorff, the $\overline{\lim}A$ consists of all Q-limit points of A and $\underline{\lim}A$ consists of all P-limit points of A.

Proposition 3. *Let (X, P, Q) and (Y, S, T) be two bitopological spaces such that $(X \times Y, P \times S, Q \times T)$ is hereditarily pairwise normal. Then*

(i) *If there is a countable subset $A \subset X$ with non-empty $\overline{\lim}A$, then every T-closed subset of Y is a countable intersection of S-open sets and*

(ii) *If there is a countable subset $B \subset X$ with non-empty $\underline{\lim}B$ then every S-closed subset of Y is a countable intersection of T-open sets.*

PROOF : We shall only prove (i). Let A be a countable set $\{x_n : n \in \mathbb{N}\}$ with $x \in \overline{\lim}A$. If A is a finite set with distinct points $a_1, \ldots, a_k$, then $x \in Q\,\mathrm{cl}\{a_j\}$, for some j, $1 \le j \le k$, so that $a_j \in P\,\mathrm{cl}\{x\}$, contradicting our assumption on x. Thus A is infinite and we shall assume $x_n \neq x_m$ whenever $n \neq m$. Let F be a T-closed subset of Y. Observe that

$$(Q \times T)\,\mathrm{cl}(A \times F) \cap \{x\} \times (X - F) = (Q\,\mathrm{cl}\,A \times T\,\mathrm{cl}\,F) \cap \{x\} \times (X - F) = \emptyset$$

since F is T-closed and

$$(P \times S)\,\mathrm{cl}(\{x\} \times (X - F)) \cap A \times F = (P\,\mathrm{cl}\{x\} \times S\,\mathrm{cl}(X - F)) \cap A \times F = \emptyset$$

since $A \cap P\,\mathrm{cl}\{x\} = \emptyset$. It follows that there is a $P \times S$-open set U and a disjoint $Q \times T$-open set V such that $A \times F \subset U$ and $\{x\} \times (X - F) \subset V$. For each $n \in \mathbb{N}$ and each $y \in F$, there is a P-open neighbourhood of x_n, $V(x_n, y)$ and an S-open neighbourhood of y, $W_n(y)$, such that $V(x_n, y) \times W_n(y) \subset U$. Let $W_n = \cup\{W_n(y) : y \in Y\}$. Clearly, $F \subset \bigcap_{n=1}^{\infty} W_n$. Suppose there is z in $\bigcap_{n=1}^{\infty} W_n - F$. Since $(x, z) \in \{x\} \times (X - F) \subset V$ and V is $Q \times T$-open, there is a Q-open set U_1 and T-open set V_1 such that $(x, z) \in U_1 \times V_1 \subset V$. By assumption on x, there is $j \in \mathbb{N}$ such that $x_j \in U_1 \cap A$. Since $z \in \bigcap_{n=1}^{\infty} W_n$, we have $z \in W_j$, so there is $y \in F$ such that $z \in W_j(y)$; but then, $(x_j, z) \in V(x_j, y) \times W_j(y) \subset U$ and $(x_j, z) \in U_1 \times V_1 \subset V$, this impossible. Thus $F = \bigcap_{n=1}^{\infty} W_n$, as required. ■

In the presence of a form of compactness and some separation, spaces for which every sequence has an empty $\overline{\lim}$ and $\underline{\lim}$ are necessarily finite.

Proposition 4. *Suppose every countable set A in (X, P, Q) has $\overline{\lim}A = \underline{\lim}A = \emptyset$. If*

(i) *(X, P, Q) is pairwise Hausdorff and pairwise compact or*

(ii) *(X, P, Q) is 2separated and 2compact* *then X is a finite set.*

PROOF : Suppose that (X, P, Q) is a pairwise Hausdorff pairwise compact space. Then both P and Q are T_1 topologies and the assumption is equivalent to the requirement that every sequence is both P-closed and Q-closed. If X were infinite and T denotes the cocountable topology on X, then $T \subset P$ and $T \subset Q$, hence T is compact which is impossible.

We now assume that $P \vee Q$ is a compact topology such that (X, P, Q) is 2separated. If X were infinite, then there is a countable set $A = \{x_n : n \in \mathbb{N}\}$ where $x_n \neq x_m$ if $n \neq m$. By $P \vee Q$-compactness, there is x which is a $P \vee Q$-cluster point of A. By assumption, $x \notin \overline{\lim} A$ and $x \notin \underline{\lim} A$. Now $x \notin \overline{\lim} A$ is equivalent to $A \cap P\,\mathrm{cl}\{x\} \neq \emptyset$ or $x \notin Q\,\mathrm{cl}\,A$, hence $x \notin \overline{\lim} A$. if and only if $A \cap P\,\mathrm{cl}\{x\} \neq \emptyset$, since x is $P \vee Q$-cluster point of A. We show that the sequence $\{x_n : n \in \mathbb{N}\}$ is eventually in $P\,\mathrm{cl}\{x\}$. If not, then there are only $x_{n(1)}, \ldots, x_{n(k)}$ in $A \cap P\,\mathrm{cl}\{x\}$ (with $n(j) < n(j+1)$, $1 \leq j \leq k-1$). By the 2separation property, since every P-neighbourhood of $x_{n(j)}$ contains x, there is a P-neighbourhood V_j of x which does not contain $x_{n(j)}$. Hence $V = \bigcap_{j=1}^{k} V_j$ is a P-neighbourhood of x which does not contain $x_{n(j)}$, $1 \leq j \leq k$. But then $x \in \overline{\lim} B$, where $B = \{x_m : m > n(k)\}$, contradicting our assumption. Thus, the sequence $\{x_n : n \in \mathbb{N}\}$ is eventually in $P\,\mathrm{cl}\{x\}$. Similarly, it is eventually in $Q\,\mathrm{cl}\{x\}$, hence eventually in $P\,\mathrm{cl}\{x\} \cap Q\,\mathrm{cl}\{x\} = (P \vee Q)\,\mathrm{cl}\{x\} = \{x\}$, which is impossible. ∎

5. ***G_δ*-diagonal theorems.** The following theorems are analogues of Šneĭder's theorem that a Hausdorff compact space is metrizable if and only if it has a G_δ-diagonal. In [9] it is proved that every pairwise Hausdorff pairwise (countably) compact space (X, P, Q) such that (X, P) and (X, Q) both have G_δ-diagonals is quasi–metrizable. Theorem 1 that follows provides another generalization of Šneĭder's theorem for pairwise Hausdorff pairwise compact spaces. It is interesting that by relaxing the pairwise Hausdorff condition and strengthening pairwise compactness another generalization of Šneĭder's theorem is obtained for pairwise regular 2separated 2compact spaces. As observed earlier the two classes spaces are essentially different.

Let (X, P, Q) be a bitopological space and let $A \subset X \times X$. We say that A is a $P \times Q$-G_δ if $A = \bigcap_{n=1}^{\infty} G_n$ where each G_n is a $P \times Q$-open set. Similarly we define the notion of a $Q \times P$-G_δ.

Proposition 5. *Let (X, P, Q) be pairwise Hausdorff space such that Δ is a $P \times Q$-G_δ. If (X, P) is a Hausdorff space and every Q-closed proper subset of X is P-compact, then P is second countable.*

PROOF : Suppose $\Delta = \bigcap_{n=1}^{\infty} G_n$ where each G_n is $P \times Q$-open. For each $x \in X$ there exists a sequence $\{U_n(x) : n \in \mathbb{N}\}$ of P-open neighbourhoods of x and a sequence $\{V_n(x) : n \in \mathbb{N}\}$ of Q-open neighbourhoods of x such that $U_n(x) \times V_n(x) \subset G_n$ for all $n \in \mathbb{N}$. It follows from the proof of [3, theorem 12] that there also exists, for each $x \in X$, a sequence $\{W_n(x) : n \in \mathbb{N}\}$ of Q-open neighbourhoods of x such that $P\,\mathrm{cl}\,W_n \subset V_n$ for all $n \in \mathbb{N}$. Note that, for each $x \in X$ and $n \in \mathbb{N}$, the subset $X - W_n(x)$ is P-compact because it is a Q-closed proper subset. Fix x. Since (X, P) is a Hausdorff space we deduce that $Q \subset P$ [3, theorem 11] and, hence,the subspace $(A_n, P \mid A_n)$ is a Hausdorff compact space that has a G_δ-diagonal where $A_n = X - W_n(x)$. By Šneĭder's theorem $P \mid A_n$ is second countable for all $n \in \mathbb{N}$. Therefore $P \mid (X - P\,\mathrm{cl}\,W_n(x))$ is also second countable for all $n \in \mathbb{N}$. Take a point in X, $y \neq x$. Then there is a $k \in \mathbb{N}$ such that $x \in X - P\,\mathrm{cl}\,W_k(y)$. Since $P \mid (X - P\,\mathrm{cl}\,W_k(y))$ is second countable we deduce that P is second countable. ■

Theorem 1. *A pairwise Hausdorff pairwise compact space (X, P, Q) is quasi-metrizable if and only if Δ is a $P \times Q$-G_δ.*

PROOF : Let (X, P, Q) be a pairwise Hausdorff pairwise compact space such that $\Delta = \bigcap_{n=1}^{\infty} G_n$ where each G_n is $P \times Q$-open. For each $x \in X$ there exist a sequence $\{U_n(x) : n \in \mathbb{N}\}$ of P-open neighbourhoods of x and a sequence $\{W_n(x) : n \in \mathbb{N}\}$ of Q-open neighbourhoods of x satisfying $U_n(x) \times P\,\mathrm{cl}\,W_n(x) \subset G_n$ for all $n \in \mathbb{N}$. Fix x. Put $A_n = X - W_n(x)$, then $(A_n, P \mid A_n, Q \mid A_n)$ is a pairwise Hausdorff pairwise compact space that A_n is P-compact. Hence, $P \mid A_n \subset Q \mid A_n$ [3, theorem 10]. Consequently, $(A_n, Q \mid A_n)$ is a Hausdorff space and, by proposition 5, $Q \mid A_n$ is second countable. In a similar way to proposition 5 we deduce that Q is second countable. Since every pairwise Hausdorff pairwise compact space is pairwise regular [3, theorem 12], the above argument also shows, interchanging the roles of P and Q, that P is second countable. Therefore the quasi-metrizalibity of (X, P, Q) follows from a well known theorem of Kelly [7, theorem 2.8]. We omit the easy proof of the converse. ■

Remark. In [1] J. Chaber proved that every countably compact space with a G_δ-diagonal is compact. Chaber's theorem suggests the following open question: Is each pairwise Hausdorff pairwise countably compact space (X, P, Q) with Δ a $P \times Q$-G_δ quasi-metrizable?. By using [9, lemma 4 b)] and a slight modification of its proof one can see that a pairwise Hausdorff pairwise countably compact space (X, P, Q), such that Δ is a $P \times Q$-G_δ, is pairwise compact if P and Q are first countable. Therefore, the above question has an affirmative answer whenever P and Q are first countable. The authors are grateful to H. P. Künzi for comments and observation related to this question.

Since every pairwise regular pairwise compact space is pairwise normal [3, theorem 13], it follows that every pairwise regular 2compact space (X, P, Q) is pairwise normal. Modifying a construction of P. Samuel [12] (see also [4] and [10] for the quasi-proximity case) it can be shown that (X, P, Q) is quasi-uniformizable, in the sense that there is a quasi-uniformity $\mathcal{U}$ on X such that $P = T(\mathcal{U})$ and $Q = T(\mathcal{U}^{-1})$, explicitly, a subbasis for $\mathcal{U}$ consists of all sets $[F, H] = X \times X - F \times H$, where F is P-closed and H is a disjoint Q-closed set. Because $P = T(\mathcal{U})$ and $Q = T(\mathcal{U}^{-1})$, it follows that every U in $\mathcal{U}$ is $Q \times P$-neighbourhood of Δ and pairwise regularity implies that every such U is a $Q \times P$-neighbourhood of $(P \times Q)\,\mathrm{cl}\,\Delta$. In fact, every pairwise regular 2compact space admits a unique compatible quasi-uniformity [10], its entourages are precisely all $Q \times P$-neighbourhoods of Δ.

Theorem 2. *A pairwise regular 2compact space (X, P, Q) is quasi–pseudometrizable if and only if $(P \times Q)\,\mathrm{cl}\,\Delta$ is a $Q \times P$-G_δ.*

PROOF : If (X, P, Q) is any bitopological space that admits a compatible quasi–pseudometric d, then $(P \times Q)\,\mathrm{cl}\,\Delta = \bigcap_{n=1}^{\infty} G_n$, where $G_n = \{(x, y) : d(x, y) < 1/n\}$ is $Q \times P$-open for all $n \in \mathbb{N}$. Conversely, assume there is a countable family $\{G_n : n \in \mathbb{N}\}$ of $Q \times P$-open sets such that $(P \times Q)\,\mathrm{cl}\,\Delta = \bigcap_{n=1}^{\infty} G_n$. We now show that the unique compatible quasi–uniformity $\mathcal{U}$ has a countable base, so that there is a quasi–pseudometric d compatible with $\mathcal{U}$, hence $P = T(d)$ and $Q = T(d^{-1})$. Fix n. For each pair $x, y \in X$ with $y \leq x$ (hence $(x, y) \in (P \times Q)\,\mathrm{cl}\,\Delta$), there is a Q-open set $W(x)$ and a P-open set $V(y)$ such that $W(x) \times V(y) \subset P\,\mathrm{cl}\,W(x) \times Q\,\mathrm{cl}\,V(y) = (P \times Q)\,\mathrm{cl}(W(x) \times V(y)) \subset G_n$, by pairwise regularity of (X, P, Q). Now $(X \times X, (P \vee Q) \times (P \vee Q))$ is compact, so there are finitely many pairs (x_j, y_j), $1 \leq j \leq m_n$, such that $(P \times Q)\,\mathrm{cl}\,\Delta \subset \cup\{W(x_j) \times V(y_j) : j = 1, \ldots, m_n\} = H'_n$. Define $H_n = \cap\{H'_k : k = 1, \ldots, n\}$ and observe that each H_n is $P \times Q$-open and $(P \times Q)\,\mathrm{cl}\,H_n \subset G_n$. Moreover, $(P \times Q)\,\mathrm{cl}\,\Delta = \bigcap_{n=1}^{\infty} (P \times Q)\,\mathrm{cl}\,H_n$. Let $\mathcal{U}$ be the (unique) compatible quasi–uniformity on (X, P, Q) and let $U \in \mathcal{U}$. We show that there is n such that $H_n \subset U$. We may assume that U is a $Q \times P$-open set; observe that U will necessarily contain $(P \times Q)\,\mathrm{cl}\,\Delta$. If $H_n \not\subset U$, then $(P \times Q)\,\mathrm{cl}\,H_n \cap X \times X - U \neq \emptyset$ for all $n \in \mathbb{N}$, hence, $\bigcap_{n=1}^{\infty} ((P \times Q)\,\mathrm{cl}\,H_n \cap X \times X - U) \neq \emptyset$ and therefore $(\bigcap_{n=1}^{\infty} (P \times Q)\,\mathrm{cl}\,H_n) \cap X \times X - U \neq \emptyset$, which is impossible since $\bigcap_{n=1}^{\infty} (P \times Q)\,\mathrm{cl}\,H_n = (P \times Q)\,\mathrm{cl}\,\Delta \subset U$. Thus, $\mathcal{U}$ has a countable base as required. ∎

Corollary. *A regular compact space (X, T) is pseudometrizable if and only if $(T \times T)\,\mathrm{cl}\,\Delta$ is a G_δ.*

If the space (X, P, Q) is 2separated we have the following "diagonal theorem" whose proof we shall omit.

Theorem 3. *A pairwise regular 2separated 2compact space (X, P, Q) is quasi–pseudometrizable if and only if $\Delta = (P \times Q)\,\mathrm{cl}\,\Delta \cap (Q \times P)\,\mathrm{cl}\,\Delta$ and $(P \times Q)\,\mathrm{cl}\,\Delta$ is a $Q \times P$-G_δ and $(Q \times P)\,\mathrm{cl}\,\Delta$ is a $P \times Q$-G_δ.*

6. Hereditary pairwise normality and quasi–(pseudo)metrization.

Theorem 4. *A pairwise Hausdorff pairwise compact space (X, P, Q) is quasi–metrizable if and only if the space $(X \times X \times X, P \times Q \times P, Q \times P \times Q)$ is hereditarily pairwise normal.*

PROOF : Assume $(X \times X \times X, P \times Q \times P, Q \times P \times Q)$ is hereditarily pairwise normal. If X is a finite set, then P and Q are the discrete topology on X, hence X is (quasi–)metrizable. We assume X is an infinite set. Then, from propositions

3 and 4, it follows that every $Q \times P$-closed subset of $X \times X$ is a $P \times Q$-G_δ or every $P \times Q$-closed subset of $X \times X$ is a $Q \times P$-G_δ. Since (X, P, Q) is pairwise Hausdorff we deduce that Δ is $P \times Q$-closed and $Q \times P$-closed. Therefore, (X, P, Q) is quasi–metrizable by theorem 1. The converse follows from the well known fact that the space $(X \times X \times X, P \times Q \times P, Q \times P \times Q)$ is quasi–metrizable whenever (X, P, Q) is quasi–metrizable. ■

Theorem 5. *A pairwise regular 2separated 2compact space (X, P, Q) admits a separating quasi–pseudometric if and only if the space $(X \times X \times X, P \times Q \times P, Q \times P \times Q)$ is hereditarily pairwise normal.*

PROOF : If X is finite, then (X, P, Q) has a compatible quasi–uniformity (the unique one) with a countable base and the conclusion follows. If X is an infinite set, then propositions 3 and 4 show that $(X \times X, Q \times P, P \times Q)$ satisfies that $(Q \times P)\,\mathrm{cl}\,\Delta$ is a $P \times Q$-G_δ or $(P \times Q)\,\mathrm{cl}\,\Delta$ is a $Q \times P$-G_δ. In either case (X, P, Q) is quasi–pseudometrizable. Moreover, if $x \neq y$, then there is V a P-open set and W a disjoint Q-open set such that $x \in V$ and $y \in W$ or $x \in W$ and $y \in V$. We then have $d(x, y) > 0$ or $d(y, x) > 0$, as required. The converse follows similarly to theorem 4. ■

The following examples show that it is essential to consider hereditary pairwise normality with respect to a "mixed product" bitopology in the preceding theorems.

Example 3. Consider the bitopological space $(\mathbf{R}, P, Q)$, where $\mathbf{R}$ denotes the set of all real numbers, P is the cofinite topology on $\mathbf{R}$ and Q is the discrete topology on $\mathbf{R}$. Then, $(\mathbf{R}, P, Q)$ is not quasi–metrizable although it is pairwise Hausdorff and pairwise compact. Thus, $(\mathbf{R} \times \mathbf{R} \times \mathbf{R}, P \times Q \times P, Q \times P \times Q)$ is not hereditarily pairwise normal. However, $(\mathbf{R} \times \mathbf{R} \times \mathbf{R}, P \times P \times P, Q \times Q \times Q)$ is hereditarily pairwise normal since $Q \times Q \times Q$ is the discrete topology.

Example 4. Let the bitopological space (Ω', P, Q), where Ω' is the space of all ordinals smaller or equal to the first uncountable ordinal Ω, P has a base of P-open sets of the form $[0, a)$, where $a \in \Omega$, together with Ω', and Q has a base of Q-open sets of the form $(a, \Omega]$, where $a \in \Omega$, together with Ω'. It is not difficult to verify that $(\Omega' \times \Omega' \times \Omega', P \times P \times P, Q \times Q \times Q)$ is hereditarily pairwise normal, however the pairwise regular 2separated 2compact space (Ω', P, Q) is not quasi–pseudometrizable.

7. The metrizability of compact ordered spaces. In [8] L. Nachbin considered ordered topological spaces. The class of compact ordered spaces $(X, T, \leq)$ is specially important for many applications. In view of the fact that the categories of compact ordered spaces and pairwise regular 2separated 2compact spaces are isomorphic (see e.g. [11]) it is possible to obtain metrization theorems for compact ordered topological spaces analogues to the G_δ and hereditary normality theorems above. We describe the correspondence that provides the equivalence of the categories and state the relevant results.

Given (X, P, Q), a pairwise regular 2separated 2compact space, let $T = P \vee Q$ and let $\leq$ be the partial order given by $x \leq y \iff y \in P\,\mathrm{cl}\{x\}$. Then

(i) $x \leq x$,

(ii) $x \leq y$ and $y \leq x \implies x = y$,
(iii) $x \leq y$ and $y \leq z \implies x \leq z$;

moreover, $[\leq]$ is $Q \times P$-closed, hence $T \times T$-closed and T is a Hausdorff compact topology.

Given $(X, T, \leq)$, T a Hausdorff compact topology and $\leq$ a partial order (satisfying (i), (ii), (iii) above) with a closed graph, let P consist of all T-open and $\leq$-decreasing sets and Q consist of all T-open and $\leq$-increasing sets. Then, (X, P, Q) is a pairwise regular 2separated 2compact space.

Finally, starting with (X, P, Q), determining $(X, T, \leq)$ and the associated bitopological space, one obtains (X, P, Q); similarly, starting with $(X, T, \leq)$, determining (X, P, Q) and the associated ordered topological space one obtains $(X, T, \leq)$.

Theorem 6. *Let $(X, T, \leq)$ be a compact ordered space. Then (X, T) is metrizable if and only if the graph of $\leq$ is a $P \times Q$-G_δ.*

Theorem 7. *Let $(X, T, \leq)$ be a compact ordered space. Then (X, T) is metrizable if and only if $(X \times X \times X, T \times T \times T, \leq \times \leq^{-1} \times \leq)$ is hereditarily normally ordered.*

Remark. The example of $(\Omega', T, \leq)$, where T is the usual topology on Ω' and $\leq$ is the usual order, shows that $\leq \times \leq^{-1} \times \leq$ cannot be replaced by $\leq \times \leq \times \leq$, in the preceding theorem.

References

[1] J. Chaber, *Condition which imply compactness in countably compact spaces*, Bull. Acad. Pol. Sci. Sér. Math. **24** (1976), 993–998.

[2] I. E. Cooke and I. L. Reilly, *On bitopological compactness*, J. London Math. Soc. **9** (1975), 518–522.

[3] P. Fletcher, H. B. Hoyle III and C. W. Patty, *The comparison of topologies*, Duke Math. J. **36** (1969), 325–331.

[4] P. Fletcher and W. F. Lindgren, *Quasi–Uniform Spaces*, Marcel Dekker, New York, 1982.

[5] G. Gruenhage, *Generalized Metric Spaces*, Handbook of Set–theoretic Topology, North–Holland, Amsterdam, (1984), 423–501.

[6] M. Katětov, *Complete normality of Cartesian products*, Fund. Math. **36** (1948), 271–274.

[7] J. C. Kelly, *Bitopological spaces*, Proc. London Math. Soc. **13** (1963), 71–89.

[8] L. Nachbin, *Topology and Order*, Van Nostrand Math. Studies, Princeton, New Jersey, 1965.

[9] S. Romaguera, A. Gutiérrez and H. P. Künzi, *Quasi–metrization of pairwise countably compact spaces*, Glasnik Mat. **23(43)** (1988), 159–167.

[10] S. Salbany, *Bitopological Spaces, Compactifications and Completions*, Mathematical Monographs of the Univ. Cape Town. No. 1. 1974.

[11] S. Salbany, *A bitopological view of topology and order*, Categorical Topology (Proc. Toledo, Ohio 1983), Heldermann, Berlin, (1984), 481–504.

[12] P. Samuel, *Ultrafilters and compactifications of uniform spaces*, Trans. Amer. Math. Soc. **64** (1948), 100–132.

[13] M. K. Singal and R. Singal, *Some more separation axioms in bitopological spaces*, Ann. Soc. Sci. Bruxelles **84** (1970), 207–230.

[14] V. Šneider, *Continuous images of Souslin and Borel sets; metrization theorems*, Dokl. Acad. Nauk. USSR 50 (1945), 77–79.

Departamento de Matemática Aplicada, Escuela de Caminos,Universidad Politénica,46071 Valencia, Spain

Departament of Mathematics, University of Zimbabwe, Box MP 167, Mount Pleasant, Harare, Zimbabwe

(*Received* September 11,1989)