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On dimension of locally pseudocompact groups and their quotients

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Dedicated to the memory of Zdeněk Frolík

Abstract. It is shown that $\dim B = \operatorname{ind} B = \operatorname{ind}_0 B$ for every quotient space G/K of a closed subgroup K in a locally pseudocompact group G, and the equality $\dim G = \dim \widehat{K} + \dim G/K$ is established. We answer a question of A.V.Arhangel'skiĭ by showing that an extremally disconnected quotient space of a closed subgroup in a pseudocompact group is finite.

Keywords: Locally pseudocompact group, covering dimension, small (large) inductive dimension, quotient space, C-embedded subset, σ-lattice of mappings, perfect k-normality.

Classification: 54F45, 22A05

By theorem of B.A.Pasynkov [10] , if G is a locally compact group and K is a closed subgroup of G, then the equalities $\dim G/K = \operatorname{ind} G/K = \operatorname{Ind} G/K$ and $\dim G = \dim K + \dim G/K$ hold. Here we prove similar equalities in case when G is a locally pseudocompact group. When passing from (locally) compact groups to (locally) pseudocompact groups, two circumstances would be mentioned. First, neither pseudocompact group G nor its quotient space G/K have to be normal spaces. Second, a closed subgroup of a pseudocompact group need not be pseudocompact [5, Theorem 2.4]. An absence of normality obliges us to define the dimension dim in terms of finite functionally open covers (see [6, p.472]). The large inductive dimension function Ind would be replaced by Ind_0 which was introduced by V.V.Filippov and studied in [9]. The function Ind_0 is defined in the following way: $\operatorname{Ind}_0 X = -1$ iff X is empty, and $\operatorname{Ind}_0 X \leqq n+1$ iff for every disjoint zero-sets F_0, F_1 of X there exist disjoint open sets O_0, O_1 and a zero-set C of X such that $F_i \subseteq O_i$ $(i = 0,1), X \setminus C = O_0 \cup O_1$ and $\operatorname{Ind}_0 C \leqq n$. (Note that O_0 and O_1 are cozero-sets of X by Lemma 7.2.12 of [6]). It is known that $\operatorname{Ind}_0 X = \operatorname{Ind} X$ for every normal space X, each closed G_δ-subset of which is perfectly k-normal [7, Proposition 1].

A useful equality $\dim B = \dim \widehat{B}$, where $B = G/K$ is the quotient space with respect to a closed subgroup K of a locally pseudocompact group G and $\widehat{B} = \widehat{G}/\widehat{K}$ is the completion of B, was established in [3]. If, in addition, the underlying space of G is normal, then $\dim G = \dim B + \dim K$ [3, Theorem 4]. Thus our Theorems 1 and 2 complete the work begun in [3], and the condition "G is normal" is deleted (obviously, a normal locally pseudocompact group is locally countably compact, and closed subgroups inherits the latter property).

In fact, Theorem 1 states a bit more: $\dim F = \operatorname{ind} F = \operatorname{Ind}_0 F$ for each zero–set F in B. An analogous equality does not hold even for closed subsets of pseudocompact groups, for every Tychonoff space embeds as a closed subset into a suitable pseudocompact group.

Theorem 2 implies that the dimension of a quotient space of a closed subgroup in a locally pseudocompact group G does not exceed the dimension of G (Corollary 2). A question of A.V.Arhangel'skiĭ is answered by showing that any extremally disconnected quotient space of a closed subgroup in a pseudocompact group is necessarily finite (Theorem 3).

In what follows all topological groups are assumed to be Hausdorff and spaces to be Tychonoff. A subset Y of a space X is said to be $\aleph_0$-dense in X provided Y meets all non–empty G_δ-subsets of X. It is important to mention that a dense C-embedded subset Y of a space X is necessarily $\aleph_0$-dense in X [8].

By $\mathrm{Fr}_X U$ we denote the boundary of a set U in a space X.

Let $f : X \to Y$ and $g : X \to Z$ be continuous mappings onto. The symbol $f \prec g$ means that there exists a continuous mapping $h : Y \to Z$ such that $g : h \circ f$. Obviously $\prec$ is a partial order relation on the family MAP(X) of all continuous mappings with the domain X. Given a family $\mathcal{F} \subseteq$ MAP(X), we say that $\mathcal{F}$ is a *σ-lattice* for X if the following conditions are fulfilled:

(L1) for any $f_1, f_2 \in \mathcal{F}$ there exists $f \in \mathcal{F}$ such that $f \prec f_1$ and $f \prec f_2$;

(L2) if $f_i \in \mathcal{F}$ and $f_{i+1} \prec f_i$ for each $i \in \mathbb{N}$, then the diagonal product $\triangle_{i=0}^{\infty} f_i$ of f_i's belongs to $\mathcal{F}$;

(L3) the diagonal product $j = \triangle\mathcal{F}$ of all mappings belonging to $\mathcal{F}$ is a homeomorphism of X onto the subspace $j(X)$ of $\triangle_{f\in\mathcal{F}} f(X)$.

Note that if $\mathcal{F}$ is a σ-lattice for X, then $\mathcal{F}$ is $\aleph_0$-directed by $\prec$, i.e., for every countable subfamily $\mathcal{Z} \subseteq \mathcal{F}$ there exists $f^* \in \mathcal{F}$ such that $f^* \prec f$ whenever $f \in \mathcal{Z}$. We say that $\mathcal{F}$ has the *factorization property* provided the following holds:

(L4) for every continuous real–valued function g on X there exists $f \in \mathcal{F}$ such that $f \prec g$.

It is clear that if a σ-lattice $\mathcal{F}$ for X has the factorization property and $g : X \to Z$ is continuous, $w(Z) \leqq \aleph_0$, then one can find $f \in \mathcal{F}$ with $f \prec g$.

The main results. Let K be a closed subgroup of a topological group G. We denote by $\widehat{G}$ and $\widehat{K}$ the group completions of G and K respectively, $\widehat{K} = \mathrm{cl}_{\widehat{G}} K$. Identify G with the corresponding subgroup of $\widehat{G}$, and consider the natural quotient mappings $p : G \to G/K$ and $\widehat{p} : \widehat{G} \to \widehat{G}/\widehat{K}$. A simple verification shows that $\widehat{p}(G)$ is a subspace of $\widehat{B} = \widehat{G}/\widehat{K}$, which is homeomorphic to $B = G/K$. Therefore, we may identify p and $\widehat{p}|_G$. The following theorem is the main result of the paper.

Theorem 1. *Let Φ be a zero–set in a quotient space G/K of a locally pseudocompact group G with respect to a closed subgroup K. Then* $\dim\Phi = \operatorname{ind}\Phi = \operatorname{Ind}_0\Phi = \dim\widehat{\Phi}$, *where* $\widehat{\Phi} = cl_{\widehat{B}}\Phi$.

Remark 1. One can assume that group G under consideration is generated by a pseudocompact neighborhood V_0 of the identity. Indeed, let H be the subgroup of G

generated by V_0. Then H is open in G and the quotient space G/K is a topological sum of spaces, each of which is homeomorphic to a quotient space of $H \cap aKa^{-1}$ in H for some $a \in G$ (see [**12**, Lemma 1]). From now to the end of the proof of Theorem 1 this assumption is supposed to be fulfilled.

To prove Theorem 1 we need four auxiliary lemmas. In the sequel the above notations are used without reservation.

Lemma 1. *Suppose that a space X has a σ-lattice $\mathcal{F}$ consisting of open mappings onto second–countable spaces, Y is $\aleph_0$-dense in X and Φ is a zero–set in Y. Then*

(a) *X is perfectly k-normal;*
(b) *Y is C-embedded in X;*
(c) *Φ is perfectly k-normal;*
(d) *Φ is C-embedded in Y and in X;*
(e) *$\widehat{\Phi} = \mathrm{cl}_X \Phi$ is a zero–subset of X;*
(f) *every zero–set in Φ is a zero–set in Y.*

PROOF : (a) Recall that a space is said to be perfectly k-normal provided the closure of each open subset is a zero–set in this space. The space X has the Souslin property by virtue of [**2**, Theorem 1]. (A slight modification must be done to transform the proof of Theorem 1 of [**2**] to that of the above statement, for A.Blaszczyk dealt with inverse spectra in [**2**]). Since $\mathcal{F}$ has properties (L1) and (L2), the sets of the form $f^{-1}(U)$ constitute a base $\mathcal{B}$ of X, where $f \in \mathcal{F}$ and U is open in $f(X)$. For a given open subset O of X one can find a countable subfamily $\gamma \subseteq \mathcal{B}$ so that $V = \bigcup \gamma$ is dense in O. Using the fact that $\mathcal{F}$ is $\aleph_0$-directed by $\prec$, we can pick $f \in \mathcal{F}$ and an open subset $U \subseteq f(X)$ so that $V = f^{-1}(U)$. Since f is an open mapping, the equality $\mathrm{cl}O = \mathrm{cl}V = f^{-1}(\mathrm{cl}U)$ holds. Obviously, $\mathrm{cl}U$ is a zero–subset of the second–countable space $f(X)$. Therefore $\mathrm{cl}O$ is a zero–subset of X, i.e., X is perfectly k-normal.

(b) Being $\aleph_0$-dense in X, the set Y is C-embedded in X by [**13**, Theorem 2].

(c) Since the space $f(X)$ is second–countable for each $f \in \mathcal{F}$, an $\aleph_0$-density of Y in X implies that $f(Y) = f(X)$. This equality enables us to conclude that the restriction of every mapping $f \in \mathcal{F}$ to Y is open as well. Define $\mathcal{F}^* = \{f|_Y : f \in \mathcal{F}\}$. Since Y is $\aleph_0$-dense in X, $\mathcal{F}^*$ is a σ-lattice of open mappings for Y. Hence Theorem 1 of [**15**] implies that $\mathcal{F}^*$ has the factorization property. Taking into account that Φ is a zero–set in Y, we can find a continuous function $g : Y \to \mathbf{R}$ such that $\Phi \equiv g^{-1}(0)$. There exists $f_0 \in \mathcal{F}$ such that $f_0 \prec g$. Clearly $\Phi = f_0^{-1} f_0(\Phi)$. Put $\mathcal{F}^*_\Phi = \{f \in \mathcal{F}^* : f \prec f_0\}$. Then $\mathcal{F}^*_\Phi$ is a σ-lattice of open mappings for Φ ; therefore Φ is perfectly k-normal by (a).

(d) Let ϕ be a continuous real–valued function defined on Φ. Since $\mathcal{F}^*_\Phi$ has the factorization property, there exist $g \in \mathcal{F}^*_\Phi$ and $\psi : g(\Phi) \to \mathbf{R}$ such that $\phi \equiv \psi \cdot g$. By the definition of $\mathcal{F}^*_\Phi$ one can find $f \in \mathcal{F}^*$ so that $f \prec f_0$ (see the above item (c)) and $g = f|_\Phi$. Then $\Phi = f^{-1} f(\Phi)$, and this in turn implies that $f(\Phi)$ is closed in $f(Y)$ (we use the fact that f is open and hence quotient). Since $f(Y)$ is second–countable, ψ extends to a continuous function $\widetilde{\psi} : f(Y) \to \mathbf{R}$. Obviously, $\widetilde{\psi} \cdot f$ is a continuous function extending ϕ over Y, so Φ is C-embedded in Y. But Y is C-embedded in X by (b), and so is Φ.

(e) Let $f_0 \in \mathcal{F}^*$ and $\Phi = f_0^{-1} f_0(\Phi)$. There exists $f \in \mathcal{F}$ such that $f_0 = f|_Y$. The set $F = f_0(\Phi)$ is closed in the second–countable space $f_0(Y) = f(X)$; hence $f^{-1}(F)$ is a zero–set in X. Now the $\aleph_0$-density of Y in X implies the equality $f^{-1}(F) = \mathrm{cl}_X \Phi$, i.e., $\mathrm{cl}_X \Phi$ is a zero–set in X.

(f) Assume that C is a zero–set in Φ and f is a continuous real–valued function defined on Φ such that $C = f^{-1}(0)$. Extend f to a continuous function $\widetilde{f} : Y \to \mathsf{R}$ and put $h = |\widetilde{f}| + |g|$, where $g : Y \to \mathsf{R}$ is a continuous function with $\Phi = g^{-1}(0)$. Clearly, $C = h^{-1}(0)$. ■

Lemma 2. *If X, Y, Φ and $\widehat{\Phi}$ are as in Lemma 1, then* $\mathrm{ind}\,\Phi = \mathrm{ind}\,\widehat{\Phi}$ *and* $\mathrm{Ind}_0\,\Phi = \mathrm{Ind}_0\,\widehat{\Phi}$. *Furthermore, if X is normal, then* $\mathrm{Ind}_0\,\widehat{\Phi} = \mathrm{Ind}\,\widehat{\Phi}$.

PROOF : We begin with the equality $\mathrm{Ind}_0\,\Phi = \mathrm{Ind}_0\,\widehat{\Phi}$. First, the inequality $\mathrm{Ind}_0\,\Phi \leqq \mathrm{Ind}_0\,\widehat{\Phi}$ will be verified. Apply an induction on $n = \mathrm{Ind}_0\,\widehat{\Phi}$. Assume that Φ_0 and Φ_1 are disjoint zero–sets in Φ. There exists a continuous real–valued function f on Φ such that $\Phi_i = f^{-1}(i), i = 0, 1$. Extend f to a continuous function g over X (use Lemma 1(d)) and define $F_i = g^{-1}(i), i = 0, 1$. Since Y is $\aleph_0$-dense in X, we have $F_i = \mathrm{cl}_X \Phi_i$ for each $i = 0, 1$. The equality $\mathrm{Ind}_0\,\widehat{\Phi} = n$ implies that there exist a zero–set $\widehat{C}$ of $\widehat{\Phi}$ with $\mathrm{Ind}_0\,\widehat{C} \leqq n - 1$ and disjoint open sets O_0, O_1 of $\widehat{\Phi}$ such that $F_i \subseteq O_i$ $(i = 0, 1)$ and $O_0 \cup O_1 = \widehat{\Phi} \setminus \widehat{C}$. Then $C = \widehat{C} \cap \Phi$ is a zero–set in Φ an, a fortiori, of Y, so $\widehat{C} = \mathrm{cl}_X C$. The inductive hypothesis yields $\mathrm{Ind}_0\,C \leqq \mathrm{Ind}_0\,\widehat{C} \leqq n - 1$. Furthermore, $\Phi_i \subseteq U_i$ and $\Phi \setminus C = U_0 \cup U_1$, where $U_i = O_i \cap \Phi$, $i = 0, 1$. Consequently $\mathrm{Ind}_0\,\Phi \leqq n$.

The reverse inequality $\mathrm{Ind}_0\,\widehat{\Phi} \leqq \mathrm{Ind}_0\,\Phi$ will be proved by induction on $n = \mathrm{Ind}_0\,\Phi$. Let F_0 and F_1 be disjoint zero–sets in $\widehat{\Phi}$. Put $\Phi_i = F_i \cap \Phi$, $i = 0, 1$. Since $\mathrm{Ind}_0\,\Phi = n$, there exist a zero–set C in Φ with $\mathrm{Ind}_0\,C \leqq n - 1$ and open disjoint sets U_0, U_1 of Φ such that $\Phi_i \subseteq U_i$ $(i = 0, 1)$ and $U_0 \cup U_1 = \Phi \setminus C$. By Lemma 1(e), $\widehat{C} = \mathrm{cl}_X C$ is a zero–set in $\widehat{\Phi}$, so the inductive hypothesis implies $\mathrm{Ind}_0\,\widehat{C} \leqq n - 1$. Obviously, U_0 and U_1 are cozero–sets in Φ (apply Lemma 7.2.12 of [6]), and hence one can find a continuous real–valued function f on Φ such that $C = f^{-1}(0)$, $U_0 = f^{-1}(\mathsf{R}_-)$ and $U_1 = f^{-1}(\mathsf{R}_+)$, where $\mathsf{R}_- = \{r \in \mathsf{R} : r < 0\}$ and $\mathsf{R}_+ = \{r \in \mathsf{R} : r > 0\}$. Extend f to a continuous function g over $\widehat{\Phi}$ (Lemma 1(d)) and define $V_0 = g^{-1}(\mathsf{R}_-)$, $V_1 = g^{-1}(\mathsf{R}_+)$. The $\aleph_0$-density of Y in X implies that $\widehat{C} = g^{-1}(0)$. It is clear that $\widehat{\Phi} \setminus \widehat{C} = V_0 \cup V_1$ and $F_i \subseteq V_i$ $(i = 0, 1)$, so $\mathrm{Ind}_0\,\widehat{\Phi} \leqq n$. Thus, the equality $\mathrm{Ind}_0\,\Phi = \mathrm{Ind}_0\,\widehat{\Phi}$ is proved.

The proof of the equality $\mathrm{ind}\,\Phi = \mathrm{ind}\,\widehat{\Phi}$ is almost identical to that just carried out. We should mention only that one can use the following easy observation. If U is an open subset of Φ, then the set $U_0 = \mathrm{Int}_\Phi \mathrm{cl}_\Phi U$ satisfies the conditions $U \subseteq U_0$ and $\mathrm{Fr}_\Phi U_0 \subseteq \mathrm{Fr}_\Phi U$ (so $\mathrm{ind}\,\mathrm{Fr}_\Phi U_0 \leqq \mathrm{ind}\,\mathrm{Fr}_\Phi U$). Moreover, $\mathrm{Fr}_\Phi U_0 = \mathrm{cl}_\Phi U_0 \cap \mathrm{cl}_\Phi(\Phi \setminus \mathrm{cl}_\Phi U_0)$ is a zero–set in Φ and in Y (Lemma 1(c)).The same is true for open subsets of $\widehat{\Phi}$.

Let X be normal. Since each zero–subset of X is perfectly k-normal (apply Lemma 1(c) with $Y = X$), the space X is hereditarily perfectly k-normal in the sense of V.V.Fedorčuk [7]. Now, $\mathrm{Ind}_0\,\widehat{\Phi} = \mathrm{Ind}\,\widehat{\Phi}$ follows from [7, Proposition 1]. ■

Let $\mathcal{P}$ be the family of all normal closed subgroups of $\widehat{G}$, which have the type G_δ in $\widehat{G}$ and are contained in the compact neighborhood $\mathrm{cl}_{\widehat{G}} V_0$ of the identity. The following lemma has well-known spectral analogues [**11**,**12**].

Lemma 3. *The quotient space $\widehat{G}/\widehat{K}$ has a σ-lattice $\widehat{\mathcal{M}}$ consisting of open mappings onto second-countable spaces.*

PROOF : By Remark 1, the locally compact group $\widehat{G}$ is generated by compact set $\mathrm{cl}_{\widehat{G}} V_0$; hence $\widehat{G}$ has Souslin property [**14**, Corollary 2]. For every $N \in \mathcal{P}$ let $\widehat{\lambda}_N$ be the quotient mapping of $\widehat{G}$ onto $\widehat{G}/\widehat{K}N$. The group $\widehat{K}N$ is closed in $\widehat{G}$, consequently there exists a natural mapping $\widehat{w}_N : \widehat{G}/\widehat{K} \to \widehat{G}/\widehat{K}N$ such that $\widehat{\lambda}_N = \widehat{w}_N \circ p$. The local compactness and the Souslin property of $\widehat{G}$ together imply that the family $\widehat{\mathcal{M}} = \{\widehat{w}_N : N \in \mathcal{P}\}$ is as required. ∎

By Theorem 6 in [**2**], the space $B = G/K$ is C^*-embedded in $\widehat{B}$, i.e., $\beta B = \beta \widehat{B}$. Using local pseudocompactness of B, we can conclude that B is C-embedded in $\widehat{B}$. Consequently, $\widehat{B}$ is a subspace of the Hewitt realcompactification υB of B; hence B is $\aleph_0$-dense in $\widehat{B}$ [**8**]. For each $N \in \mathcal{P}$ let $w_N = \widehat{w}_N|_B$ and $\mathcal{M} = \{w_N : N \in \mathcal{P}\}$. Then the σ-lattice $\mathcal{M}$ for B has the factorization property (see Lemma 3 and Theorem 1 in [**15**]).

Lemma 4. *Suppose there exists a zero-set Φ in B which has a finite dimension (in the sense of* dim, ind *or* Ind_0*). Then one can find $N \in \mathcal{P}$ so that* $\mathrm{ind}\,\widehat{p}(N) = 0$.

PROOF : Since $\mathcal{M}$ has the factorization property, there exist $N_0 \in \mathcal{P}$ and a closed subset F of $w_{N_0}(B)$ such that $\Phi = w_{N_0}^{-1}(F)$. All fibers of the mapping $w_0 = w_{N_0}$ are homeomorphic to the set $P = \widehat{p}(N_0) \cap B$. Hence $w_0^{-1}(x) \cong P \subseteq \Phi$ for each $x \in F$. The fact that Φ is C-embedded in $\widehat{\Phi} = \mathrm{cl}_{\widehat{B}} \Phi$ (Lemmas 3 and 1(d)) implies $\dim \Phi = \dim \widehat{\Phi}$ and Lemma 2 yields $\mathrm{Ind}_0 \Phi = \mathrm{Ind}_0 \widehat{\Phi} = \mathrm{Ind}\, \widehat{\Phi}$. Here Theorem 7.1.8 of [**6**] and the normality of $\widehat{B}$ are used.

Assume that $\dim \Phi < \infty$. Since $\widehat{B}$ is normal and $\widehat{p}(N_0) \subseteq \widehat{\Phi}$, the inequality $\dim \widehat{p}(N_0) \leqq \dim \widehat{\Phi}$ holds [**6**, Theorem 7.1.8]. Clearly $\widehat{p}(N_0)$ is homeomorphic to the quotient space $\widehat{K}N_0/\widehat{K}$ of a closed subgroup $\widehat{K}$ in locally compact group $\widehat{K}N_0$; hence there exists a compact normal subgroup R of type G_δ in $\widehat{K}N_0$ such that $R \subseteq N_0$ and $\widehat{p}(R)$ is zero-dimensional [**12**, Theorem 1]. Let π be the quotient mapping of $\widehat{G}$ onto $\widehat{G}/N_0$. The obvious equality $\widehat{K}N_0 = \pi^{-1}\pi(K)$ implies that $\widehat{K}N_0$ is a closed G_δ-subgroup of $\widehat{G}$ (note that π is a perfect mapping onto second-countable space $\widehat{G}/N_0$). Therefore R is of type G_δ in $\widehat{G}$. There exists $N \in \mathcal{P}$ such that $N \subseteq N_0 \cap R$. It is clear that $\widehat{p}(N) \subseteq \widehat{p}(R)$; hence $\dim \widehat{p}(N) = 0$.

Now assume that $\mathrm{ind}\, \Phi < \infty$. One can find $N^* \in \mathcal{P}$ so that $\mathrm{ind}\,(\widehat{p}(N^*) \cap B) = 0$ Indeed, if $\mathrm{ind}\, \Phi = 0$, then the inequality $\mathrm{ind}\, P \leqq \mathrm{ind}\, \Phi$ (see [**6**, Theorem 7.1.1]) implies the above assertion. Otherwise we can apply induction on $\mathrm{ind}\, \Phi$ together with Lemmas 3 and 1(c). It remains to show that if $N \in \mathcal{P}$, $P = \widehat{p}(N) \cap B$ and $\mathrm{ind}\, P = 0$, then $\mathrm{ind}\, \widehat{p}(N) = 0$. Obviously $P = w_N^{-1} \widehat{\lambda}_N(e)$, where e is the identity of $\widehat{G}$. Consequently, P is a zero-set in B and P is C-embedded in $\widehat{P} = \widehat{p}(N)$ by

Lemma 1(d). Let $\mathcal{B}$ be a base of P at the point $p(e)$ consisting of clopen subsets of P. Then the closures in $\widehat{P}$ of elements of $\mathcal{B}$ are also clopen and constitute a base of $\widehat{P}$ at $p(e)$. Hence $\operatorname{ind}(p(e), \widehat{P}) = 0$. However, being a quotient space of the group $\widehat{K}N$, $\widehat{P}$ is homogeneous. Thus, $\operatorname{ind}\widehat{P} = 0$.

The case $\operatorname{Ind}_0 \Phi < \infty$ is trivial: an easy induction with the help of Lemmas 1 and 3 gives the inequality $\operatorname{ind}\Phi \leqq \operatorname{Ind}_0 \Phi$ and the fact just proved implies an existence of $N \in \mathcal{P}$ as required. ■

PROOF of Theorem 1: By Lemma 3, the space $B = G/K$ has a σ-lattice of open mappings onto second–countable spaces. Therefore Lemmas 2 and 1 together imply the equalities $\operatorname{ind}\Phi = \operatorname{ind}\widehat{\Phi}$ and $\operatorname{Ind}_0 \Phi = \operatorname{Ind}_0 \widehat{\Phi}$ for each zero–set Φ in B, where $\widehat{\Phi} = \mathrm{cl}_{\widehat{B}}\Phi$. The quotient space $\widehat{B} = \widehat{G}/\widehat{K}$ is normal because the group $\widehat{G}$ is locally compact (see [**12**]). Hence Lemma 2 implies $\operatorname{Ind}_0 \widehat{\Phi} = \operatorname{Ind}\Phi$. Since Φ is dense and C-embedded in $\widehat{\Phi}$ (Lemma 1(b)), Corollary 7.1.18 in [**6**] implies that $\dim\Phi = \dim\widehat{\Phi}$. It remains to note that $\widehat{\Phi}$ is a zero–set in $\widehat{B}$ (Lemma 1(e)), and to apply the equality $\dim\widehat{\Phi} = \operatorname{ind}\widehat{\Phi} = \operatorname{Ind}\widehat{\Phi}$, which be proved below (informally, it is contained in [**12**]).

Assume that one of the numbers $\dim\widehat{\Phi}$, $\operatorname{ind}\widehat{\Phi}$ is finite. Since $\dim\Phi = \dim\widehat{\Phi}$ and $\operatorname{ind}\Phi = \operatorname{ind}\widehat{\Phi}$, Lemma 4 implies that there exists a closed normal subgroup $N^* \in \mathcal{P}$ of $\widehat{G}$ such that $\operatorname{ind}\widehat{p}(N^*) = 0$, where $\widehat{p} : \widehat{G} \to \widehat{G}/\widehat{K}$ is the quotient mapping. One can assume that $\widehat{G}$ is a projective–Lie group in the sense of [**9**], because every locally compact group contains an open projective–Lie subgroup (see [**16**]). By Theorem 1 of [**12**] the space $\widehat{B} = \widehat{G}/\widehat{K}$ is the limit of a well–ordered spectrum $S = \{\widehat{B}_\alpha, \varphi_{\beta,\alpha} : \alpha < \beta < \tau\}$, where mappings $\varphi_{\beta,\alpha}$'s are open and "onto", a mapping $\varphi_{\alpha+1,\alpha}$ is a locally trivial fibering with a fiber $M_{\alpha+1}$, a compact manifold $(\alpha < \tau)$, and B_0 is a second–countable manifold. An existence of an $N^* \in \mathcal{P}$ with $\operatorname{ind}\widehat{p}(N^*) = 0$ implies that the spectrum S can be chosen so that all fibers $M_{\alpha+1}$'s are zero–dimensional, i.e., finite. The proof of Theorem 2 of [**12**] implies that the limit projection $\varphi_0 : \widehat{B} \to \widehat{B}_0$ is a locally trivial fibering with fibers homeomorphic to the Cantor cube D^τ. Since $\widehat{\Phi}$ is a zero–set in $\widehat{B}$, the same is true for $\Phi_0 = \widehat{p}^{-1}(\widehat{\Phi})$ in $\widehat{G}$. Consequently there exists $N_0 \in \mathcal{P}$ such that $N_0 \subseteq N^*$ and $\Phi_0 = \pi_0^{-1}\pi_0(\Phi_0)$, where $\pi_0 : \widehat{G} \to \widehat{G}/N_0$. One can start a "decomposition of $\widehat{B}$ into the spectrum S" with quotient space $\widehat{B}_0 = \widehat{G}/N_0\widehat{K}$. Then the limit projection $\varphi_0 : \widehat{G}/\widehat{K} \to \widehat{G}/N_0\widehat{K}$ has the property $\widehat{\Phi} = \varphi_0^{-1}\varphi_0(\widehat{\Phi})$. Thus, the restriction of φ_0 to $\widehat{\Phi}$ is a locally trivial fibering over a locally compact second–countable space $F = \varphi_0(\widehat{\Phi})$ with fibers homeomorphic to D^τ. Now the equality $\dim\widehat{\Phi} = \operatorname{ind}\widehat{\Phi} = \operatorname{Ind}\widehat{\Phi}$ follows from Lemma 6 of [**12**]. ■

Corollary 1. $\dim G = \operatorname{ind} G = \operatorname{Ind}_0 G = \dim\widehat{G}$ *for each locally pseudocompact group* G.

Remark 2. The conclusion of Corollary 1 cannot be extended to all closed subsets of G even if G is pseudocompact. Indeed, every Tychonoff space embeds in a pseudocompact topological group as a closed subspace. It is also useful to remember that every precompact group embeds into a pseudocompact group as a closed subgroup (apply the construction given in the proof of Theorem 2.4 of [5]). Consequently, a closed subgroup of a pseudocompact group need not be pseudocompact.

Theorem 2. *Let K be a closed subgroup of a locally pseudocompact group G. Then* $\dim G = \dim \widehat{K} + \dim G/K$*, where $\widehat{K}$ is the completion of the group K.*

PROOF : The completion $\widehat{G}$ of the group G is locally compact, whence follows the equality $\dim \widehat{G} = \dim \widehat{K} + \dim \widehat{G}/\widehat{K}$ (see [10,17]). Theorem 1 and Corollary 1 together imply $\dim G = \dim \widehat{G}$ and $\dim G/K = \dim \widehat{G}/\widehat{K}$, so we are done. ■

Corollary 2. *The dimension of a quotient space of a locally pseudocompact group G does not exceed the dimension of G. Furthermore, if K,H are closed subgroups of G and $K \subseteq H$, then* $\dim G/H \leqq \dim G/K$.

Corollary 3. *A quotient space of a zero-dimensional pseudocompact group is zero-dimensional.*

Let K be a closed subgroup of a pseudocompact group G. By Theorem 6 in [3] the Čech–Stone compactification of the quotient space G/K is homeomorphic to the quotient homogeneous space $\widehat{G}/\widehat{K}$, where $\widehat{G}$ and $\widehat{K}$ are the completions of G and K resp., $\widehat{K} = \mathrm{cl}_{\widehat{G}} K$. On the other hand, no infinite extremally disconnected compact space is homogeneous (see [1] or [4,p.69]). Since extremal disconnectedness is preserved when passing to the Čech–Stone compactification, we have proved the following.

Theorem 3. *An extremally disconnected quotient space of a pseudocompact group is finite.*

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