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Global branching for discontinuous problems

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In memory of Svatopluk Fučík and Rudolf Švarc

Abstract. The paper deals with a class of elliptic problems with discontinuous nonlinearities both in bounded and unbounded domains. The existence of a global branch of positive solutions is proved and an application to Plasma Physics is discussed.

Keywords: Global bifurcation, elliptic free boundary problems, plasma physics

Classification: 35B32, 35R35, 82A45

1. Introduction. The main purpose of this paper is to establish the existence of global branch of positive solutions for some elliptic problems with discontinuous nonlinearities.

In §2 we deal with problems like

\[(1a) \quad -\Delta u = f(u - a) \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega\]

where \(\Delta\) is the Laplace operator, \(a \in \mathbb{R}\), \(\Omega\) is a bounded domain in \(\mathbb{R}^N\) with suitable symmetry and \(f = f(s)\) has a discontinuity at \(s = 0\).

Taking \(a\) as the bifurcation parameter, we will prove (Theorems 1 and 1') that there is a global branch \(S\) of pairs \((a, u)\) with \(u > 0\) solution of \((1a)\), bifurcating from \(a = 0, u = 0\) and having a turning point at some value \(a_0 > 0\), in such a way that \((1a)\) has at least two positive solutions for all \(a \in ]0, a_0[\).

In §3 we study problems like

\[(2a) \quad -\Delta u = f(u - a) \quad \text{in} \quad \mathbb{R}^N, \quad u(x) \to 0 \quad \text{as} \quad |x| \to \infty\]

with \(N \geq 3\).

Problem \((2a)\) is approximated by Dirichlet problems such as \((1a)\) with \(\Omega = B(R) = \{x \in \mathbb{R}^N : |x| < R\}\) where \(|x|\) denotes the Euclidean norm.

By a limiting procedure based upon an a priori estimate of the free boundary \(\{u = a\}\), we show (Theorem 8) that \((2a)\) possesses an unbounded branch \(S\) of positive solutions bifurcating from \((0, 0)\).

Moreover, we prove (Theorem 10) that for all \(a > 0\) there is a solution \(u\) of \((2a)\) such that \((a, u) \in S\).
It is well known that several problems in Plasma Physics give rise to equations with discontinuous nonlinearities, such as (1a) or (2a) (see [2], [8]). As an example, an application in this frame is shortly discussed in the sequel.

We also recall that problems with discontinuous nonlinearities have been studied by several points of view: see e.g. [1], [5], [7], [13] for a general set up and [4], [6], [9] for problems in Vortex Theory.

2. Problems in a bounded domain. In this section we study problem (1a) with \( \Omega \) bounded domain in \( \mathbb{R}^N, N \geq 2 \), with smooth boundary \( \partial \Omega \).

Even if our results hold true for any Steiner symmetric set \( \Omega \) (see Theorem 1' below) we prefer to start dealing with the case in which \( \Omega \) is the ball \( B(R) \). Actually, this is the kind of results we need in the following section where we study problem (2a) on all \( \mathbb{R}^N \).

Let us denote by \( \lambda(R) \) the first eigenvalue of \(-\Delta\) on \( B(R) \) with zero Dirichlet boundary condition and let \( \phi \) be such that

\[-\Delta \phi = \lambda(R) \phi, \quad \int_{B(R)} \phi^2 = 1, \quad \phi > 0.\]

Consider the boundary value problem

\[(3a-R) \quad -\Delta u = f(u - a) \text{ in } B(R), \quad u = 0 \text{ on } \partial B(R)\]

with \( a \in \mathbb{R} \) and \( f : \mathbb{R} \to \mathbb{R} \) satisfying

- (f1) \( f(s) = 0 \ \forall s \leq 0; \)
- (f2) \( f \in C^{0,\alpha}(\mathbb{R}^+, \mathbb{R}^+) \) and is non-decreasing;
- (f3) there exists \( d > 0, c_1 < \lambda(R) \) and \( c_2 \leq 0 \), such that \( d \leq f(s) \leq c_1 s + c_2 \) for all \( s > 0 \).

By a positive solution of (3a-R) we mean an \( u(x) \in C_0^1(\overline{B(R)}) \) such that: (i) \( u > 0 \) in \( B(R) \); (ii) letting \( T(a) = \{x \in B(R) : u(x) = a\}, u \in C^2(B(R) - T(a))\); and (iii) \( u \) solves pointwise (3a-R) on \( B(R) - T(a) \).

Notice that, by (f1) and the maximum principle, any positive solution \( u \) which is \( \not\equiv 0 \) is strictly positive: in fact, \( u(x) > a \) for some \( x \in B(R) \); moreover the discontinuity is not influent for \( a \leq 0 \) and, taking into account (f3), (3a-R) has nontrivial, positive, classical solutions for such range of parameters.

Set \( r = |x| \ (x \in \mathbb{R}^N), \ E = C(\overline{B(R)}) \) with norm \( \cdot \, |_{\infty, R} \); \( V(b) = \{u \in E : |u|_{\infty, R} < b\} \); for \( u \) radial, \( u' = \frac{du}{dr} \); \( \Sigma(R) = \{(a, u) \in \mathbb{R}^+ \times E : u \text{ is a positive solution of (3a-R)}\} \); \( \Sigma_0(R) = \{u : u \text{ is a positive solution of (3a-R) for } a = 0\} \).

**Theorem 1.** Let (f1-2-3) hold. Then there exists a global branch \( S(R) \subseteq \text{cl}(\Sigma(R)) \) (i.e. a closed, connected component of \( \text{cl}(\Sigma(R)) \)) such that:

(i) \( (0, 0) \in S(R) \) and if \( (a, 0) \in S(R) \) then \( a = 0; \)
(ii) \( S(R) \) is bounded in \( \mathbb{R}^+ \times E \) and \( S(R) \cap \Sigma_0(R) \neq \emptyset; \)
(iii) if \((a, u) \in S(R)\), with \(0 < |u|_{\infty, R} \text{ small}\), then \(a > 0\). As a consequence, there is \(a_0 > 0\) such that for all \(a \in (0, a_0]\) \((3a - R)\) has at least two distinct positive solutions with \((a, u) \in S(R)\).

(iv) any \(u \in S(R)\) is radial, \(u'(r) < 0\) \(\forall r > 0\) and \(|T(a)| = \text{meas}[T(a)] = 0\).

**Proof:** The proof is divided in 2 steps. For simplicity of notations the dependence on \(R\) is understood and omitted during all the proof, but where a precisation is worthwhile.

**Step 1.** (Smooth approximation of \((3a - R)\)). For \(\epsilon > 0\), let \(f_\epsilon \in C^{0,\alpha}(\mathbb{R}, \mathbb{R})\) be defined by setting

\[
f_\epsilon(s) = f(s) \quad \text{for} \quad s \leq -\epsilon \quad \text{and} \quad s > 0; \quad f_\epsilon(s) = f(0+)(\frac{s}{\epsilon} + 1) \quad \text{for} \quad -\epsilon \leq s \leq 0;
\]

and consider the smooth problems

\[
(4a - \epsilon) \quad -\Delta u = f_\epsilon(u - a) \text{ in } B(R), \quad u = 0 \text{ on } \partial B(R)
\]

Let \(G = (-\Delta)^{-1}\) with zero Dirichlet boundary conditions, set \(\Phi(\epsilon, a, u) = u - Gf_\epsilon(u - a)\) and denote by \(\Sigma(\epsilon)\) the set of \((a, u) \in \mathbb{R}^+ \times E\) such that \(\Phi(\epsilon, a, u) = 0\). If \((a, u) \in \Sigma(\epsilon)\), then, by regularity theory, \(u\) is a positive, classical solution of \((4a - \epsilon)\).

First of all we note that \(\exists a^* = a^*(R) > 0\) and \(b^* = b^*(R) > 0\) such that \(\Sigma(\epsilon) \subset [0, a^*] \times \mathcal{V}(b^*)\) for all \(\epsilon > 0\). In fact, by (f3) there is \(a^* > 0\) such that \(f_\epsilon(u - a) < c_1 u\) for all \(a \geq a^*\). If \(u\) is any (positive) solution of \((4a - \epsilon)\) with \(a > a^*\), one has:

\[
\lambda(R) \int_{B(R)} u \phi = \int_{B(R)} f_\epsilon(u - a) \phi < c_1 \int_{B(R)} u \phi
\]

a contradiction.

We need:

**Lemma 2.** There is a global branch \(S(\epsilon) \subset \Sigma(\epsilon)\) such that \((\epsilon, 0) \in S(\epsilon)\) and if \((a, 0) \in S(\epsilon)\) then \(a = \epsilon\).

**Proof:** We claim that

\[
(5) \quad \text{ind}(\Phi(\epsilon, a, .), 0) = 1 \quad \forall \ a > \epsilon
\]

\[
(6) \quad \text{ind}(\Phi(\epsilon, a, .), 0) = 0 \quad \forall \ a^* < \epsilon
\]

Here \(\text{ind}\) denotes the Leray-Schauder index.

To see (5) we take \(0 < \epsilon < a\) and consider the homotopy \(H(t, u) = u - tGf_\epsilon(u - a), \ t \in [0, 1]\). \(H\) is admissible on \(\mathcal{V}(r), \ r > 0\) small enough, because, otherwise there are sequences \(|u_n|_{\infty} \to 0\) and \(t_n \in [0, 1]\) such that \(H(t_n, u_n) = 0\). Letting \(v_n = \frac{u_n}{|u_n|_{\infty}}\) it follows

\[
-\Delta v_n = t_n \frac{f_\epsilon(u_n - a)}{u_n} v_n
\]
Then (7) and \(|v_n|_\infty = 1\) imply \(v_n \to \bar{v}\) in \(E\) and \(|\bar{v}|_\infty = 1\). But for \(n\) large \(u_n(x) < a - \epsilon\), hence \(f_\epsilon(u_n - a) \equiv 0\); therefore, passing to the limit into (7) one finds \(-\Delta \bar{v} = 0\), a contradiction, and (5) follows.

Next let \(a < \epsilon\). We prove (6) showing that there is \(b > 0\) such that \(\Phi(\epsilon,a,u) \neq 0\) for all \(u \in V(b)\). In fact, otherwise, there is a sequence \(u_n \in E\), \(u_n > 0\), such that \(u_n \to 0\) and satisfies \(-\Delta u_n = f_\epsilon(u_n - a)\). Letting \(u_n = t_n \phi + w_n\) with \(\int_\Omega \phi w_n = 0\), it follows, for \(n\) large

\[
\lambda(R)t_n \phi - \Delta w_n = f_\epsilon(u_n - a) = f(0+)(\frac{u_n - a}{\epsilon} + 1).
\]

Multiplying by \(\phi\), integrating and letting \(t_n \to 0\) one finds

\[
0 = f(0+)(1 - \frac{a}{\epsilon}) \int_\Omega \phi,
\]

a contradiction, proving (6).

By standard arguments in global bifurcation theory (see, e.g. \[3\]), (5) and (6) yield the existence of a global branch of positive solutions of (4a-c) emanating from \((\epsilon,0)\).  

**Remarks 3.**

(a) As remarked above, \(S(\epsilon) \subset [0,a^*] \times V(b^*)\). Then, by regularity, it follows readily that \((a,u) \in S(\epsilon)\) are bounded in \(C^{1,\alpha}\), uniformly in \(\epsilon\).

(b) By the well known result of \[10\], the positive solutions of (4a-\(\epsilon\)) are radial: \(u = u(r)\) and \(u'(r) < 0, \forall r > 0\).

**Step 2.** (Limit as \(\epsilon \to 0\)). To obtain the branch \(S\) of solutions of (3a-R) we shall let \(\epsilon \to 0\) and show that \(S(\epsilon)\) converges (in a suitable sense) to \(S\). This will be obtained, as in \[6\], by means of the following topological lemma:

**Lemma 4** ([14, Thm.9.1]). Let \(X\) be a metric space and let \(S_n\) be a sequence of connected subsets of \(X\). Let

(i) \(\liminf(S_n) \neq \emptyset\);

(ii) \(\cup S_n\) is precompact.

Then \(S =: \limsup(S_n)\) is (non empty) compact and connected.

In our case we take \(X = \mathbb{R}_+ \times E, \epsilon = \frac{1}{n}\) and \(S_n = S(\frac{1}{n})\). Lemma 2 implies that \(S_n\) is connected and \((0,0) \in \liminf(S_n)\). Moreover, using Remark 3-(a) it is easy to see that (ii) holds. By Lemma 4, it follows that \(S = S(R) \neq \emptyset\) is compact and connected. In addition there results \(S(R) \subset [0,a^*] \times V(b^*)\).

Using (f1), Remark 3-(a) and regularity, it follows readily that for any \(u \in S\) there results \(u \in C^2(B(R) - T(a)) \cap C^1(B(R))\).

As a consequence of Remark 3-(b), one has that \(u = u(r)\) and \(u'(r) \leq 0, \forall r > 0\), for any \(u \in S(R)\). To show that \(u'(r) < 0, \forall r > 0\) one uses the maximum principle applied to \(u'(r)\), similarly than in [6]. As a consequence \(|T(a)| = 0\) and (iv) follows.

The same argument used for (6) allows us to prove (iii). Roughly, if \(-\Delta u_n = f(u_n)\) with \(u_n \in S\) and \(|u_n|_\infty \to 0\), it follows
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\[ \lambda(R) \int_{B(R)} u_n \phi = \int_{B(R)} f(u_n) \phi \]

Passing to the limit one finds \( 0 = \int_{B(R)} f(0^+) \phi \), a contradiction.

By (iii) and \( S(R) \subset [0,a^*] \times V(b^*) \) it follows that \( S \cap \Sigma_0 \neq \emptyset \), yielding (ii).

This completes the proof of Theorem 1. \( \blacksquare \)

As anticipated before, Theorem 1 holds in greater generality.

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N(N \geq 2) \), with smooth boundary \( \partial \Omega \). We suppose that \( \Omega \) is Steiner symmetric with respect to the hyperplane \( \{ x_1 = 0 \} \), say. For shortness, we will simply say that \( \Omega \) is Steiner symmetric. For \( u \in H^1_0(\Omega) \), \( u(x) > 0 \), we denote by \( u^*(x) \) the Steiner symmetrization (with respect to \( \{ x_1 = 0 \} \)) of \( u \). For definitions concerning symmetrization, see, for ex., [12, § II.1-f].

Let \( \lambda(\Omega) \) be the first eigenvalue of \( -\Delta \) on \( H^1_0(\Omega) \), and suppose that \( f \) satisfies (f1-2) and (f3) with \( \Omega \) (resp. \( \lambda(\Omega) \)) instead of \( B(R) \) (resp. \( \lambda(R) \)). Moreover, let \( S(\Omega), \Sigma(\Omega) \) and \( \Sigma_0(\Omega) \) denote the sets \( S(R), \Sigma(R) \) and \( \Sigma_0(R) \), respectively, with \( \Omega \) instead of \( B(R) \).

**Theorem 1'**. Let \( \Omega \) be Steiner symmetric and suppose (f1-2-3) hold. Then there exists a global branch \( S(\Omega) \subset \text{cl}(\Sigma(\Omega)) \) (i.e. a closed, connected component of \( \Sigma(\Omega) \)) such that:

(i) \( (0,0) \in S(\Omega) \) and if \( (a,0) \in S(\Omega) \) then \( a = 0 \);

(ii) \( S(\Omega) \) is bounded in \( \mathbb{R}^+ \times E \) and \( S(\Omega) \cap \Sigma_0(\Omega) \neq \emptyset \);

(iii) if \( (a,u) \in S(\Omega) \), with \( 0 < |u|_\infty \) small, then \( a > 0 \). As a consequence, there is \( a_0 > 0 \) such that for all \( a \in ]0,a_0[ \) (1a) has at least two distinct solutions with \( (a,u) \in S(\Omega) \).

(iv) any \( u \in S(\Omega) \) is Steiner symmetric and \( \partial u/\partial x_1 < 0 \ \forall x_1 > 0 \) and \( |T(a)| = \text{meas}[T(a)] = 0 \).

The proof of Theorem 1' is the same as that of Theorem 1 (with obvious changes) and is left to the reader.

**Remarks 5.** (i) Theorem 1' should be compared with the multiplicity results of [5], which are variational in nature.

(ii) The preceding arguments provide a complete proof of some results stated in [2], where we refer for applications to the Grad-Shafranov equation.

(iii) For a global result concerning (1a) in a multivalued sense, see [1].

**Example 6.** Following [8] we consider an ionized gas confined in an electrically insulated cylinder having as cross section a bounded domain \( \Omega \subset \mathbb{R}^2 \).

Denote by \( v \) the temperature of the gas, by \( \sigma = \sigma(v) \) its electrical conductivity and by \( \delta \) its temperature of discharge. Since the gas is ionized, \( \sigma \) has a simple discontinuity at \( v = \delta \): precisely we suppose that

\[ \sigma(v) = 0 \quad \text{for} \quad v \leq \delta, \]

\[ \sigma(v) > 0 \quad \text{and continuous for} \quad v > \delta. \]
Assuming that the electric field $\mathcal{E}$ is constant and directed along the axis of the cylinder, and up to a normalization of the constants, such as the thermal conductivity and $|\mathcal{E}|$, one is led to the equations

(8) \[-\Delta v = 0 \quad \text{in} \quad \{x \in \Omega : v(x) \leq \delta\}\]

(9) \[-\Delta v = \sigma(v) \quad \text{in} \quad \{x \in \Omega : v(x) > \delta\}\]

together with the boundary condition

(10) \[v = v_0 \quad \text{on} \quad \partial\Omega\]

where $v_0 \in \mathbb{R}$. Let us assume that $v_0 \leq \delta$, otherwise the problem is unaffected by the discontinuity.

In order to evidence the dependence on $\delta$ and $v_0$, it is convenient to set $u = v - v_0$ and $a = \delta - v_0 (\geq 0)$.

Introducing the Heaviside function $H : \mathbb{R} \rightarrow \mathbb{R}$ defined by

\[H(s) = 0 \quad \forall \ s \leq 0, \quad H(s) = 1 \quad \forall \ s > 0,\]

we set

(11) \[f(s) = H(s)\sigma(s + \delta).\]

With these notations, (8-9-10) become:

\[-\Delta u = f(u - a) \quad \text{in} \quad \Omega, \ u = 0 \quad \text{on} \quad \partial\Omega.\]

This is exactly problem (1a) and the preceding bifurcation results apply.

3. Global Bifurcation for Problems in $\mathbb{R}^N$. In this section we deal with global branching for problem (2a).

As usual, (2a) is approximated by Dirichlet problems (3a-R). The existence of a global branch of positive solutions of (2a) will be established with another application of Lemma 4, letting $R \rightarrow \infty$. The meaning of positive solution of (2a) is the same given for those of (3a-R).

We note explicitly that in all this section $N \geq 3$.

Here we suppose that $f$ satisfies (f1-2) and

(f4) There exist $c \geq d > 0$ such that $d \leq f(s) \leq c$ for all $s > 0$.

Fixed $a, R > 0$, let $(a, u_R) \in S(R)$. Set $T_R(a) = \{x \in \mathbb{R}^N : u_R(r) = a\}$ and denote by $\rho = \rho(R, a)$ the radius of the sphere $T_R(a)$. For $v \in H_0^1(B(R))$, respectively $u \in D^{1,2}(\mathbb{R}^N)$, we set

\[\|v\|^2_R = \int_{B(R)} |\nabla v|^2 \quad \text{and} \quad \|u\|^2 = \int_{\mathbb{R}^N} |\nabla u|^2.\]

In order to pass to the limit as $R \rightarrow \infty$ we need to estimate $\rho(R, a)$. 

Lemma 7. If \((f_1-2-4)\) hold then \(\rho = \rho(R, a)\) satisfies:

\[
kd^2\rho^{N+2} \leq \int_{\{u_R > a\}} |\nabla u|^2 \leq kc^2\rho^{N+2},
\]

where \(k = \frac{\omega_{N-1}}{N^2(N+2)}\) and \(\omega_{N-1}\) denotes the measure of the unit sphere \(\partial B(1)\).

**Proof:** Being a radial solution of \((3a-R)\) \(u = u_R(r)\) satisfies

\[
\frac{d^2u}{dr^2} + \frac{N - 1}{r} \frac{du}{dr} + f(u - a) = 0,
\]

namely

\[
(r^{N-1}u')' = -r^{N-1}f(u - a).
\]

By this and the fact that \(f = 0\) for \(s \leq 0\), there results

\[
u_R(r) = a + \int_r^\rho s^{1-N}ds \int_0^s t^{N-1}f(u(t) - a)dt, \quad \text{for} \quad 0 \leq r \leq \rho;
\]

\[
u_R(r) = \frac{a}{1 - \left(\frac{\rho}{R}\right)^{N-2} - \left(\frac{\rho}{R}\right)^{N-2}}, \quad \text{for} \quad \rho \leq r \leq R.
\]

with \(\rho = \rho(R, a)\) determined by

\[
u'\left(\rho^-\right) = \nu'\left(\rho^+\right).
\]

By (13) we deduce:

\[
\int_{\{u_R > a\}} |\nabla u_R|^2 = \omega_{N-1} \int_0^\rho s^{1-N}ds \int_0^s t^{N-1}f(u_R(t) - a)dt|^2
\]

Using (14) and (f4) the lemma follows.

We denote by \(\Sigma\) the set of pairs \((a, u) \in \mathbb{R}^+ \times D^{1,2}(\mathbb{R}^N)\) such that \(u\) is a positive solution of \((2a)\) and by \(T(a)\) the set \(\{x \in \mathbb{R}^N : u(x) = a\}\).

We are now in position to state:

**Theorem 8.** Let \(N \geq 3\) and \((f_1-2-4)\) hold. Then there is a global, unbounded branch \(S \subset cl(\Sigma)\) such that:

(i) \((0, 0) \in S;\)

(ii) if \((a, u) \in S\) then \(u\) is a radial, positive solution of \((1a)\), \(u'(r) < 0 \forall r > 0\) and \(|T(a)| = 0\).

**Proof:** Since the proof is very similar to that of [6], based again on Lemma 4, we will be sketchy.

First, any solution \(u_R \in S(R)\) can be extended on all \(\mathbb{R}^N\) setting \(u_R = 0\) for \(r > R\). Fixed an integer \(j \gg 1\), let \(X_j = \{(a, u) \in \mathbb{R}^+ \times D^{1,2}(\mathbb{R}^N) : a^2 + \|u\|^2 \leq j^2\}\). Taken a sequence \(R_n \uparrow \infty\), we set \(S_{n,j} = S(R_n) \cap X_j\). We claim that (i) and (ii)
of Lemma 4 hold true for $S_n = S_{n,j}$. In fact, (i) is trivially verified. As for (ii), let us take a sequence $(a_h, u_h) \in \bigcup_{n \in \mathbb{N}} S_{n,j}$. This means that $u_h$ is a solution of $(3a_h - R_n(h))$ for some $R_n(h)$. Set $\rho(h) = \rho(R_n(h), a_h)$. Since $a_h \leq j$, and $\|u_h\| \leq j$, the left hand side of (12) implies:

$$kd^2 \rho(h)^{N+2} \leq \int_{\{u_h > a_h\}} |\nabla u_h|^2 \leq \|u_h\|^2$$

Hence there exists $\rho^* > 0$ such that $\rho(h) \leq \rho^*$ for all $h$, and $\{u_h > a_h\} \subset \{r < \rho^*\}$. From this and since $-\Delta u = f(u - a) = 0$ on $\{u < a\}$, it follows as in [6] that $u_h$ converges, up to a subsequence, in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. This shows that (ii) holds. Applying Lemma 4 we find a non-empty, closed, connected set $S_j = \limsup(S_{n,j})$.

Next, note that for $a = 0$ problem $(3a - R)$ has positive solutions $u \in \Sigma_0(R)$ and obviously one has $\rho(R, 0) = R$. Hence, using the left hand side of (12) one finds $\|u\|^2 \geq kd^2 R^{N+2}$ for all $u \in \Sigma_0(R)$. Since each $S(R)$ is connected, it follows that $\forall j$ there is $n(j)$ such that, for all $n \geq n(j)$ there exists $(a_n, u_n) \in S(R_n)$ such that $a_n^2 + \|u_n\|^2 = j^2$. The preceding compactness argument shows that, up to a subsequence, $(a_n, u_n)$ converges to some $(a, u) \in S_j$ with $a^2 + \|u\|^2 = j^2$. Therefore set $S = \bigcup_{j \in \mathbb{N}} S_j$ is unbounded, yielding the searched global branch.

The required properties of the solutions listed in (ii) follow by standard arguments as in [6].

In order to control the behaviour of the branch $S$ we state:

**Lemma.** Let $u$ be any radial solution of (2a) and let $\rho(a) = \{r : u(r) = a\}$. Then there results

$$\sqrt{\frac{N(N-2)a}{c}} \leq \rho(a) \leq \sqrt{\frac{N(N-2)a}{d}}.$$

**Proof:** First of all we remark that formulas $(13' - 13'')$ hold true for $u(r)$ with 0 instead of $\rho/R$.

Using $(13'')$ one finds

$$\rho^{1-N} \int_0^\rho t^{N-1} f(u(t) - a) dt = (N-2) \frac{1}{\rho} a$$

and (15) follows.

**Theorem 10.** Let $N \geq 3$ and (f1-2-4) hold. Then $\forall a > 0$ any positive, radial solution $u$ of (1a) satisfies

$$\frac{a^{(N+2)/2} K}{c^{(N-2)/2}} \left[ \frac{d^2 N - 2}{c^2 N + 2} + 1 \right] \leq \|u\|^2 \leq \frac{a^{(N+2)/2} K}{d^{(N-2)/2}} \left[ \frac{c^2 N - 2}{d^2 N + 2} + 1 \right]$$

where

$$K = \frac{\omega_{N-1}}{N} [(N-2)N]^{N/2}.$$
Hence $\forall a > 0$, \((2a)\) possesses a positive, radial solution $u$, with \((a,u) \in S$.  

**Proof**: As remarked in Lemma 9, $u(r)$ has the form (13'-13'') with 0 instead of $\rho/R$. In particular: (i) Lemma 7 holds with $u$ instead of $u_R$; and (ii) one has: 

$$ \int_{\{u < a\}} |\nabla u|^2 = \omega_{N-1} \int_0^\infty [(2 - N)a\rho^{N-2}]r^{1-N} dr = \omega_{N-1}(N - 2)a^2 \rho^{N-2} $$

Then: 

$$ ||u||^2 = \int_{\{u > a\}} |\nabla u|^2 + \int_{\{u < a\}} |\nabla u|^2 = \int_{\{u > a\}} |\nabla u|^2 + \omega_{N-1}(N - 2)a^2 \rho^{N-2}. $$

By (12) it follows: 

$$ kd^2 \rho^{N+2} + \omega_{N-1}(N - 2)a^2 \rho^{N-2} \leq ||u||^2 \leq kc^2 \rho^{N+2} + \omega_{N-1}(N - 2)a^2 \rho^{N-2}. $$

Using the estimates for $\rho$ found in Lemma 9 we obtain (16). 

Lastly, suppose that, for some $a > 0$, \((2a)\) has no solutions $u$, with $(a,u) \in S$. Since, by Theorem 8, $S$ is connected, unbounded and bifurcates from \((0,0)\), there would exist $A > 0$ and sequences $(a_n,u_n) \in S$ with $a_n \uparrow A$ and $||u_n|| \to \infty$. This contradicts the right hand side of (16). $\blacksquare$

**Remarks 11.** (i) If $N = 2$ problem \((1a)\) can have no (radial) solutions. This is the case when $f(s) = c > 0 \forall s > 0$.  

(ii) Using the same arguments as above, one can find the following estimate for the $L^\infty$ norm $|u|_\infty$ of any radial solution of \((1a)\):

$$ a[1 + \frac{dN - 2}{c2}] \leq |u|_\infty = u(0) \leq [1 + \frac{cN - 2}{d2}]a. $$

(iii) A global bifurcation result for vortex rings in an ideal fluid is proved in [6]. Such a paper does not provide any bound of the type we prove in Theorem 10, and the behaviour of the branch cannot be controlled (see, [6, §4]). Unfortunately our approach relies on the special symmetry of the problem \((2a)\) and we are not able to extend the arguments to cover the case of vortex rings, where a cylindrical (rather than radial) symmetry arises. An extension of the Amick & Turner results describing the asymptotic behaviour of the bifurcating branch would be an interesting question to pursue.

**Example 12.** The application discussed in Example 6 extends to the case of an isotropic, ionized gas filling all of $\mathbb{R}^3$. One is led to the free boundary problem (8-9) with $\Omega$ substituted by $\mathbb{R}^3$. A natural boundary condition to be added is now

$$ v \longrightarrow v_0 \text{ as } |x| \longrightarrow \infty. $$

If $f$ is given by (11) and keeping the same notations as in Example 6, (8-9-17) give rise to a problem of the form \((2a)\), and the preceding bifurcation results apply.

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