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On the group of isometries of the Urysohn universal metric space

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Dedicated to the memory of Zdeněk Frolík

Abstract. We show that the group mentioned in the title, equipped with the topology of pointwise convergence, is a universal topological group with a countable base.

Keywords: topological group, Urysohn space

Classification: 22A05, 54E40

Does there exist a universal topological group with a countable base, i.e. such a group G that every topological group with a countable base is isomorphic (topologically or algebraically) to a subgroup of G ? A positive answer to this question of S. Ulam was given in [1]: the group $\text{Aut } Q$ of all autohomeomorphisms of the Hilbert cube, equipped with the compact-open topology, is universal. In the present paper we apply Katětov's construction [2] of Urysohn universal metric spaces to give another example of a universal topological group with a countable base.

Let us say that a separable metric space M is Urysohn iff for any finite metric space X , any subspace $Y \subseteq X$ and any isometric embedding $f: Y \rightarrow M$ there exists an isometric embedding $\bar{f}: X \rightarrow M$ which extends f . There exists a unique (up to an isometry) complete Urysohn space [2], [3], and there exist non-complete Urysohn spaces [2]. For a metric space M , let $\text{Is } M$ be the topological group of isometries of M onto itself, equipped with the topology of pointwise convergence (which coincides with the compact-open topology on $\text{Is } M$). If M is separable, the group $\text{Is } M$ has a countable base.

Theorem. *Let U be the complete Urysohn separable metric space. Then $\text{Is } U$ is a universal topological group with a countable base, i.e. every (Hausdorff) topological group with a countable base is isomorphic to a subgroup of $\text{Is } U$.*

The proof is based on Katětov's paper [2]. Recall some definitions from [2]. For a metric space (X, d) let $E(X)$ be the set of all functions $f: X \rightarrow \mathbb{R}$ such that $|f(p) - f(q)| \leq d(p, q) \leq f(p) + f(q)$ whenever $p, q \in X$. For $f, g \in E(X)$ put $d^E(f, g) = \sup\{|f(p) - g(p)|: p \in X\}$. Then $(E(X), d^E)$ is a metric space. If $Y \subseteq X$ and $f \in E(Y)$, define $f^* \in E(X)$ as follows: for $p \in X$, let $f^*(p) = \inf\{d(p, q) + f(q): q \in Y\}$. The mapping $f \mapsto f^*$ is an isometric embedding of $E(Y)$ in $E(X)$, so we can identify $E(Y)$ with a subspace of $E(X)$. Let $E(X, \omega) = \bigcup\{E(Y): Y \subseteq X, Y \text{ is finite}\} \subseteq E(X)$. There is a natural isometric embedding $p \mapsto f_p$ of X in $E(X, \omega)$, where $f_p(q) = d(p, q)$ for $p, q \in X$, so X can be identified

with a subspace of $E(X, \omega)$. Put $X_0 = X$, $X_{n+1} = E(X_n, \omega)$. We regard each X_n as a subspace of X_{n+1} , so we get a chain $X_0 \subset X_1 \subset \dots$ of metric spaces. There is a natural metric on $X_\omega = \bigcup X_n$ which extends the metric on X_n for every n .

Every isometry $\varphi \in \text{Is } X$ extends uniquely to an isometry $E(\varphi) \in \text{Is } E(X)$ [2, Fact 1.6]. Let $\varphi^* \in \text{Is } E(X, \omega)$ be the restriction of $E(\varphi)$ to $E(X, \omega)$. The mapping $\varphi \mapsto E(\varphi)$ from $\text{Is } X$ to $\text{Is } E(X)$ need not be continuous; however, it follows from [2, Fact 1.7] that the mapping $\varphi \mapsto \varphi^*$ from $\text{Is } X$ to $\text{Is } E(X, \omega)$ is an isomorphic embedding of topological groups. For $\varphi = \varphi_0 \in \text{Is } X$ let $\varphi_{n+1} = (\varphi_n)^* \in \text{Is } X_{n+1}$. There is a unique isometry φ_ω of X_ω which extends φ_n for every n .

Lemma 1. *The topological group $\text{Is } X$ is isomorphic to a subgroup of $\text{Is } X_\omega$.*

PROOF : The mapping $\varphi \mapsto \varphi_\omega$ from $\text{Is } X$ to $\text{Is } X_\omega$ is an isomorphic embedding of topological groups. ■

If X is separable, then X_ω is Urysohn [2]. The completion of X_ω is also Urysohn [3], [2, lemma 3.3] and hence isometric to U . This proves

Lemma 2. *If X is separable, the topological group $\text{Is } X_\omega$ is isomorphic to a subgroup of $\text{Is } U$.*

Lemma 3. *Every topological group with a countable base is isomorphic to a subgroup of $\text{Is } X$ for some separable metric space X .*

Actually every topological group is isomorphic to a subgroup of $\text{Is } X$ for some metric space X . For metrizable groups this is obvious: if G is a metrizable group, there exists a left-invariant metric d on G compatible with its topology. For any $g \in G$ the left shift $x \mapsto gx$ is an isometry of (G, d) , and thus we obtain an isomorphic embedding of G in $\text{Is}(G, d)$. This proves lemma 3. ■

The theorem follows immediately from lemmas 1, 2, 3.

Question 1. *Are the topological groups $\text{Is } U$ and $\text{Aut } Q$ isomorphic?*

Question 2. *Let m be an uncountable cardinal. Does there exist a universal topological group of weight m ? Is $\text{Aut } I^m$ such a group?*

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