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Jürgen Schu

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Iterative approximation of fixed points of nonexpansive mappings with starshaped domain

JÜRGEN SCHU

Abstract. Let E be a reflexive Banach Space, which possesses a weakly sequentially continuous duality mapping and A be a closed, bounded subset of E, which is starshaped with respect to zero. Then for each nonexpansive self-mapping T of A the iteration process $z_{n+1} = \lambda_{n+1}T(z_n)$ converges strongly to some fixed point of T, if (λ_n) satisfies certain conditions.

Keywords: fixed points, nonexpansive mappings on starshaped domains, iteration processes

Classification: 47H10

1. Introduction.

In [2] B. Halpern introduced the process $z_{n+1} = \lambda_{n+1}T(z_n)$ for approximation of a fixed point of a nonexpansive self-mapping defined on the unit ball of a Hilbert Space.

Later it was shown by S. Reich ([5], Theorem 3.1), that in case of a smooth Opial Space E, admitting a duality mapping which is weakly sequentially continuous at zero, the sequence (z_n) converges strongly to a fixed point for every nonexpansive self-mapping T of a closed, bounded and convex subset A of E, containing zero, if (λ_n) equals $\left(1 - \frac{1}{(n+2)^c}\right)$ with $c \in (0,1)$.

We intend to show, that this result remains valid, if we demand A to be merely starshaped with respect to zero instead of being convex, and assume that E is reflexive and possesses a duality mapping, which is weakly sequentially continuous on the whole of E.

Conventions. Throughout this paper all normed spaces are assumed to be real Banach Spaces.

Let $(E, \|.\|)$ be a normed space; $\emptyset \neq A \subset E$; $T : A \longrightarrow E$; $x_0 \in A$.
We call $(E, \|.\|)$ *smooth* iff $\|.\|$ is Gateaux–differentiable on $E \setminus \{0\}$ and A is called *starshaped* with respect to x_0 iff for all $x \in A$ and $\lambda \in [0,1]$ we have $\lambda x + (1-\lambda)x_0 \in A$. T is said to be *nonexpansive* iff $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in A$. For abbreviation we denote the fixed point set of T by $Fix(T)$. The weak (weak*, strong) convergence of a sequence (x_n) to some element x is indicated by $(x_n) \rightharpoonup x$ $\left((x_n) \overset{*}{\rightharpoonup} x,\ \lim(x_n) = x\right)$.

Let us now recall the definition of a duality mapping.

A function $\mu : \mathbf{R}^+ \longrightarrow \mathbf{R}^+$ is said to be a *gauge function* iff μ is continuous, strictly increasing, $\mu(0) = 0$ and $\lim_{x \longrightarrow \infty} \mu(x) = \infty$.

The related set-valued *duality mapping* $J_E : E \longrightarrow 2^{E^*}$ is given by $J_E(0) := \{0\}$ and $J_E(x) := \{u \in E^* | u(x) = \|u\| \, \|x\| \text{ and } \|u\| = \mu(\|x\|)\}$ for all $x \in E \setminus \{0\}$.

$J : E \longrightarrow E^*$ is said to be a duality mapping iff $J(x) \in J_E(x)$ for all $x \in E$.

For convenience in all proofs of section 2, we will assume, without loss of generality, that J_E respectively J is normalized, i.e. μ =id. Note, that for a smooth normed space J_E is always singlevalued, in which case we regard J_E as a mapping from E to E^*.

Finally we call J *weakly sequentially continuous* in $x \in E$ iff for all $(x_n) \in E^{\mathbf{N}}$ $(x_n) \rightharpoonup x$ implies that $(J(x_n)) \overset{*}{\rightharpoonup} J(x)$.

2. Main result.

Before stating our above mentioned theorem, we have to give several lemmas.

Lemma 1. *Let $(E, \|.\|)$ be a normed space; $x, y \in E$; $\alpha, \beta \in \mathbf{R}$; $\|(1+\alpha)x - (1+\beta)y\| \leq \|x - y\|$.*
Then $u(\alpha x - \beta y) \leq 0$ for all $u \in J_E(x - y)$.

PROOF : For $u \in J_E(x-y)$ we have $u(\alpha x - \beta y) = u((1+\alpha)x - (1+\beta)y) - u(x-y) \leq \|u\| \, \|(1+\alpha)x - (1+\beta)y\| - \|x-y\|^2 \leq \|x-y\| \, \|x-y\| - \|x-y\|^2 = 0$. ■

Lemma 2. *Let $(E, \|.\|)$ be a smooth normed space; $x, y \in E$; $\alpha > \beta \geq 0$; $J_E(x - y)(\alpha x - \beta y) \leq 0$.*
Then $J_E(y - x)(x) \geq 0$.

PROOF : From our assumption $0 \geq J_E(x-y)((\alpha+\beta)(x-y) + (\alpha y - \beta x)) = (\alpha+\beta)\|x-y\|^2 + J_E(x-y)(\alpha y - \beta x)$, hence $J_E(x-y)(\beta x - \alpha y) \geq (\alpha+\beta)\|x-y\|^2$. Define $\gamma := \frac{\beta}{\alpha+\beta}$. Then, since $\alpha > \beta \geq 0$, $\gamma \in [0, \frac{1}{2})$ and therefore $1 - 2\gamma \in (0, 1]$. Now we obtain

$$\|x-y\|^2 \leq J_E(x-y)(\gamma x - (1-\gamma)y) = \frac{1}{2} J_E(x-y)((2\gamma-1)(x+y) + (x-y)) =$$
$$\frac{1}{2}\|x-y\|^2 + \frac{1}{2}(2\gamma-1)J_E(x-y)(x+y), \text{ hence}$$
$$\|x-y\|^2 \leq (1-2\gamma)J_E(y-x)(x+y), \text{ where } 1-2\gamma > 0.$$

Therefore $J_E(y-x)(x+y) \geq 0$ and consequently $\|x-y\|^2 \leq J_E(y-x)(x+y)$, from which we conclude, that

$$2J_E(y-x)(x) = J_E(y-x)((y+x) - (y-x)) = J_E(y-x)(y+x) - \|x-y\|^2 \geq 0.$$

■

Lemma 3. *Let $(E, \|.\|)$ be a normed space with a weakly sequentially continuous duality mapping $J : E \longrightarrow E^*$; $(x_n) \in E^{\mathbf{N}}$; $x \in E$; $(x_n) \rightharpoonup x$;*

$$(*) \qquad J(x_m - x_n)(x_n) \geq 0 \text{ for all } m \geq n.$$

Then $\lim(x_n) = x$.

Remark. Since J is weakly sequentially continuous, E is a smooth normed space (see e.g. [**1**]).

PROOF : Fix $n \in \mathbb{N}$. Then $(x_m - x_n)_m \rightharpoonup x - x_n$, hence $(J(x_m - x_n))_m \overset{*}{\rightharpoonup} J(x - x_n)$ and with the help of $(*)$ we get

$$0 \leq \lim_{m \longrightarrow \infty} J(x_m - x_n)(x_n) = J(x - x_n)(x_n).$$

Therefore

$$J(x - x_n)(x) = J(x - x_n)(x - x_n) + J(x - x_n)(x_n) \geq \|x - x_n\|^2,$$

from which the result follows, because $\lim_{n \longrightarrow \infty} J(x - x_n)(x) = 0$. ∎

Remark. Combining Lemma 2 with Lemma 3 one immediately obtains a convergence lemma of G. Müller and J. Reinermann ([**4**], Lemma 2.5).

Lemma 4. *Let* $(E, \|.\|)$ *be a smooth normed space;* $\emptyset \neq A \subset E$; $T : A \longrightarrow E$ *nonexpansive;* $x, y \in A$; $\lambda \in (0,1)$; $x = \lambda T(x)$; $y = T(y)$.
Then $J_E(y - x)(x) \geq 0$.

PROOF : If we define $\alpha := \frac{1}{\lambda} - 1$ and $\beta := 0$ we observe, that $\alpha > \beta \geq 0$ and

$$\|(1+\alpha)x - (1+\beta)y\| = \|\frac{1}{\lambda}x - y\| = \|Tx - Ty\| \leq \|x - y\|.$$

The result follows from Lemma 1 and Lemma 2. ∎

Lemma 5. *Let* $(E, \|.\|)$ *be a smooth normed space possessing a duality mapping* $J : E \longrightarrow E^*$, *which is weakly sequentially continuous at zero;* $\emptyset \neq A \subset E$; $T : A \longrightarrow E$ *nonexpansive;* $(x_n) \in A^{\mathbb{N}}$; $(\lambda_n) \in (0,1)^{\mathbb{N}}$; $x \in A$; $x = Tx$; $x_n = \lambda_n T(x_n)$ *for all* $n \in \mathbb{N}$; $(x_n) \rightharpoonup x$.
Then

(1) $\lim(x_n) = x$
(2) $J(y - x)(x) \geq 0$ *for all* $y \in Fix(T)$.

PROOF : Lemma 4 tells us, that $J(x - x_n)(x_n) \geq 0$ for all $n \in \mathbb{N}$ and, as already seen in the proof of Lemma 3, this implies, that $\|x - x_n\|^2 \leq J(x - x_n)(x)$ for $n \in \mathbb{N}$. Since $(x - x_n) \rightharpoonup 0$ and J is weakly sequentially continuous at zero, we obtain $\lim \|x - x_n\| = 0$. To prove (2) let y be an arbitrary fixed point of T.

Again by Lemma 4 $J(y - x_n)(x_n) \geq 0$ for $n \in \mathbb{N}$.

Since E is smooth, J is strong–weak* continuous (see e.g. [**1**]) and so $\lim(y - x_n) = y - x$ implies, that $(J(y - x_n)) \overset{*}{\rightharpoonup} J(y - x)$.

This, together with $\lim(x_n) = x$, shows, that $\lim_{n \longrightarrow \infty} J(y - x_n)(x_n) = J(y - x)(x)$ and so $J(y - x)(x) \geq 0$. ∎

Lemma 6. *Let $(E, \|.\|)$ be a normed space with a weakly sequentially continuous duality mapping $J : E \longrightarrow E^*$; $\emptyset \neq A \subset E$ closed; $T : A \longrightarrow E$ nonexpansive; $(x_n) \in A^{\mathbf{N}}$; $x \in E$; $(\lambda_n) \in (0,1)^{\mathbf{N}}$ strictly increasing; $\lim(\lambda_n) = 1$; $x_n = \lambda_n T(x_n)$ for all $n \in \mathbf{N}$; $(x_n) \rightharpoonup x$.*
Then

(1) $\lim(x_n) = x$ *and* $T(x) = x$
(2) $J(y-x)(x) \geq 0$ *for all* $y \in Fix(T)$.

PROOF : Defining $\mu_n := \frac{1}{\lambda_n} - 1$ for $n \in \mathbf{N}$, we observe, that for $m > n$, $\mu_n > \mu_m \geq 0$ and $\|(1+\mu_n)x_n - (1+\mu_m)x_m\| = \|Tx_n - Tx_m\| \leq \|x_n - x_m\|$.
We now apply Lemma 1 and Lemma 2 to derive, that $J(x_m - x_n)(x_n) \geq 0$ for $m > n$ and from Lemma 3 we conclude, that $\lim(x_n) = x \in \bar{A} = A$.
Since T is continuous $Tx = \lim(Tx_n) = \lim(\frac{1}{\lambda_n} x_n) = x$.
Part two of our claim now follows from Lemma 5. ∎

Lemma 7. *Let $(E, \|.\|)$ be a reflexive Banach Space with a weakly sequentially continuous duality mapping $J : E \longrightarrow E^*$; $\emptyset \neq A \subset E$ closed, bounded and starshaped with respect to zero; $T : A \longrightarrow A$ nonexpansive.*
Then there exists $z \in A$ such that $T(z) = z$ and $J(y-z)(z) \geq 0$ for all $y \in Fix(T)$.

PROOF : Set $\lambda_n := 1 - \frac{1}{n+1} \in (0,1)$ and $T_n := \lambda_n T$ for $n \in \mathbf{N}$.
Then $\|T_n x - T_n y\| \leq \lambda_n \|x - y\|$ and, because A is starshaped with respect to zero, $T_n(A) = \lambda_n T(A) \subset \lambda_n A + (1-\lambda_n)\{0\} \subset A$. The classical contraction principle therefore delivers for each $n \in \mathbf{N}$, $x_n \in A$, such that $x_n = T_n(x_n) = \lambda_n T(x_n)$.
Since A is bounded and E is reflexive, there exists $z \in E$ and $\varphi : \mathbf{N} \longrightarrow \mathbf{N}$ strictly increasing, such that $(x_{\varphi_n}) \rightharpoonup z$. Because $x_{\varphi_n} = \lambda_{\varphi_n} T(x_{\varphi_n})$ we are allowed to apply Lemma 6 to (x_{φ_n}), from which the result follows. ∎

Definition 8. (see B. Halpern [**2**])
A sequence (λ_n) is said to fulfill condition (H) iff

(1) $(\lambda_n) \in (0,1)^{\mathbf{N}}$ strictly increasing and $\lim(\lambda_n) = 1$
(2) there is $(\beta_n) \in \mathbf{N}^{\mathbf{N}}$ nondecreasing, such that $\lim(\beta_n(1-\lambda_n)) = \infty$ and $\lim\left(\frac{1-\lambda_{n+\beta_n}}{1-\lambda_n}\right) = 1$.

In the course of the proof of Theorem 3, [**2**] B. Halpern actually showed, that the following holds.

Theorem 9. *Let $(E, \|.\|)$ be a normed space; $\emptyset \neq A \subset E$ bounded and starshaped with respect to zero; $T : A \longrightarrow A$ nonexpansive; $(\lambda_n) \in (0,1)^{\mathbf{N}}$ satisfying condition (H); $(x_n) \in A^{\mathbf{N}}$; $x_n = \lambda_n T(x_n)$ for all $n \in \mathbf{N}$; $z_0 \in A$; $z_{n+1} := \lambda_{n+1} T(z_n)$ for all $n \in \mathbf{N}_0$; assume further, that (x_n) converges strongly to some $q \in E$.*
Then $\lim(z_n) = q$.

Note, that (z_n) is well-defined, because $T(A) \subset A$ and A is starshaped with respect to zero.
Now we are able to show our main result.

Theorem 10. *Let $(E, \|.\|)$ be a reflexive Banach Space with a weakly sequentially continuous duality mapping $J : E \longrightarrow E^*$; $\emptyset \neq A \subset E$ closed, bounded and starshaped with respect to zero; $T : A \longrightarrow A$ nonexpansive; $(\lambda_n) \in (0,1)^{\mathbb{N}}$ satisfying condition (H); $z_0 \in A$; $z_{n+1} := \lambda_{n+1}T(z_n)$ for all $n \in \mathbb{N}_0$.*
Then there exists $z \in A$, such that $T(z) = z$ and $\lim(z_n) = z$.

PROOF : From Lemma 7 we obtain $z \in A$ such that $T(z) = z$ and

$$(*) \qquad J(y - z)(z) \geq 0 \text{ for all } y \in Fix(T).$$

As shown in the proof of Lemma 7 there is $(x_n) \in A^{\mathbb{N}}$ with $x_n = \lambda_n T(x_n)$ for all $n \in \mathbb{N}$.
Consider an arbitrary strictly increasing mapping $\varphi : \mathbb{N} \longrightarrow \mathbb{N}$ now.
Since A is bounded and E is reflexive, we find some strictly increasing $\psi : \mathbb{N} \longrightarrow \mathbb{N}$ and $x \in E$, such that $(x_{\varphi_{\psi_n}}) \rightharpoonup x$.
If we apply Lemma 6 to $(x_{\varphi_{\psi_n}})$, we get $\lim(x_{\varphi_{\psi_n}}) = x$, $T(x) = x$ and

$$(**) \qquad J(y - x)(x) \geq 0 \text{ for all } y \in Fix(T).$$

Because $Tx = x$, $(*)$ delivers $J(x - z)(z) \geq 0$ and since $Tz = z$, $(**)$ shows us, that $J(z - x)(x) \geq 0$, hence $J(x - z)(-x) \geq 0$. Adding both inequalities one gets $0 \leq J(x - z)(z - x) = -\|x - z\|^2 \leq 0$, hence $x = z$ and therefore $\lim(x_{\varphi_{\psi_n}}) = z$. This shows, that $\lim(x_n) = z$ and applying Theorem 9 we are done. ∎

Remark. A result of J.-P. Gossez and E. Lami Dozo ([3], Theorem 1) states, that every normed space, which possesses a weakly sequentially continuous duality mapping, is also an Opial Space (i.e. $(x_n) \rightharpoonup x$ and $y \neq x$ always implies that $\liminf \|x_n - x\| < \liminf \|x_n - y\|$).

Therefore Theorem 10 reduces to a special case of the result of S. Reich, already mentioned in the beginning ([5], Theorem 3.1), if we additionally demand A to be convex. The proof of S. Reich, however, does not carry over to starshaped domains. Note for example, that for a nonexpansive self–mapping T of a closed, bounded and starshaped subset of an Opial Space, $id - T$ is not necessarily demiclosed. But demiclosedness of $id - T$ is essential to the proof of Theorem 3.1 from [5].

For a result concerning the weak convergence of the sequence given by $z_{n+1} := \lambda_{n+1}T(z_n)$ in case of a Hilbert Space and under different assumptions, we refer to [6], Theorem 8.

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RWTH Aachen, Lehrstuhl C für Mathematik, Templergraben 55, D-5100 Aachen, FRG

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