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ALEXANDER ŠOSTAK, D.DZHAJANBAJEV

Dedicated to the memory of Zdeněk Frolík

Abstract. The aim of this paper is to extend the spectral approach for the study of uniform properties of fuzzy sets in (Hutton's) fuzzy uniform spaces. The notions of completeness spectrum and precompactness spectrum are introduced and studied. In particular, the relations between these spectra and the compactness spectrum of a fuzzy set in a fuzzy topological space introduced earlier by the first author are discussed.

Keywords: fuzzy uniform space, completeness, precompactness, compactness spectrum

Classification: 54A40

The spectral approach developed by the first author has proved to be an effective tool for the investigation of different topological properties of fuzzy topological spaces(see [14] - [19] e.a.). The aim of this and some subsequent papers is to extend the spectral approach for the study of uniform properties of fuzzy sets in fuzzy uniform spaces. In this paper, we define the completeness spectrum and the precompactness spectrum of a fuzzy set in a fuzzy uniform space and study basic properties of such spectra. The theory of completeness and precompactness developed here is in some aspects analogous to the classical theory of completeness and precompactness in (ordinary) uniform spaces (see e.g.[2], [4]). However, as the reader will notice, there are also essential special features distinguishing this spectral theory from its classical prototype.

Terminology and notation which is standard for the Fuzzy Topology is accepted in the paper. We emphasize that the expression "a fuzzy topological space" is always used in Chang's sense [3]. If M is a fuzzy subset of a set X, i.e. $M \in I^X$ ($I := [0,1]$), then $M^c := 1 - M$ denotes its complement. A fuzzy set $M \in I^X$ is called normed, if $\sup_{x\in X} M(x) = 1$. For a family of fuzzy sets $\mathcal{U} \subset I^X$ let $\bigvee \mathcal{U} := \bigvee\{U : U \in \mathcal{U}\}$ denote its union, $\wedge \mathcal{U} := \wedge\{U : U \in \mathcal{U}\}$ denote its intersection and let $\mathcal{U}^c := \{U^c : U \in \mathcal{U}\}$. Following [12] we say that fuzzy sets M and N are quasicoincident and write MqN, if there exists a point $x \in X$ such that $M(x) + N(x) > 1$. An open fuzzy set U is called a q-neighborhood of a fuzzy point x^t, if $x^t qU$ [12]. For $M, N \in I^X$ let $M \tilde{\subset} N := \inf_x M^c(x) \vee N(x)$ denote the fuzzy inclusion of the fuzzy set M into the fuzzy set N (see e.g. [14] - [16]. The closed fuzzy unit interval [6] is denoted $\mathcal{F}(I)$.

In Section 0 we expose briefly the principial features of the theory of fuzzy uniform spaces developed by Hutton [7]. Some facts about fuzzy filters [8], [10] used in the paper are also discussed in Section 0.

0. Preliminaries.

Fuzzy uniform spaces. Fuzzy uniform spaces were first defined by Hutton [7]. For the convenience of the reader we reproduce here some definitions and facts from [7] which are essential for the subject of our paper.

Let X be a set and let $\mathcal{D}$ denote the family of all mappings $U : I^X \to I^X$ satisfying the next two conditions:

($\mathcal{D}$1) $U(M) \geqslant M$ for each $M \in I^X$

($\mathcal{D}$2) $U(\bigvee_{i\in\mathcal{J}} M_i) = \bigvee_{i\in\mathcal{J}} U(M_i)$ for every family $\{M_i : i \in \mathcal{J}\} \subset I^X$.

A *fuzzy uniformity* on a set X is a nonempty subfamily $\mathcal{U} \subset \mathcal{D}$ satisfying the next four axioms:

($FU1$) if $U \in \mathcal{U}$, $V \geqslant U$, and $V \in \mathcal{D}$, then $V \in \mathcal{U}$;

($FU2$) if $U, V \in \mathcal{U}$, then $U \wedge V \in \mathcal{U}$;

($FU3$) for each $V \in \mathcal{U}$ there exists $U \in \mathcal{U}$ s.t. $U \circ U \leqslant V$;

($FU4$) if $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$ (a mapping $U^{-1} : I^X \to I^X$ is defined by $U^{-1}(M) = \wedge\{N : U(N^c) \leqslant M^c$ for each $M \in I^X\}$).

A pair $(X,\mathcal{U})$, where X is a set and $\mathcal{U}$ is a fuzzy uniformity on it, is called a *fuzzy uniform space*.

A subfamily $\mathcal{B} \subset \mathcal{U}$ is called a *base* of the fuzzy uniformity $\mathcal{U}$, if for each $U \in \mathcal{U}$ there exists $V \in \mathcal{B}$ such that $V \leq U$; a subfamily $\mathcal{P} \subset \mathcal{U}$ is called a *subbase* of the fuzzy uniformity $\mathcal{U}$, if $\mathcal{B} := \{V_1 \wedge \cdots \wedge V_n : V_i \in \mathcal{U},\ n \in \mathbb{N}\}$ is its base.

Let $(X,\mathcal{U}_X)$ and $(Y,\mathcal{U}_Y)$ be fuzzy uniform spaces. A mapping $f : X \to Y$ is called uniformly continuous, if $V \in \mathcal{U}_Y$ implies $f^{-1} \circ V \circ f \in \mathcal{U}_X$. In other words this means that for each $V \in \mathcal{V}_Y$ there exists $U \in \mathcal{U}_X$ such that $U(M) \leqslant f^{-1}(V(f(M)))$ for all $M \in I^X$.

Fuzzy uniform spaces and uniformly continuous mappings between them form a category; we denote it by HFU and call by the category of Hutton fuzzy uniform spaces (to distinguish them from essentially different Lowen fuzzy uniform spaces and from the approach to fuzzy uniformities developed in [4]).

Let $(X,\mathcal{U})$ be a fuzzy uniform space. For each $M \in I^X$ let $IntM := \vee\{N :$ there exists $U \in \mathcal{U}$ s.t. $U(N) \leqslant M\}$. Then $\tau_{\mathcal{U}} := \{M \in I^X : M = IntM\}$ is a fuzzy topology on X; it is caled the fuzzy topology induced by the fuzzy uniformity $\mathcal{U}$.

More details about (Hutton) fuzzy uniform spaces can be found in [7],[9],[1],[13].

Fuzzy filters. A family $\mathcal{F} \subset I^X \setminus \{0\}$ is called a *fuzzy filter* on a set X, if (1) $F_1, F_2 \in \mathcal{F}$ implies $F_1 \wedge F_2 \in \mathcal{F}$, and (2) if $F_1 \in \mathcal{F}$ and $F_2 \geqslant F_1$, $F_2 \in I^X$, then $F_2 \in \mathcal{F}$ [8], [10].

Let (X,τ) be a fuzzy topological space. A family $\Phi \subset I^X$ will be called a *closed fuzzy filter*, if $\Phi = \mathcal{F} \cap \tau^c$ for some fuzzy filter $\mathcal{F}$ on X.

Somewhat modifying Lowen's terminology [10], a (closed) fuzzy filter will be called an α-filter, where $\alpha \in I$, if $\sup_x F(x) \geqslant \alpha$ for each $F \in \mathcal{F}$. If $\mathcal{F}$ is an α-filter and $\alpha' \in (0,\alpha)$, then $\mathcal{F}$ is obviously an α'-filter, too.

It is not difficult to show that for each fuzzy α-filter $\mathcal{F}$ there exists an α-filter Φ which is the maximal one among all fuzzy α-filters containing $\mathcal{F}$. If Φ is a maximal fuzzy α-filter and $A \vee B \in \Phi$, then either $A \in \Phi$ or $B \in \Phi$.

Proposition. *Let $\mathcal{F}$ be a maximal fuzzy α-filter on X and $M_1, M_2 \in I^X$. If $M_1 q F$ and $M_2 q F$ for all $F \in \mathcal{F}$, then also $(M_1 \wedge M_2) q F$ for all $F \in \mathcal{F}$.*

PROOF : Assume that $(M_1 \wedge M_2)\not{q}F$ for some $F \in \mathcal{F}$. Then, obviously, $F = F \wedge (M_1 \wedge M_2)^c = F \wedge (M_1^c \vee M_2^c) = (F \wedge M_1^c) \vee (F \wedge M_2^c)$ and by the maximality condition of $\mathcal{F}$ it follows that either $F_1 = F \wedge M_1^c$ or $F_2 = F \wedge M_2^c$ belongs to $\mathcal{F}$. However, this contradicts the assumption that $M_i q F'$ for each $F' \in \mathcal{F}$ and i=1,2. ■

1. Completeness spectrum.

Let $(X, \mathcal{U})$ be a fuzzy uniform space.

Definition 1.1. A fuzzy set $M \in I^X$ is called *U-small*, where $U \in \mathcal{U}$, if there exists a point $x \in X$ such that $M \leqslant U(x)$. A nonempty family of fuzzy sets $\mathcal{F} \subset I^X \setminus \{0\}$ is called a *(closed) fuzzy Cauchy filter* or, briefly, a *(closed) K-filter*, if $\mathcal{F}$ is a fuzzy filter (resp. a closed fuzzy filter) and for each $U \in \mathcal{U}$ there exists a U-small element $F \in \mathcal{F}$. A family $\omega \subset I^X \setminus \{1\}$ is called an *(open) fuzzy Cauchy ideal* or, briefly, an *(open) K-ideal*, if ω^c is a K-filter (resp. a closed K-filter).

Proposition 1.2. *If M is U-small, then $\overline{M}$ is $(U \circ U)$-small.*

PROOF : Take a point $x \in X$ such that $M \leqslant U(x)$. Then $\overline{M} \leqslant U(M) \leqslant (U \circ U)(x)$. ■

Corollary 1.3. *If $\mathcal{F}$ is a K-filter in X, then $\overline{\mathcal{F}} := \{\overline{F} : F \in \mathcal{F}\}$ is a closed K-filter in X.*

Definition 1.4. By the *completeness spectrum* of a fuzzy set $M \in I^X$ we call the set $Cpl(M)$ consisting of all $\beta \in I$ such that for every open K-ideal ω satisfying the inequality $M \tilde{\subset} \vee \omega \geqslant \beta$, it follows that $\sup\{M \tilde{\subset} \vee \omega_0 : \omega_0 \subset \omega, |\omega_0| < \aleph_0\} \geqslant \beta$. The completeness degree of a fuzzy set M is the number $cpl(M) := \inf(I \setminus Cpl(M))$ (here and later $\inf \emptyset := 1$).

The proofs of the next four propositions are straightforward and therefore omitted.

Proposition 1.5. *$0 \in Cpl(M)$ and $cpl(M) \in Cpl(M)$ for each fuzzy set M.*

Proposition 1.6. *If (β_n) is an increasing sequence converging to β and $(\beta_n) \subset Cpl(M)$, then $\beta \in Cpl(M)$.*

Proposition 1.7. *If $M, N \in I^X$, then $Cpl(M \vee N) \supset Cpl(M) \cap Cpl(N)$.*

Proposition 1.8. *$\beta \in Cpl(M)$ iff for each closed K-filter satisfying the inequality $M^c \tilde{\supset} \wedge \mathcal{F} \geqslant \beta$ it follows that $\sup\{M^c \tilde{\subset} \wedge \mathcal{F}_0 : |\mathcal{F}_0| < \aleph_0, \mathcal{F}_0 \subset \mathcal{F}\} \geqslant \beta$.*

Proposition 1.9. *If $M, N \in I^X$ and besides $N \in \tau^c_{\mathcal{U}}$, then $Cpl(M) \subset Cpl(M \wedge N)$ and hence $cpl(M) \leqslant cpl(M \wedge N)$. In particular, if $N \leqslant M$ and $N \in \tau^c_{\mathcal{U}}$, then $Cpl(M) \subset Cpl(N)$ and hence $cpl(M) \leqslant cpl(N)$.*

PROOF : Let $\beta \in Cpl(M)$ and $(M \wedge N) \tilde{\subset} \vee \omega \geqslant \beta$ for some open K-ideal ω. It is easy to notice that $M \wedge N \tilde{\subset} \vee \omega = M \tilde{\subset} N^c \vee \omega \geqslant \beta$ and that $\omega' := \{N^c \vee U : U \in \omega\}$ is also an open K-ideal. Hence $\sup\{M \tilde{\subset} \vee \omega_0' : \omega_0' \subset \omega, |\omega_0'| < \aleph\} \geqslant \beta$ and therefore $\sup\{M \tilde{\subset} \vee \omega_0 : \omega_0 \subset \omega, |\omega_0| < \aleph_0\} \geqslant \beta$. ■

Theorem 1.10. *For each $i \in \mathcal{J}$ let $(X_i, \mathcal{U}_i)$ be a fuzzy uniform space, M_i be a fuzzy subset of X_i and $M = \sqcap M_i$ be the product of these fuzzy sets (considered as the fuzzy subset of the product fuzzy uniform space $(X, \mathcal{U}) = \sqcap(X_i, \mathcal{U}_i)$). If $(\beta, \alpha] \subset Cpl(M_i)$ for each $i \in \mathcal{J}$, then $(\beta, \alpha] \subset Cpl(M)$, too. Hence, in particular, $cpl(M) \geqslant \inf_i cpl(M_i)$.*

PROOF : It suffices to show that if $\beta < \alpha$ and $(\beta, \alpha] \subset \bigcap_i Cpl(M_i)$, then $\alpha \in Cpl(M)$. Assume that there exist $\epsilon > 0$ and a closed K-filter $\mathcal{F}$ in X such that $\sup\{M^c \breve{\supset} \wedge \mathcal{F}_0 : \mathcal{F}_0 \subset \mathcal{F}, |\mathcal{F}_0| < \aleph_0\} \leqslant \alpha - \varepsilon$ or, equivalently, such that $\sup_x(M \wedge F_1 \wedge \cdots \wedge F_n)(x) \geqslant \alpha^c + \varepsilon$ for each finite subfamily $\{F_1, \ldots, F_n\} \subset \mathcal{F}$. Then $\mathcal{F}$ is an $(\alpha^c + \varepsilon)$-filter an hence $\mathcal{F}$ is contained in a maximal $(\alpha^c + \varepsilon)$-filter Φ such that $M \in \Phi$. For each $i \in \mathcal{J}$ consider now the family $\Phi_i = \{\overline{p_i F} : F \in \Phi\}$, where $p_i : X \to X_i$ is the corresponding projection. It is easy to notice that Φ_i is a closed K-filter on X_i and $\sup_{x_i}(M_i \wedge (\wedge \Phi_i^0))(x_i) \geqslant \alpha^c + \varepsilon$ for each finite $\Phi_i^0 \subset \Phi_i$ or, equivalently, $\sup\{M_i^c \breve{\supset} \wedge \Phi_i^0 : \Phi_i^0 \subset \Phi_i, |\Phi_i^0| < \aleph_0\} \leqslant \alpha - \varepsilon$. Without loss of generality we can assume that $\alpha - \varepsilon > \beta$ and hence $\alpha - \varepsilon/2 \in Cpl(M_i)$; therefore $M_i^c \breve{\supset} \wedge \Phi_i \leqslant \alpha - \varepsilon/2$. However, this means that there exists a point $x_i \in X_i$ such that $(M_i \wedge (\wedge \Phi_i))(x_i) =: t_i \geqslant \alpha^c + \varepsilon/4$. Now, to finish the proof, it is sufficient to show that $x^t \in M \wedge (\wedge \Phi)$, where x^t is the fuzzy point with the support $x = (x_i)_{i \in \mathcal{J}}$ and the value $t := \inf_i t_i$: this would imply that $M \wedge (\wedge \Phi)(x) \geqslant t \geqslant \alpha^c + \varepsilon/4 > \alpha^c$ and hence $M^c \breve{\supset} \wedge \mathcal{F} < \alpha$, i.e. $\alpha \in Cpl(M)$.

Since obviously $x^t \in M$, we have to show only that $x^t \in \wedge \mathcal{F}$.

Let $O = \bigwedge_{i=1}^{n} p_i^{-1}(O_i)$ be a standard q-neighborhood of x^t, where $O_i \in \tau_{\mathcal{U}_i}$, $i = 1, \ldots, n$. It is clear that O_i is a q-neighborhood of the fuzzy point x_i^t and $x_i^t \in \wedge \Phi_i$. Therefore $O_i q F_i$ for each $F_i \in \Phi_i$ and hence $(p_i^{-1}(O_i)) q F$ for each $F \in \Phi$. By maximality of Φ and by Proposition in Section 0, it follows that OqF for each $F \in \Phi$ and hence $x^t \in \wedge \overline{\Phi} \leqslant \wedge \mathcal{F}$.

Our next aim is to show that under certain conditions a result in a known sense inverse to the previous theorem holds. ∎

Theorem 1.11. *For each $i \in \mathcal{J}$ let $(X_i, \mathcal{U}_i)$ be a fuzzy uniform space, M_i be its normed fuzzy subset, and let $M = \sqcap M_i$ be the product of these fuzzy sets. Then $Cpl(M) \subset \bigcap_i Cpl(M_i)$.*

PROOF : Fix $i \in \mathcal{J}$ and let $M_* := \sqcap M_{i'}' : i' \neq i\}$, $X_* := \sqcap\{X_{i'}' : i' \neq i\}$; then obviously $X = X_i \times X_*$, $M = M_i \times M_*$. Take $\beta \in Cpl(M)$ and consider a closed K-filter $\mathcal{F}_i$ in X_i satisfying the inequality $M_i^c \breve{\supset} \wedge \mathcal{F}_i \geqslant \beta$. Since, according to (1.5) and (1.6) one can assume that $\beta \in (0,1)$ and since M_* being a product of normed fuzzy sets is normed itself, there exists a point $x_* \in X_*$ such that $M_*^c(x_*) < \beta$. Let $\mathcal{F}_* = \{A : A$ is a closed fuzzy set in X_* such that $x_* \leqslant A\}$; it is easy to notice that $\mathcal{F}_*$ is a closed K-filter in X_* and $M_*^c \breve{\supset} \wedge \mathcal{F}_* < \beta$. The family $\mathcal{F}_* \times \mathcal{F}_i := \{F_* \times F_i : F_* \in \mathcal{F}_*, F_i \in \mathcal{F}_i\}$ is obviously a base of a closed K-filter $\mathcal{F}$ on X.

It easily follows now that

$$M^c \tilde{\supset} \wedge \mathcal{F} = (M_i \times M_*)^c \tilde{\supset} \wedge (\mathcal{F}_i \times \mathcal{F}_*) = (M_i^c \tilde{\supset} \wedge \mathcal{F}_i) \vee (M_*^c \tilde{\supset} \wedge \mathcal{F}_*) \geqslant M_i^c \tilde{\supset} \wedge \mathcal{F}_i \geqslant \beta$$

and hence $\sup\{M^c \tilde{\supset} \wedge \mathcal{F}^0 : \mathcal{F}^0 \subset \mathcal{F}, |\mathcal{F}^0| < \aleph_0\} \geqslant \beta$. Taking into account that $M_*^c \tilde{\supset} \wedge \mathcal{F}_*^0 \leqslant M_*^c \tilde{\supset} \wedge \mathcal{F}_* < \beta$ for each finite $\mathcal{F}_*^0 \subset \mathcal{F}_*$, we conclude that $\sup\{M_i^c \tilde{\supset} \wedge \mathcal{F}_i^0 : \mathcal{F}_i^0 \subset \mathcal{F}_i, |\mathcal{F}_i^0| < \aleph_0\} \geqslant \beta$ and hence $\beta \in Cpl(M_i)$. ■

From Theorems(1.10) and (1.11) the next corollary follows.

Corollary 1.12. *Under the assumptions of (1.11)* $cpl(M) = \inf_i cpl(M_i)$.

Remark 1.13. It is easy to construct an example showing that the statements of (1.11) and (1.12) do not generally hold for non-normed fuzzy sets M_i.

Examples 1.14. *Completeness spectra of fuzzy sets in ordinary uniform spaces.*

In this subsection $(X, \mathcal{U})$ is an ordinary uniform space and M is its fuzzy subset. Notice first that from (1.3) it follows that

(1.14.1) The space $(X, \mathcal{U})$ is complete iff $Cpl(X, \mathcal{U}) = [0, 1]$.

Patterned after the proof of Theorem 6.1 in [15] one can easily establish the following fact:

(1.14.2) If the sets $M^{-1}[\gamma, 1]$ are complete for all $\gamma > \beta^c$ ($\beta \in I$), then $cpl(M) \geqslant \beta$ and besides M is uppersemicontinuous, then the sets $M^{-1}[\gamma, 1]$ are complete for all $\gamma > \beta^c$.

The statements (1.14.3) - (1.14.6) are easy corollaries of (1.14.2).

(1.14.3) If the sets $M^{-1}[\gamma, 1]$ are complete for all $\gamma > 0$, then $cpl(M) = 1$.

(1.14.4) If the space $(X, \mathcal{U})$ is complete and M is uppersemicontinuous, then $cpl(M) = 1$.

(1.14.5) If M is uppersemicontinuous, then $Cpl(M) = [0, cpl(M)]$.

(1.14.6) If M is uppersemicontinuous and $cpl(M) = 1$, then the subspace $M^{-1}(0, 1]$ is a union of countably many complete subspaces.

We finish this section with some concrete examples. One can easily justify them basing on the previous statements.

(1.14.7) If X is not complete and $M = a$, then $Cpl(M) = [0, a^c]$, $cpl(M) = a^c$.

(1.14.8) Let X be non-complete, $X = X_1 \cup X_2$, $X_1 \cap X_2 = 0$ and $0 \leqslant a_1 < a_2 \leqslant 1$. Let the fuzzy set $M \in I^X$ be defined by the equality $M = a_1 X_1 + a_2 X_2$ (i.e. $M(x) = a_i$, iff $x \in X_i$, $i = 1, 2$). Then $Cpl(M) = [0, a_1^c]$, $cpl(M) = a_1^c$, if X_1 is not complete and $Cpl(M) = [0, a_2^c]$, $cpl(M) = a_2^c$ otherwise.

(1.14.9) Let X be complete, $X = X_1 \cup X_2$, $X_1 \cap X_2 = 0$ and both X_1 and X_2 be non-complete. If M is defined as in (1.14.8), then $Cpl(M) = [0, a_2^c] \cup [a_1^c, 1]$ and $cpl(M) = a_2^c$.

2. Precompactness spectrum.

Let $(X, \mathcal{U})$ be a fuzzy uniform space and M be its fuzzy subset.

Definition 2.1. By the *precompactness spectrum* of a fuzzy set M we call the set $Pc(M)$ consisting of all $\beta \in I$ such that $\sup\{M \tilde{\subset} U(X_0) : X_0 \subset X, |X_0| < \aleph_0\} \geq \beta$.

The number $pc(M) = \sup Pc(M)$ is called the *precompactness degree* of the fuzzy set M.

Directly from this definition, one can establish the following easy facts.

Proposition 2.2. *$Pc(M) = [0, pc(M)]$ for each fuzzy set M. (The case $pc(M) = 0$ is not excluded!)*

Proposition 2.3. *$pc(M) = \inf_{U \in \mathcal{U}} \sup\{M \tilde{\subset} U(X_0) : X_0 \subset X, |X_0| < \aleph_0\}$ for each fuzzy set M.*

Proposition 2.4. *If $N \leqslant M$, $(N \in I^X)$, then $Pc(N) \supset Pc(M)$.*

Proposition 2.5. *$Pc(M \vee N) = Pc(M) \cap Pc(N)$ for any $M, N \in I^X$.*

Proposition 2.6. *If $(X, \mathcal{U})$, $(Y, \mathcal{V})$ are fuzzy uniform spaces, $M \in I^X$, and $f : X \to Y$ is a uniformly continuous mapping, then $Pc(M) \subset Pc(f(M))$.*

PROOF : Take $\beta \in Pc(M)$ and let $\varepsilon > 0$, $V \in \mathcal{V}$. Since f is uniformly continuous, there exists $U \in \mathcal{U}$ such that $f(U) \subset V$. Choose a finite subset $X_0 \subset X$ for which $M \tilde{\subset} U(X_0) \geqslant \beta - \varepsilon$. It follows now easily that $\beta - \varepsilon \leqslant f(M) \tilde{\subset} f(U)(Y_0) \leqslant f(M) \tilde{\subset} V(Y_0)$, where $Y_0 = f(X_0)$, and hence $\beta \in Pc(f(M))$. ■

Theorem 2.7. *For each $i \in \mathcal{J}$ let $(X_i, \mathcal{U}_i)$ be a uniform space and M_i be its fuzzy subset. Let $M = \sqcap M_i$ be the product of these fuzzy sets (considered as the fuzzy subset of the product fuzzy uniform space $(X, \mathcal{U}) = \sqcap(X_i, \mathcal{U}_i)$). Then $Pc(M) \supset \bigcap_i Pc(M_i)$ and hence $pc(M) \geqslant \inf_i pc(M_i)$. If, besides, all M_i are normed, then $Pc(M) = \bigcap_i Pc(M_i)$ and hence $pc(M) = \inf_i pc(M_i)$.*

PROOF : Assume that $\beta \in Pc(M_i)$ for every $i \in \mathcal{J}$ and take some $U \in \mathcal{U}$ and $\varepsilon > 0$. From the definition of the product fuzzy uniformity, it follows that there exists a finite subset $\mathcal{J}_0 \subset \mathcal{J}$ and $U_i \in \mathcal{U}_i$ for each $i \in \mathcal{J}_0$ such that $\bigwedge_{i \in \mathcal{J}_0} p_i^{-1}(U_i) \leqslant U$, where $p_i : X \to X_i$ are the corresponding projections. Now for each $i \in \mathcal{J}_0$ fix a finite set $A_i \subset X_i$ such that $M_i \tilde{\subset} U_i(A_i) \geqslant \beta - \varepsilon$, and hence, obviously, $p_i^{-1}(M_i) \tilde{\subset} p_i^{-1}(U_i(A_i)) \geqslant \beta - \varepsilon$, too. Let $A = (\prod_{i \in \mathcal{J}_0} A_i) \times \{x_*\}$, where x_* is an arbitrary (but fixed) point of $X_* := \sqcap\{X_i : i \in \mathcal{J}_0\}$. Since obviously $p_i^{-1}(U_i)(A_i) = (p_i^{-1}(U_i))(A)$ for each $i \in \mathcal{J}_0$, it follows now that $M \tilde{\subset} U(A) \geqslant M \tilde{\subset} \bigwedge_{i \in \mathcal{J}_0} (p_i^{-1}(U_i))(A) \geqslant \bigwedge_{i \in \mathcal{J}_0} (p_i^{-1}(M_i) \tilde{\subset} p_i^{-1}(U_i(A_i)) \geqslant \beta - \varepsilon$ and hence $\beta \in Pc(M)$. This completes the proof of the first part of the theorem. To prove the second part, notice that $p_i(M) = M_i$ for each $i \in \mathcal{J}$ (in case all M_i are normed) and use Proposition (2.5). ■

Proposition 2.8. *If (X, τ) is a completely regular fuzzy topological space ([7], [9]), then a fuzzy uniformity $\mathcal{U}$ on X exists such that $\tau_{\mathcal{U}} = \tau$ and $pc(X, \mathcal{U}) = 1$.*

This statement is a corollary of Remark (2.11) below and Propositions (4.8) and (5.2) of Artico and Moresco [1]. However, for the convenience of the reader, we shall give here a direct and effective proof, too.

PROOF : Following [7] for each $\varepsilon > 0$, consider the mappings B_ε, $B_\varepsilon^{-1} : I^{\mathcal{F}(I)} \to I^{\mathcal{F}(I)}$ defined by $B_\varepsilon(U) = \wedge\{\varrho_{s-\varepsilon} : U \leqslant 1-\lambda_s\}$ and $B_\varepsilon^{-1}(U) = \wedge\{\lambda_{s+\varepsilon} : U \leqslant 1-\varrho_s\}$, where λ_s, $\varrho_s : \mathcal{F}(I) \to I (s \in I)$ are the elements of the standard subbase of the fuzzy unit interval. It is known that $\{B_\varepsilon, B_\varepsilon^{-1} : \varepsilon \in (0,1]\}$ is the subbase of the standard fuzzy uniformity on $\mathcal{F}(I)$ and $\{f^{-1}(B_\varepsilon), f^{-1}(B_\varepsilon^{-1}) : f \in C(X, \mathcal{F}(I)), \varepsilon > 0\}$ (where $C(X,\mathcal{F}(I))$ is the set of all continuous functions from the fuzzy topological space X into $\mathcal{F}(I)$) is a subbase of a fuzzy uniformity $\mathcal{U}$ on X inducing τ (see e.g.[7], Theorem 1.7). Therefore it is sufficient to show that $pc(X,\mathcal{U}) = 1$.

From (3.1) below it follows that $pc(\mathcal{F}(I)) \geqslant c(\mathcal{F}(I))$ and hence e.g. by. Theorem 3.20 of [15] $pc(\mathcal{F}(I)) = 1$. Therefore, for any $U \in \mathcal{U}$ and $\varepsilon > 0$ there exist $f_1, f_2 \in C(X,\mathcal{F}(I))$ and $\varepsilon_1, \varepsilon_2 > 0$ such that $f_1^{-1}(B_{\varepsilon_1}) \wedge f_2^{-1}(B_{\varepsilon_2}^{-1}) \leqslant U$. Choose finite subsets A_1 and A_2 of $\mathcal{F}(I)$ for which $B_{\varepsilon_1}(A_1) \geqslant (1-\varepsilon)\mathcal{F}(I)$, $B_{\varepsilon_2}^{-1}(A_2) \geqslant (1-\varepsilon)\mathcal{F}(I)$ and let C_1, C_2 be finite subsets of X such that $f_i(C_i) = A_i \cap f_i(X)$, $i = 1,2$, and $C = C_1 \cup C_2$. Then obviously $f_1^{-1}(B_{\varepsilon_1}) \wedge f_2^{-1}(B_{\varepsilon_2}^{-1})(C) = f_1^{-1}(B_{\varepsilon_1})(C) \wedge f_2^{-1}(B_{\varepsilon_2}^{-1})(C) = f_1^{-1}(B_{\varepsilon_1}(f_1(C))) \wedge f_2^{-1}(B_{\varepsilon_2}^{-1}(f_2(C))) \geqslant f_1^{-1}((1-\varepsilon)\mathcal{F}(I)) \wedge f_2^{-1}((1-\varepsilon)\mathcal{F}(I)) \geqslant (1-\varepsilon)X$ and hence $U(A) \geqslant (1-\varepsilon)X$. ∎

Corollary 2.9. *If (X,τ) is a completely regular fuzzy topological space and $M \in I^X$, then a fuzzy uniformity $\mathcal{U}$ on X exists such that $\tau_{\mathcal{U}} = \tau$ and $pc(M),\mathcal{U}) = 1$.*

Examples 2.10. *Precompactness spectra of fuzzy sets in ordinary uniform spaces.*
Let $(X,\mathcal{U})$ be an ordinary uniform space and M be its fuzzy subset. It is easy to check the next facts.

(2.10.1) The space $(X,\mathcal{U})$ is precompact iff $Pc(X) = [0,1]$.

(2.10.2) $\beta \in Pc(M)$ iff for all $\gamma > \beta^c$ the sets $M^{-1}(\gamma,1]$ are precompact.

(2.10.3) $Pc(M) = [0,1]$ iff for all $\gamma > 0$ the sets $M^{-1}(\gamma,1]$ are precompact.

Remark 2.11. Artico and Moresco call a fuzzy uniform space $(X,\mathcal{U})$ precompact, if for each $U \in \mathcal{U}$ the set $\{U(M) : M \in I^X\}$ is finite. It is easy to notice that if X is precompact, then for each $U \in \mathcal{U}$ there exists a finite set $X_0 \subset X$ such that $U(X_0) = X$ and hence $pc(X) = 1$. However, the converse does not hold: there exists a non-precompact [1] fuzzy uniform space, the precompactness degree of which is 1. This can be illustrated by the next example:

Example 2.11.1. (cf.Example10 in [1]). Let X be a set and let $U : I^X \to I^X$ map every $M \in I^X$ into the constant function $c_M = \sup_x M(x)$. Let $\mathcal{U}$ denote the fuzzy uniformity having $\{U\}$ as a base. Then $pc(X,\mathcal{U}) = 1$, but $(X,\mathcal{U})$ is not precompact in the sense of [1].

3. On compactness spectra of fuzzy sets in fuzzy uniform spaces.

In [14], [15], the notion of compactness spectrum of a fuzzy set in a fuzzy topological space was introduced and studied. The aim of this section is to establish some relations between the compactness spectrum of a fuzzy set in a fuzzy uniform space and its completeness and precompactness spectra.

Theorem 3.1. *For every fuzzy set M in a fuzzy uniform space $(X,\mathcal{U})$ $C(M) \subset Cpl(M) \cap Pc(M)$ and $C(M) \cap (\frac{1}{2},1] = Cpl(M) \cap Pc(M) \cap (\frac{1}{2},1]$.*

PROOF : The first statement is obvious. To prove the second statement, take $\beta \in Cpl(M) \cap Pc(M)$, $\beta > \frac{1}{2}$, and assume that $\beta \notin C(M)$. Then a family $\omega \subset \tau_{\mathcal{U}}$ and a number $\varepsilon > 0$ exist such that $M \tilde{\subset} \vee \omega \geqslant \beta$ but $M \tilde{\subset} \vee \omega_0 \leqslant \beta - \varepsilon$ for each finite subfamily ω_0 of ω; from the second inequality it follows that $\sup(M \wedge (\wedge \omega_0^c))(x) \geqslant \beta^c + \varepsilon$ for each finite $\omega_0 \subset \omega$, and hence the family $\{M\} \cup \omega^c$ is a base of a closed $(\beta^c + \varepsilon)$-filter $\mathcal{F}$. Let Φ be a maximal $(\beta^c + \varepsilon)$-filter containing $\mathcal{F}$. We shall show that Φ is a K-filter.

Really, let $U \in \mathcal{U}$; then there exists a finite subset X_0 of X such that $M \tilde{\subset} U(X_0) \geqslant \beta - \frac{\varepsilon}{2}$. It easily follows now that $U(X_0)(x) \geqslant \beta - \frac{\varepsilon}{2}$ whenever $M^c(x) < \beta - \frac{\varepsilon}{2}$ for a point $x \in X$. However, this implies easily that $\sup(M \vee U(X_0))(x) \geqslant \beta - \frac{\varepsilon}{2} \geqslant \beta^c + \varepsilon$. (Without loss of generality we assume that $\beta - \varepsilon > \frac{1}{2}$). Since Φ is a maximal $(\beta^c + \varepsilon)$-filter, we conclude that $U(X_0) \in \Phi$ and hence $U(a) \in \Phi$ for some point $a \in X_0$, and therefore Φ is a K-filter. Then by (1.3) $\overline{\Phi}$ is also a K-filter and hence $\mathcal{V} := \overline{\Phi}^c$ is an open K-ideal. On the other hand, $\mathcal{V} \supset \omega$ and hence $M \tilde{\subset} \vee \mathcal{V} \geqslant M \tilde{\subset} \vee \omega \geqslant \beta$. Since $\beta \in Pc(M)$, there exists a finite subfamily $\mathcal{V}_0 \subset \mathcal{V}$ such that $M \tilde{\subset} \vee \mathcal{V}_0 > \beta - \varepsilon$. It is easy to conclude now that $\sup_x M \wedge (\wedge \overline{\Phi}_0)(x) < \beta^c + \varepsilon$, where $\overline{\Phi}_0 = \mathcal{V}_0^c$. However, this contradicts the assumption that $\overline{\Phi}$ is a $(\beta^c + \varepsilon)$-filter. The obtained contradiction completes the proof. ■

Theorem 3.1′. *Let M be a fuzzy set in a fuzzy uniform space $(X, \mathcal{U})$ and $pc(M) = 1$. Then $cpl(M) = c(M)$.*

PROOF : is quite analogous to the proof of (3.1). The only difference is that now it is impossible to assume that $\beta > \frac{1}{2}$, but on the other hand, when choosing X_0 for a given $U \in \mathcal{U}$, we can ensure a stronger condition, namely, the inequality $M \tilde{\subset} U(X_0) > 1 - \delta$, where $\delta = \min\{\beta^c + \frac{\varepsilon}{2}, \beta - \varepsilon\}$. ■

Theorem 3.2. *Let (X, τ) be a fuzzy completely regular topological space and $M \in I^X$. Then $c(M) = \inf\{cpl(M, \mathcal{U}) : \tau_{\mathcal{U}} = \tau\}$.*

PROOF : From (3.1) it follows that $c(M) \leqslant cpl(M, \mathcal{U})$ for each $\mathcal{U}$ satisfying $\tau_{\mathcal{U}} = \tau$. Conversely, from (2.8) and (3.1') it follows that there exists a uniformity $\mathcal{V}$ such that $\tau_{\mathcal{V}} = \tau$ and $c(M) = cpl(M, \mathcal{V})$ and hence $c(M) \geqslant \inf\{cpl(M, \mathcal{U}) : \tau_{\mathcal{U}} = \tau\}$. ■

Remark 3.3. We think that the reader has noticed the classical prototypes of the results in this Section. Namely Theorem (3.1), as well as Theorem (3.1'), contains in itself a well-known Weil's Theorem stating that a uniform space is compact iff it is complete and precompact, and Theorem (3.2) contains in itself a well-known statement that a completely regular topological space is compact iff its topology is induced by some complete uniformity.

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