Ladislav Bican; Renata Binderová
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A note on flat modules

LADISLAV BICAN, RENATA BINDEROVÁ

Abstract. The Chase’s theorem on flatness of direct product of flat modules is generalized to the class of modules possessing a set of generators every element of which is annihilated by a given right ideal I such that the factormodule $R/I$ is flat.

Keywords: flat module, finite $I$ - presentation

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Throughout this note $R$ stands for associative ring with identity and all modules are unitary left or right modules. The terminology and notations will be as in [1] or [2]. The properties of flat modules presented there are used without references.

1. Definition. Let $I$ be a right ideal of a ring $R$ and $RL$ be a submodule of a free left module $R^m$. We say that $L$ is finitely $I$ - presented if there is an exact sequence

$$0 \rightarrow U \rightarrow R^p \rightarrow L \rightarrow 0 \tag{1}$$

of left modules such that the inverse image $f^{-1}(I^m)$ of the subgroup $I^m \cap L$ of $L$ is of the form $RZ + P$, where $RZ$ is a finitely generated submodule of $U$. In this case the sequence $(1)$ is said to be a finite $I$ - presentation of $L$.

2. Remark For $I = 0$ we clearly get the ordinary notion of a finitely presented module. It is well known that any rank finite free presentation of a finitely presented module is a finite presentation and so we are going to prove similar result for finitely $I$ - presented modules.

3. Lemma. Let $I$ be a right ideal of $R$ and $RL$ be a finitely $I$ - presented submodule of $R^m$. Then every rank finite free presentation of $L$ is a finite $I$ - presentation of $L$.

Proof: Let $(1)$ be a finite $I$ - presentation of $L$ and $O \rightarrow V \rightarrow R^p \rightarrow L \rightarrow O$ be another free presentation of $L$. For $R^p = \bigoplus_{i=1}^{p}Rx_i$ and $R^q = \bigoplus_{j=1}^{q}Ry_j$ define the homomorphism $\psi : R^p \rightarrow R^q$ by setting $\psi(x_i) = w_i$ where $f(x_i) = g(w_i), i = 1, \ldots, p$. Choosing elements $z_j \in R^p$ such that $f(z_j) = g(y_j), j = 1, \ldots, q$, we have $g(\psi(z_j) - y_j) = f(z_j) - g(y_j) = 0$ and consequently $\psi(z_j) = y_j + v_j$ for some $v_j \in V$. By the hypothesis $f^{-1}(I^m) = RZ + P$ where $RZ = \langle z_1, \ldots, z_n \rangle$. Setting $RW = \langle \psi(z_1), \ldots, \psi(z_n), v_1, \ldots, v_q \rangle$ we obviously have $W \subseteq V$ and so $W + I^q \subseteq g^{-1}(I^m)$. On the other hand, for $x \in g^{-1}(I^m), x = \sum_{j=1}^{q}r_jz_j$, it is $f\left(\sum_{j=1}^{q}r_jz_j\right) = g(x) \in I^m$ and so $y = \sum_{j=1}^{q}r_jz_j = \sum_{i=1}^{n}s_ix_i + d, d \in P$. Summarizing we have $x = \psi(y) - \sum_{j=1}^{q}r_jv_j = \sum_{i=1}^{n}s_i\psi(z_i) - \sum_{j=1}^{q}r_jv_j + \psi(d) \in W + I^q$ as desired. \[\square\]
4. Definition. For a right ideal \( I \) of \( R \) define \( \mathcal{M}(I) \) to be the class of all right \( R \)-modules \( M \) having a set of generators \( \{ m_\alpha | \alpha \in A \} \) such that \( I \subseteq (O : m_\alpha) \) for each \( \alpha \in A \).

5. Theorem. The following conditions are equivalent for a right ideal \( I \) of \( R \):
   
   (a) \( R/I \) is flat and if \( \{ M_c | c \in C \} \subseteq \mathcal{M}(I) \) is arbitrary then \( \prod_{c \in C} M_c \) is flat;
   
   (b) For any collection \( \{ B_c | c \in C \} \) of sets the module \( ((R/I)^{(B_c)})^C \) is flat;
   
   (c) For any set \( C \) the cartesian power \( (R/I)^C \) is flat;
   
   (d) Every finitely generated left submodule of a free module of finite rank is finitely \( I \)-presented;
   
   (e) Every finitely generated left ideal of \( R \) is finitely \( I \)-presented.

Proof: The implications (a) \( \Rightarrow \) (b) \( \Rightarrow \) (c) and (d) \( \Rightarrow \) (e) are obvious.

(b) \( \Rightarrow \) (a). Every \( M_c \in \mathcal{M}(I) \), \( c \in C \) has a free presentation \( O \to K_c \to (R/I)^{(B_c)} \to M_c \to O \) for a suitable set \( B_c \). Then we have the exact sequence \( O \to \prod_{c \in C} K_c \to ((R/I)^{(B_c)})^C \to \prod_{c \in C} M_c \to O \) and it is easy to see that \( \left( \prod_{c \in C} K_c \right) J = \left( \prod_{c \in C} K_c \right) \cap ((R/I)^{(B_c)})^C J \) for every (finitely generated) left ideal \( J \) of \( R \).

(c) \( \Rightarrow \) (d). Let \( RL = (u_1, \ldots, u_p) \) be a finitely generated left submodule of \( R^m \), \( u_i = (u_{i1}, \ldots, u_{im}), i = 1, \ldots, p, F = \bigoplus_{i=1}^p Rx_i \) be a free left \( R \)-module and (1) be the free presentation of \( L \) with \( f \) given by \( f(x_i) = u_i, i = 1, \ldots, p \). Taking \( x \in K = f^{-1}(I^m) \) arbitrarily, we have a unique expression \( x = \sum_{i=1}^p a_i(x)x_i \) and so \( a_i \in R^K, i = 1, \ldots, p \). Further, \( f(x) = \sum_{i=1}^p a_i(x)u_i = (\sum_{i=1}^p a_i(x)u_{ij})_{j=1}^m \), which yields \( \sum_{i=1}^p a_i(x)u_{ij} \in I \) for each \( j = 1, \ldots, m \). Defining \( \bar{a}_i \in (R/I)^K \) naturally by \( \bar{a}_i(x) = a_i(x) + I \) we have \( \sum_{i=1}^p \bar{a}_i(x)u_{ij} = 0 \) in \( (R/I)^K \) for each \( j = 1, \ldots, m \). By flatness there are \( \bar{b}_k \in (R/I)^K \) and \( r_{ki} \in R \) such that \( \bar{a}_i = \sum_{k=1}^n \bar{b}_kr_{ki} \) and \( \sum_{i=1}^p r_{ki}u_{ij} = 0 \) for all \( k = 1, \ldots, n, j = 1, \ldots, m \). This yield \( \sum_{i=1}^n r_{ki}u_{ij} = 0 \) for each \( k = 1, \ldots, n \) and \( a_i = \sum_{k=1}^n b_kr_{ki} + c_i, i = 1, \ldots, p \), where \( b_k(x) \) is any representative of \( \bar{b}_k(x) \) and \( c_i \in I^K \). Setting

\[
  z_k = \sum_{i=1}^p r_{ki}x_i \in F
\]

and \( RZ = (z_1, \ldots, z_n) \), we have \( Z \subseteq U \) since \( f(z_k) = \sum_{i=1}^p r_{ki}u_i = 0 \) and consequently \( Z + IF \subseteq K \). Conversely, for \( x \in K \) we have \( x = \sum_{i=1}^p a_i(x)x_i = \sum_{i=1}^p \left( \sum_{k=1}^n b_k(x)r_{ki} + c_i(x) \right)x_i = \sum_{k=1}^n b_k(x)z_k + \sum_{i=1}^p c_i(x)x_i \in Z + IF \) and (1) is a finite \( I \)-presentation of \( L \).

(e) \( \Rightarrow \) (b). Let \( v_1, \ldots, v_p \in ((R/I)^{(B_c)})^C \) be elements such that \( \sum_{i=1}^p v_iu_i = 0, u_i \in R \). Denote \( L = \sum_{i=1}^p Ru_i \) the left ideal of \( R \) and (1) be its free presentation with \( F = \bigoplus_{i=1}^p Rx_i \) and \( f(x_i) = u_i \). By the hypothesis \( L \) is finitely \( I \)-presented and so by lemma 3 \( f^{-1}(I) \subseteq RZ + IF \), where \( RZ = (z_1, \ldots, z_n) \subseteq V \) and \( z_k \) are of the form (2). Take \( c \in C \) arbitrarily. Then \( v_i(c) \) lies in \( (R/I)^{(B_c)} \) and so...
A note on flat modules 199

vi(c) = (d(c) + I) for some d(c) £ R, £ Bc, with d(c) \not\in I for a finite number of \alpha's, only. So, let A \subseteq B_c be the finite set of all \alpha \in B_c for which d(c) \not\in I for some i = 1, \ldots, p. Now \sum_{i=1}^p d(c)u_i = (\sum_{i=1}^p d(c)u_i + I) \alpha = 0 and so \sum_{i=1}^p d(c)u_i \in I for each \alpha \in A. Consequently \sum_{i=1}^p d(c)x_i = \sum_{i=1}^p d(c)x_i \in f^{-1}(I) for each \alpha \in A and we can write \sum_{i=1}^p d(c)x_i = \sum_{k=1}^n q^{(c)}x_k + \sum_{i=1}^p h^{(c)}x_i with h^{(c)} \in I. Using (2) we get \sum_{i=1}^p d(c)x_i = \sum_{k=1}^n q^{(c)}x_k + \sum_{k=1}^n h^{(c)}x_i and consequently d(c) = \sum_{k=1}^n q^{(c)}rk_i + h^{(c)} \alpha, \alpha \in A, i = 1, \ldots, p. For every k = 1, \ldots, n set w^{(c)}(c) = q^{(c)} + I if \alpha \in A and w^{(c)}(c) = I otherwise. Then w^{(c)}(c) \in (R/I)^{(B_c)} and since \sum_{k=1}^n w^{(c)}(c)r_{k_i} = \sum_{k=1}^n (q^{(c)} + I)r_{k_i} = d(c) + I for each \alpha \in A and \sum_{k=1}^n w^{(c)}(c)r_{k_i} = I for \alpha \in B_c \setminus A, we see that \sum_{k=1}^n w^{(c)}(c)r_{k_i} = v_i(c) and hence \sum_{k=1}^n w^{(c)}(c)r_{k_i} = v_i, i = 1, \ldots, p. Moreover, by (2) it is \sum_{i=1}^p r_{k_i}u_i = f(z_k) = 0 which shows that ((R/I)^{(B_c)})^c is flat and the proof is complete.

At the end of this note we list some conditions equivalent to the flatness of a homomorphic image of a given cyclic flat right R - module.

6. Proposition. Let I \subseteq J be right ideals of R, R/I flat. The following conditions are equivalent:

(a) R/J is flat;
(b) For every left ideal L of R the equality JL + I = (J \cap L) + I holds;
(c) For each v \in J there are y \in J and u \in I with v = yv + u;
(d) For each v \in J there exists a homomorphism f : R \to J with f(v) = v + u for some u \in I;
(e) For any elements v_1, \ldots, v_n \in J there exists a homomorphism f : R \to J with f(v_i) = v_i + u_i for some u_i \in I, i = 1, \ldots, n;
(f) For all elements a_i, q_i \in R, i = 1, \ldots, m with \sum_{i=1}^m a_i q_i \in J there exist elements p_i \in R such that p_i - a_i \in J for each i = 1, \ldots, n and \sum_{i=1}^m p_i q_i \in I;
(g) For all elements a_i, q_{ij} \in R, i = 1, \ldots, m, j = 1, \ldots, n with \sum_{i=1}^m a_i q_{ij} \in J there exist elements p_i \in R such that p_i - a_i \in J for each i = 1, \ldots, m and \sum_{i=1}^m p_i q_{ij} \in I for each j = 1, \ldots, n.

Proof: (a) \Rightarrow (b). Obvious.

(b) \Rightarrow (c). Setting L = Rv for v \in J we have v \in J \cap L \subseteq JL + I, so that v = \sum_{k=1}^n j_k r_{k_i} v + u where u \in I and y = \sum_{k=1}^n j_k r_{k_i} \in J.

(c) \Rightarrow (d). Definining f : R \to J by f(1) = y we have f(v) = yv = v - u.

(d) \Rightarrow (e). The case m = 1 is clear and we shall induct on m. Taking g : R \to J with g(v_{m+1}) = v_{m+1} - s_{m+1}, s_{m+1} \in I, we have g(v_i) = v_i - s_i, s_i \in J, i = 1, \ldots, m. There is t : R \to I with t(s_{m+1}) = s_{m+1}, R/I being flat. Setting z_i = (1-t)(s_i), the induction hypothesis gives the existence of h : R \to J with h(z_i) - z_i \in I. An easy computation now shows that f = 1 - (1 - h)(1-t)(1-g) has all desired properties.

(e) \Rightarrow (g). There is f : R \to J with f(\sum_{i=1}^m a_i q_{ij}) = \sum_{i=1}^m a_i q_{ij} - u_j, u_j \in I, j = 1, \ldots, n. Now the elements p_i = a_i - f(a_i), i = 1, \ldots, m, have desired properties.

(g) \Rightarrow (f). Obvious.

(f) \Rightarrow (b). For 1.v \in J \cap L there is p \in R with 1 - p \in J and pv \in I showing that v = (1 - p)v + pv \in JL + I.
(b) ⇒ (a). Every element \( v \in J \cap L \subseteq JL + I \) can be written in the form \( v = x + i, x \in JL, i \in I \). But then \( i = v - x \in J \cap L \cap I = I \cap L = IL \subseteq JL \) gives \( v \in JL \).

7. Corollary. Let \( I \) be a two-sided ideal of \( R, R/I \) right flat. The following conditions for a right ideal \( J \) of \( R \) containing \( I \) are equivalent:

(a) \( R/J \) is a flat \( R \)-module;
(b) For each \( v \in J \) there is a homomorphism \( \bar{f} : R/I \to J/I \) with \( \bar{f}(v+I) = v+I \);
(c) \( R/J \) is a flat \( R/I \)-module;
(d) For each left ideal \( L \) of \( R \) containing \( I \) it holds \( J \cap L \subseteq JL + I \).

Proof: (a) ⇒ (b). By proposition 6 there is \( f : R \to J \) with \( f(v) = v + u \) for some \( u \in I \). Since \( f(I) \subseteq I \), \( f \) induces naturally \( \bar{f} : R/I \to J/I \) which has the desired property.

(b) ⇒ (a). Let \( \bar{f}(1 + I) = y + I \). Defining \( f : R \to J \) by \( f(1) = y \) we have \( f(v) = v + u, u \in I \), and proposition 6 applies.

The equivalence of the conditions (b) and (c) is well-known. Assuming (d) we have \( J/I.L/I = (JL + I)/I = (J \cap L)/I = (J/I) \cap (L/I) \) which is equivalent to (c) while the converse implication follows by proposition 6.(b).

References


Dept. of Mathematics, Charles University, Sokolovská 83, 18600 Prague, Czechoslovakia

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