Eugen Viszus
On the regularity up to the boundary for higher order quasilinear elliptic systems

Commentationes Mathematicae Universitatis Carolinae, Vol. 31 (1990), No. 2, 295--306

Persistent URL: http://dml.cz/dmlcz/106859

Terms of use:

© Charles University in Prague, Faculty of Mathematics and Physics, 1990

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project DML-CZ: The Czech Digital Mathematics Library http://project.dml.cz
On the regularity up to the boundary for higher order quasilinear elliptic systems

EUGEN VISZUS

Abstract. The partial regularity up to the boundary of weak solutions to the Dirichlet problem for higher order quasilinear elliptic systems is proved. The proof of partial regularity is direct.

Keywords: Quasilinear elliptic system, weak solution, partial regularity up to the boundary

Classification: 35J60, 35B65

1. Introduction. Using the direct method from [3], [4], we shall prove partial regularity of weak solutions up to the boundary.

We shall consider the following problem:

\[ \sum_{j=1}^{N} \sum_{|\alpha|=m_i} (-1)^{|\alpha|} D^\alpha(A_{ij}^\alpha (x, \delta(u)) D^\beta u^j) = 0, \quad x \in \Omega, \quad i = 1, \ldots, N, \]

\[ D^\alpha u^i \mid_{\Gamma} = 0, \quad i = 1, \ldots, N, \quad |\alpha| \leq m_i - 1, \]

where \( n \geq 2, \Omega = Q(0, b) \cap \{ x \in \mathbb{R}^n : x_n > 0 \} \),
\( Q(y, a) = \{ x \in \mathbb{R}^n : |x_i - y_i| < a, \quad i = 1, \ldots, N \}, \quad a > 0, \)
\( \Gamma = Q(0, b) \cap \{ x \in \mathbb{R}^n : x_n = 0 \}, \quad m_i \geq 1, \quad m_i \) is integer for \( i = 1, \ldots, N, \)
\( \delta(u) = \{ D^\alpha u^i : |\alpha| \leq m_i - 1, \quad i = 1, \ldots, N \}. \)

Let us denote \( \kappa = \sum_{i=1}^{N} \left( \frac{n+m_i-1}{m_i-1} \right) \). We suppose that

\[ \left\{ \begin{array}{l}
A_{ij}^{\alpha \beta} \text{ are uniformly continuous on } \overline{\Omega} \times \mathbb{R}^\kappa \\
|A_{ij}^{\alpha \beta}| \leq L \text{ on } \overline{\Omega} \times \mathbb{R}^\kappa, \quad L > 0.
\end{array} \right. \]  

\[ \sum_{i,j=1}^{N} \sum_{|\alpha|=m_i} A_{ij}^{\alpha \beta} (x, \xi) \xi_i^{\alpha} \xi_j^{\beta} \geq \nu \| \xi \|^2, \quad \nu > 0 \]

for all \((x, \xi) \in \overline{\Omega} \times \mathbb{R}^\kappa\) and \( \xi \in \mathbb{R}^\vartheta, \vartheta = \sum_{i=1}^{N} \left( \frac{n+m_i-1}{m_i} \right) \). By a weak solution of the problem (1.1), (1.2) we mean a function \( u \in H^{m_i}(\Omega) (H^{m_i}(\Omega) = H^{m_1} \times \cdots \times H^{m_N}(\Omega), \quad H^{m_i}(\Omega) - \text{Sobolev space for } i = 1, \ldots, N, \quad u = (u^1, \ldots, u^N) - \text{ see [8]} \) such that

\[ \sum_{i,j=1}^{N} \sum_{|\alpha|=m_i} \int_{\Omega} A_{ij}^{\alpha \beta} (x, \delta(u)) D^\beta u^j D^\alpha \varphi^i \, dx = 0 \]

for all \( \varphi \in H_0^{m_i}(\Omega) \) and \( u \) satisfies (1.2) in the sense of traces.

The main result of this paper is
Theorem 1.1. Let (1.3), (1.4) be satisfied and let \( u \in H^m(\Omega) \) be the weak solution to the problem (1.1), (1.2). Then there exists \( \Omega_0 \subset (\Omega \cup \Gamma) \) (open in \( \Omega \cup \Gamma \)) such that \( u \in C^{m-1,\mu}_{\text{loc}}(\Omega_0) \), \( \mu \in (0,1) \) and \( \dim_H((\Omega \cup \Gamma) \setminus \Omega_0) \leq n - p, \quad p > 2. \) (\( \dim_H \) - Hausdorff dimension, \( n \geq 3 \)).

This theorem generalizes the result of [1] and [4]. In [1] partial regularity up to the boundary is proved by another indirect approach for systems of second order. In [4], interior regularity is proved by direct approach for systems of higher order. Our proof is interesting from the methodical point of view too.

2. The interior regularity. In this part we shall formulate some assertions, which we do not prove. The proofs of the assertions are analogous to those in [4] or [3].

Theorem 2.1 \((L^p\)-estimate in the interior). Let (1.3), (1.4) be satisfied and let \( u \in H^m(\Omega) \) be the weak solution to the problem (1.1). Then there exists \( p > 2 \) such that \( u \in C^m_\Omega, \mu \), \( \mu = 1 - \frac{n}{p} \).

Moreover, there exists a constant \( c_1 = c_1(n, N, m, L, \nu) \) such that for all \( x_0 \in \Omega \) and \( 0 < R < \min\{\text{dist}(x_0, \partial \Omega), 2\} \) the following inequality holds:

\[
(2.1) \left( \int_{Q(x_0, R)} |D^m u|^p \, dx \right)^{2/p} \leq c_1 \int_{Q(x_0, 2R)} |D^m u|^2 \, dx.
\]

By \( \frac{1}{\Omega} \int \Omega f \, dx \) we mean the integral mean value of \( f \) in \( \Omega \) and \( D^m u = \{D^\alpha u^i : |\alpha| = m_i, \quad i = 1, \ldots, N\} \).

By using the Sobolev’s lemma we get

Corollary 2.1. Let the assumptions of Theorem 2.1 be satisfied and let \( n = 2 \). Then \( u \in C^{m-1,\mu}_{\text{loc}}(\Omega), \quad \mu = 1 - \frac{n}{p} \).

Let for \( u^i \in H^{m_i}(Q(x_0, R)), \quad i = 1, \ldots, N, \) the polynomials \( P^i(x) = P^i(x_0, R, u^i, x), \quad x \in Q(x_0, R), \) be such that \( \deg(P^i) \leq m_i - 1 \) and \( \int_{Q(x_0, R)} D^\alpha (u^i - P^i) \, dx = 0 \) for all multiindices \( \alpha : |\alpha| = m_i - 1 \).

Let us denote \([P^i_{x_0, R}] = 1 + \sum_{i=1}^N \sum_{|\alpha| < m_i} |c^i_\alpha|\), where \( c^i_\alpha \) are coefficients of polynomial \( P^i, \quad i = 1, \ldots, N \).

The crucial point in the proof of regularity (for \( n \geq 3 \)) is

Lemma 2.1. Let the assumptions of Theorem 2.1 be satisfied. Then there exists a constant \( c_2 = c_2(n, N, m, L, \nu) \) such that for all \( x_0 \in \Omega \) and \( 0 < \rho < R < \min\{\text{dist}(x_0, \partial \Omega), 1\} \) the inequality

\[
(2.2) \int_{Q(x_0, \rho)} |D^m u|^2 \, dx \leq c_2 \int_{Q(x_0, R)} |D^m u|^2 \, dx \cdot \left\{ \left( \frac{\rho}{R} \right)^n + \chi(x^0, R) \right\}
\]

holds. Here \( \chi(x_0, R) = \left\{ \omega(c_3[R^2 + R^{2-n} \int_{Q(x_0, R)} |D^m u|^2 \, dx]) \right\}^{1-2/p}, \) where \( \omega \) is the modulus of continuity of the functions \( A^0_{ij}, \quad c_3 = c_3([P^i_{x_0, R}]) \).

Using Lemma 2.1 and the method of induction we could prove
Theorem 2.2. Let the assumptions of Theorem 2.1 be satisfied. Then there exists an open set $\Omega_0 \subset \Omega$ such that $u \in C^{m-1,\mu}_{\text{loc}}(\Omega_0)$, $0 < \mu < 1$ and $\dim_H(\Omega \setminus \Omega_0) \leq n-p$, $p > 2$.

3. Regularity up to the boundary. In this part we shall prove the regularity of weak solutions to (1.1), (1.2) "near" the boundary $\Gamma$. For points $x^0 \in \Gamma$ we shall prove the assertions analogous to those in Part 2. The assertions in Parts 2 and 3 will imply Theorem 1.1.

In our proofs we shall use

Lemma 3.1 (proved in [9]). Let $M = \{u \in H^{1,p}(Q(x^0,R)) : u = 0$ on $S$ with $\text{meas}(S) \geq c_1[\text{meas}(Q(x^0,R))]$, $c_1 > 0\} 1 \leq p < \infty$.

Then there exists a constant $c = c(n, p, c_1) > 0$ such that

$$
\int_{Q(x^0,R)} |u|^p \, dx \leq c R^p \int_{Q(x^0,R)} |\nabla u|^p \, dx, \quad u \in M.
$$

Lemma 3.2 (Cacciopoli's inequality). Let (1.3), (1.4) be satisfied and let $u \in H^m(\Omega)$ be the weak solution to (1.1), (1.2). Then there exists a constant $c' = c'(n, N, m, L, \nu)$ such that for all $x^0 \in \Gamma$ and $0 < R < \frac{1}{2} \text{dist}(x^0, \partial \Omega \setminus \Gamma)$ the inequality

$$
(3.1) \int_{Q'(x^0, R)} |D^m u|^2 \, dx \leq \frac{c'}{R^2} \int_{Q'(x^0,2R)} |D^{m-1} u|^2 \, dx
$$

holds. ($Q'(x^0, r) = Q(x^0, r) \cap (\Omega \cup \Gamma)$, $x^0 \in \Omega \cup \Gamma$, $r > 0$.)

Proof: Let $x^0 \in \Gamma$, $0 < R < \frac{1}{2} \text{dist}(x^0, \partial \Omega \setminus \Gamma)$. If $\eta \in C_0^\infty(Q(x^0, 2R))$, $0 \leq \eta \leq 1$, $\eta = 1$ in $Q(x^0, R)$ and $|D^\alpha \eta| \leq c_1 R^{-|\alpha|}$, $|\alpha| \leq a$, $a = \max\{m_i\}_{i=1}^N$, choosing $\varphi^i = u^i \eta^{2\alpha}$ ($i = 1, \ldots, N, \tilde{\eta}$ is the restriction of $\eta$ to $Q'(x^0, 2R)$) in (1.5), we get easily, using the formula of Leibniz:

$$
(3.2) \sum_{i,j=1}^N \sum_{|\alpha|=m_i} \sum_{|\beta|=m_j} \int_{Q'(x^0, 2R)} A_{ij}^{\alpha\beta}(x, \delta(u)) D^\beta U^j D^\alpha U^i \, dx =
$$

$$
= \sum_{i,j=1}^N \sum_{|\alpha|=m_i} \sum_{|\beta|=m_j} \int_{Q'(x^0,2R)} A_{ij}^{\alpha\beta}(x, \delta(u)) \left( \sum_{\gamma<\alpha} L_{\alpha\gamma}(\tilde{\eta}) D^\gamma u^i \right) \left( \sum_{\delta<\beta} M_{\beta\delta}(\tilde{\eta}) D^\delta u^j \right) \, dx +
$$

$$
+ \sum_{i,j=1}^N \sum_{|\alpha|=m_i} \sum_{|\beta|=m_j} \int_{Q'(x^0,2R)} A_{ij}^{\alpha\beta}(x, \delta(u)) D^\alpha U^i \left( \sum_{\delta<\beta} M_{\beta\delta}(\tilde{\eta}) D^\delta u^j \right) \, dx -
$$

$$
- \sum_{i,j=1}^N \sum_{|\alpha|=m_i} \sum_{|\beta|=m_j} \int_{Q'(x^0,2R)} A_{ij}^{\alpha\beta}(x, \delta(u)) D^\beta U^j \left( \sum_{\gamma<\alpha} L_{\alpha\gamma}(\tilde{\eta}) D^\gamma u^i \right) \, dx,
$$
where \( U^i = \tilde{\eta}^i \cdot u^i, \ i = 1, \ldots, n \) and \( L_{\alpha \gamma} (\tilde{\eta}), M_{\alpha \gamma} (\tilde{\eta}) \) are polynomials which involve derivatives of order \( \leq |\alpha| - |\gamma| \) of \( \tilde{\eta} \) and being such that \( |L_{\alpha \gamma} (\tilde{\eta})|, |M_{\alpha \gamma} (\tilde{\eta})| \leq c'_2 R^{|\gamma| - |\alpha|} \). Using estimates of \( L_{\alpha \gamma}, M_{\alpha \gamma} \) and (1.3), (1.4), it follows from (3.2) that

\[
\sum_{i=1}^N \sum_{|\alpha| = m_i} \int_{Q'(x^0, R)} |D^\alpha u^i|^2 \, dx \leq c'_3 \sum_{i=1}^N \sum_{|\alpha| = m_i} \sum_{\gamma < \alpha} R^{2(|\gamma| - |\alpha|)} \int_{Q'(x^0, 2R)} |D^\gamma u^i|^2 \, dx.
\]

The function \( u \in H^m(\Omega) \) satisfies (1.2) and it may be extended by zero from \( Q'(x^0, 2R) \) into \( Q(x^0, 2R) \). We denote this extension by \( \tilde{u} \). It is clear that \( \tilde{u} \in H^m(Q(x^0, 2R)) \). Using Lemma 3.1 we obtain the estimate:

\[
\sum_{i=1}^N \sum_{|\alpha| = m_i} \sum_{\gamma < \alpha} R^{2(|\gamma| - |\alpha|)} \int_{Q'(x^0, 2R)} |D^\gamma \tilde{u}|^2 \, dx \leq c'_4 R^{-2} \int_{Q(x^0, 2R)} |D^{m-1} \tilde{u}|^2 \, dx.
\]

This inequality and (3.3) imply (3.1).

**Lemma 3.3.** Let the assumptions of Lemma 3.2 be satisfied. Then there exists a constant \( c'_5 = c'_5(n, N, m, L, \nu) \) such that for all \( x^0 \in \Gamma \) and \( 0 < R < \frac{1}{2} \min\{\text{dist}(x^0, \partial \Omega \setminus \Gamma), 2\} \), the inequality

\[
(3.4) \quad \int_{Q'(x^0, R)} |D^m u|^2 \, dx \leq c'_5 \left\{ \int_{Q'(x^0, 2R)} |D^m u|^q \, dx \right\}^{2/q}, \quad q = \frac{2n}{n + 2}
\]

holds.

**Proof:** Dividing both sides of (3.1) by \( \text{meas}(Q'(x^0, R)) \) we have

\[
(3.5) \quad \int_{Q'(x^0, R)} |D^m u|^2 \, dx \leq c'_6 R^{-n-2} \int_{Q'(x^0, 2R)} |D^{m-1} u|^2 \, dx.
\]

Let \( \tilde{u} \) be the extension of \( u \) by zero from \( Q'(x^0, 2R) \) to \( Q(x^0, 2R) \). Using the Sobolev lemma and Lemma 3.1 we obtain

\[
\int_{Q'(x^0, 2R)} |D^{m-1} u|^2 \, dx =
\]

\[
= \int_{Q(x^0, 2R)} |D^{m-1} \tilde{u}|^2 \, dx \leq c'_7 R^{-2} \left\{ R^q \int_{Q(x^0, 2R)} |D^m \tilde{u}|^q \, dx \right\}^{2/q}
\]

From (3.5) and this estimate it follows that

\[
\int_{Q'(x^0, R)} |D^m u|^2 \, dx \leq c'_8 R^{-n-2} \left\{ \int_{Q'(x^0, 2R)} |D^m u|^q \, dx \right\}^{2/q}
\]

This estimate implies (3.4).
Lemma 3.4. Let (1.3), (1.4) be satisfied and let \( u \in H^m(\Omega) \) be a weak solution to (1.1), (1.2). Then there exists a constant \( c^* = c^*(n, N, m, L, \nu) \) such that for all \( x^0 \in \Omega \cup \Gamma \) and \( 0 < R < \frac{1}{6} \min \{ \text{dist}(x^0, \partial \Omega \setminus \Gamma), 6 \} \) the estimate

\[
(3.6) \quad \int_{Q'(x^0, R)} |D^m u|^2 \, dx \leq c^* \left\{ \int_{Q'(x^0, 6R)} |D^m u|^q \, dx \right\}^{2/q}, \quad q = \frac{2n}{n + 2}
\]

holds.

PROOF: Is is known (from proof of Theorem 2.1) that there exists a constant \( c_1^* \) such that for all \( x^0 \in \Omega \) and \( R \) satisfying the inequality \( 6R < \min \{ \text{dist}(x^0, \partial \Omega \setminus \Gamma), 6 \} \) the estimate

\[
\int_{Q(x^0, R)} |D^m u|^2 \, dx \leq c_0^* \left\{ \int_{Q(x^0, 2R)} |D^m u|^q \, dx \right\}^{2/q} \leq c_1^* \left\{ \int_{Q(x^0, 6R)} |D^m u|^q \, dx \right\}^{2/q}
\]

holds.

Now, let \( x^0 \in \Omega \) and \( 0 < R < \frac{1}{6} \min \{ \text{dist}(x^0, \partial \Omega \setminus \Gamma), 6 \} \). There are two possibilities:

a) \( 2R < \delta \), where \( \delta = \text{dist}(x^0, x^1) \), \( x^1 \in \Gamma \), \( x^1 \) - projection of \( x^0 \) on \( \Gamma \).

b) \( 2R \geq \delta \).

a) If \( 2R < \delta \), then

\[
\int_{Q'(x^0, R)} |D^m u|^2 \, dx \leq c_0^* \left\{ \int_{Q'(x^0, 6R)} |D^m u|^q \, dx \right\}^{2/q} \leq c_1^* \left\{ \int_{Q(x^0, 6R)} |D^m u|^q \, dx \right\}^{2/q}.
\]

This estimate follows from the interior estimate.

b) If \( 2R \geq \delta \) then

\[
\int_{Q'(x^0, R)} |D^m u|^2 \, dx \leq c_4^* (R + \delta)^n c_5^* \left\{ \int_{Q'(x^1, 2(R + \delta))} |D^m u|^q \, dx \right\}^{2/q} \leq c_4^* \left( \frac{(R \delta)^n}{R^n} \right) c_5^* \left( \frac{(6R)^n}{2(R + \delta)^n} \right) \left\{ \int_{Q'(x^0, 6R)} |D^m u|^q \, dx \right\}^{2/q} \leq c_6^* \left\{ \int_{Q'(x^0, 6R)} |D^m u|^q \, dx \right\}^{2/q}.
\]

Putting \( c^* = \max \{ c_1^*, c_2^*, c_3^*, c_4^*, c_5^*, c_6^* \} \) we have (3.6).

Remark 3.1. We know that the weak solution \( u \in H^m(\Omega) \) to (1.1), (1.2) may be extended by zero from \( \Omega \) into \( Q(0, b) \).

The extension \( \bar{u} \) belongs to \( H^m(Q(0, b)) \), and it is clear that one has for all \( x^0 \in Q(0, b) \setminus (\Omega \cup \Gamma) \) and \( 0 < R < \frac{1}{6} \text{dist}(x_0, \partial Q(0, b)) \)

\[
\int_{Q(x^0, R)} |D^m \bar{u}|^2 \, dx \leq \int_{Q(x^1, R)} |D^m \bar{u}|^2 \, dx \leq c_5^* \left\{ \int_{Q(x^1, 2R)} |D^m \bar{u}|^q \, dx \right\}^{2/q} \leq c_7^* \left\{ \int_{Q(x^0, 6R)} |D^m \bar{u}|^q \, dx \right\}^{2/q}
\]

(\( x^1 \) - projection of \( x^0, x^1 \in \Gamma \)).

Now we may prove
**Theorem 3.1.** Let \((1.3), (1.4)\) be satisfied and let \(u \in H^m(\Omega)\) be the weak solution to the problem \((1.1), (1.2)\). Then there exist \(p > 2\) and a constant \(c_8 = c_8(n, N, m, L, \nu)\) such that \(u \in H^{m,p}(Q'(x^0, R))\) for all \(x^0 \in \Omega \cup \Gamma\) and for \(0 < R < \frac{1}{6} \min\{\text{dist}(x^0, \partial\Omega \setminus \Gamma), 6\}\).

Moreover, the estimate

\[
\left\{ \int_{Q'(x^0, R)} |D^m u|^p \, dx \right\}^{2/p} \leq c_8^* \int_{Q'(x^0, 6R)} |D^m u|^2 \, dx
\]

holds.

**Proof:** We shall use the following

**Lemma 3.5 ([4, Proposition 5.1]).** Let \(Q \subset \mathbb{R}^n\) be a cube, \(g \in L^s(Q), s > 1\), \(g(x) \geq 0\) on \(Q\). Let the inequality

\[
\int_{Q(x^0, R)} g^s \, dx \leq b \left( \int_{Q(x^0, 6R)} g \, dx \right)^s + \theta \int_{Q(x^0, 6R)} g^s \, dx
\]

be satisfied for all \(x^0 \in Q\) and \(R < \min \{ \frac{1}{b} \text{dist}(x^0, \partial Q), R_0 \}\) where \(b > 1, R_0 > 0, 0 \leq \theta < 1\) are constants. Then \(g \in L^p_{\text{loc}}(Q)\) for \(p \in [s, s + \varepsilon)\) and

\[
\left( \int_{Q(x^0, R)} g^p \, dx \right)^{1/p} \leq c \left( \int_{Q(x^0, 6R)} g^s \, dx \right)^{1/s},
\]

where \(Q(x^0, 6R) \subset Q, R < R_0\). The constants \(c, \varepsilon\) depend on \(b, \theta, s, n\).

Let \(\tilde{u} \in H^m(Q(0, b))\) be the extension of \(u \in H^m(\Omega)\) by zero from \(\Omega\) to \(Q(0, b)\). Let us put \(g = |D^m \tilde{u}|^q, q = \frac{2n}{n+2}, s = \frac{2}{q} > 1\).

It is clear that \(g \in L^s(Q(0, b))\). Lemma 3.4, Remark 3.1 and Lemma 3.5 imply that there exists \(r > \frac{2}{q}\) such that \(g \in L^r_{\text{loc}}(Q(0, b))\). Putting \(p = qr > 2\), it is clear that \(|D^m \tilde{u}| \in L^p_{\text{loc}}(Q(0, b))\) and for all \(x^0 \in Q(0, b)\) and \(0 < R < \frac{1}{6} \min\{\text{dist}(x^0, \partial Q(0, b)), 6\}\) the inequality

\[
\left\{ \int_{Q(x^0, R)} |D^m \tilde{u}|^p \, dx \right\}^{\frac{1}{p}} \leq c_8^* \left\{ \int_{Q(x^0, 6R)} |D^m \tilde{u}|^2 \, dx \right\}^{\frac{1}{2}}
\]

holds. The assertion of the theorem follows.

**Corollary 3.1.** Let the assumptions of Theorem 3.1 be satisfied and let \(n = 2\). Then \(u \in C^{m-1, \mu}_{\text{loc}}(\Omega \cup \Gamma)\), \(\mu = 1 - \frac{n}{p}\).

**Proof:** Theorem 3.1 and Sobolev's lemma imply the result.

For \(n \geq 3\) we have
Lemma 3.6. Let (1.3), (1.4) be satisfied and let $u \in H^m(\Omega)$ be the weak solution to the problem (1.1), (1.2). Then for all $x^0 \in \Gamma$ and $0 < \rho < R < \min \{ \text{dist}(x^0, \partial \Omega \setminus \Gamma), 1 \}$ the estimate

$$
\int_{Q'(x^0, \rho)} |D^m u|^2 \, dx \leq c'_{10} \int_{Q'(x^0, R)} |D^m u|^2 \, dx \left\{ \left( \frac{\rho}{R} \right)^n + \chi(x^0, R) \right\}
$$

holds.

$c'_{10} = c'_{10}(n, N, m, L, \nu), \chi(x^0, R) = \left\{ \omega \left( c'_{11} \left[ R^2 + R^{2-n} \int_{Q'(x^0, R)} |D^m u|^2 \, dx \right] \right) \right\}^{1-2/p}, c'_{11} = c'_{11}(n, N, m), p > 2, \omega$ is defined in Lemma 2.1.

Proof: Let $x^0 \in \Gamma, 0 < R < \min \{ \text{dist}(x^0, \partial \Omega \setminus \Gamma), 1 \}$. Put $A^{\alpha \beta}_{ij0} = A^{\alpha \beta}_{ij}(x^0, \theta), \theta$ - the zero-vector in $\mathbb{R}^k$ and let $v \in H^m(Q'(x^0, R_6))$ be the weak solution to the Dirichlet problem

$$
\sum_{j=1}^N \sum_{|\beta|=m_j} (-1)^{|\alpha|} D^{\alpha}(A^{\alpha \beta}_{ij0} D^\beta v^j) = 0, \quad i = 1, \ldots, N \quad \text{in} \ Q'(x^0, \frac{R}{6}),
$$

Then the inequality

$$
\int_{Q'(x^0, \rho)} |D^m v|^2 \, dx \leq c'_{12} \left( \frac{\rho}{R} \right)^n \int_{Q'(x^0, \frac{R}{6})} |D^m v|^2 \, dx, \quad 0 < \rho < \frac{R}{6},
$$

holds. (This fact may be proved by the method in [10, Lemma 4.2.11].)

Putting $w = (u - v) \in H^m_0(Q'(x^0, R_6))$, we have

$$
\sum_{i,j=1}^N \sum_{|\alpha|=m_i} \sum_{|\beta|=m_j} \int_{Q'(x^0, \frac{R}{6})} A^{\alpha \beta}_{ij0} D^\beta w^j D^{\alpha} \Phi^i \, dx =
$$

$$
= \sum_{i,j=1}^N \sum_{|\alpha|=m_i} \sum_{|\beta|=m_j} \int_{Q'(x^0, \frac{R}{6})} [A^{\alpha \beta}_{ij0} - A^{\alpha \beta}_{ij}(x, \delta(u))] D^\beta w^j D^{\alpha} \Phi^i \, dx,
$$

$\Phi \in H^m_0(Q'(x^0, \frac{R}{6})).$

The inequality (3.10) implies

$$
\int_{Q'(x^0, \rho)} |D^m u|^2 \, dx \leq c'_{13} \left\{ \left( \frac{\rho}{R} \right)^n \int_{Q'(x^0, \frac{R}{6})} |D^m v|^2 \, dx + \int_{Q'(x^0, \frac{R}{6})} |D^m w|^2 \, dx \right\}.
$$
If we put $\Phi = w$ in (3.11) using (1.4) and the Cauchy-Schwartz inequality, we have
\[ (3.13) \int_{Q'(z^0, \frac{R}{6})} |D^m w|^2 \, dx \leq c'_{14} \int_{Q'(z^0, \frac{R}{6})} \left( \sum_{i,j=1}^{N} \sum_{|\alpha|=m_i, |\beta|=m_j} |A_{ij0}^{\alpha\beta} - A_{ij}^{\alpha\beta}(x, \delta(u))| \right)^2 |D^m u|^2 \, dx. \]

From (1.3) it is clear that there exists a function $\omega = \omega(t)$, $\omega$ is increasing, continuous, concave, bounded, $\lim_{t \to 0^+} \omega(t) = \omega(0) = 0$, such that
\[ \sum_{i,j=1}^{N} \sum_{|\alpha|=m_i, |\beta|=m_j} |A_{ij}^{\alpha\beta}(x, p) - A_{ij}^{\alpha\beta}(y, q)| \leq \omega(|x - y|^2 + |p - q|^2), \]
\[ x, y \in \Omega, \quad p, q \in \mathbb{R}^k \]

Using this fact, we have from (3.13)
\[ (3.14) \int_{Q'(z^0, \frac{R}{6})} |D^m w|^2 \, dx \leq c'_{15} \int_{Q'(z^0, \frac{R}{6})} \omega^2 |D^m u|^2 \, dx, \]
\[ \omega = \omega(|x - x^0|^2 + \sum_{i=1}^{N} \sum_{|\alpha| \leq m_i - 1} |D^\alpha u_i|^2). \]

Now we shall obtain some estimates. The method of estimating is analogous to that in [3, Lemma 2.2] or [4, Lemma 3.2].

We estimate the right-hand side of (3.14) using Hölder inequality, Theorem 3.1, and boundedness of $\omega$. For $p > 2$ we obtain
\[ (3.15) \int_{Q'(x^0, \frac{R}{6})} \omega^2 |D^m u|^2 \, dx \leq c'_{16} \int_{Q'(x^0, R)} |D^m u|^2 \, dx \left( \int_{Q'(x^0, R)} \omega \, dx \right)^{1-2/p}. \]

Jensen inequality and Lemma 3.1 imply
\[ (3.16) \int_{Q'(x^0, R)} \omega \, dx \leq \omega(c'_{17}[R^2 + R^{2-n}] \int_{Q'(x^0, R)} |D^m u|^2 \, dx), \quad c'_{17} = c'_{17}(n, N, m). \]

Now (3.14), (3.15), (3.16) imply
\[ (3.17) \int_{Q'(x^0, \frac{R}{6})} |D^m w|^2 \, dx \leq\]
\[ \leq c'_{18} \int_{Q'(x^0, R)} |D^m u|^2 \, dx \left\{ \omega(c'_{17}[R^2 + R^{2-n}] \int_{Q'(x^0, R)} |D^m u|^2 \, dx) \right\}^{1-2/p}. \]
From (3.12), (3.17) we have (for $0 < \rho < \frac{R}{6}$)

$$
(3.18) \quad \int_{Q'(x^0, \rho)} |D^m u|^2 \, dx \leq c_{19} \int_{Q'(x^0, R)} |D^m u|^2 \, dx \left\{ \left( \frac{\rho}{R} \right)^n + \chi(x^0, R) \right\},
$$

$$
\chi(x^0, R) = \omega^{1-2/p}, \quad p > 2.
$$

For $\frac{R}{6} \leq \rho < R$ the inequality (3.8) is clear: $c_{19}' = \max\{6^n, c_{19}\}$.

Now we can prove

**Theorem 3.2.** Let (1.3), (1.4) be satisfied and let $u \in H^m(\Omega)$ be the weak solution to the problem (1.1), (1.2). Let us put

$$
\Gamma_1 = \left\{ x \in \Gamma : \lim_{R \to 0^+} R^{2-n} \int_{Q'(x, R)} |D^m u|^2 \, dy = 0 \right\}.
$$

Then for all $\bar{x} \in \Gamma_1$ there exists $\delta > 0$ such that

$$
u \in C^{m-1, \mu}(\Omega'(\bar{x}, \delta)), \quad \mu \in (0, 1).
$$

**PROOF:** For $\bar{x} \in \Gamma$ and $0 < R < \min\{\text{dist}(\bar{x}, \partial \Omega) \setminus \Gamma), 1\}$ we put

$$
\Psi(\bar{x}, R) = R^{2-n} \int_{Q'(\bar{x}, R)} |D^m u|^2 \, dx.
$$

Let $c_{10}^* = \max\{c_2, c_{10}'\}$. It follows from (3.8) that for $0 < \tau < 1$

$$
(3.19) \quad \Psi(\bar{x}, \tau R) \leq c_{10}^* \Psi(\bar{x}, R) \tau^2 \{1 + \chi(\bar{x}, R) \cdot \tau^{-n}\}.
$$

Now let $\bar{x} \in \Gamma_1$ and $\varepsilon_0 > 0$, $R' < 1$ be chosen by such a way that $\Psi(\bar{x}, R) < \varepsilon_0$ for $0 < R < R'$. It follows from the construction of $[P_{\bar{x}, R}]$ on $Q'(\bar{x}, R)$ that $[P_{\bar{x}, R}] \leq c_{11}^* \Psi(\bar{x}, R) + 2$ for $0 < R < R' < 1$. This fact implies that

$$
\sup_{0 < R < \text{dist}(\bar{x}, \partial \Omega) \setminus \Gamma)} [P_{\bar{x}, R}] < +\infty \quad \text{for all } \bar{x} \in \Gamma_1.
$$

Let now $0 < \mu < 1$ and choose $\tau$ in such a way that

$$
(3.20) \quad 2c_{10}^* \cdot \tau^{2-2\mu} = 1.
$$

For $M \geq 8$ denote $c_3(M)$ the constant of Lemma 2.1. Let $\varepsilon > 0$. Then there exists $R_1 > 0$ such that $R^2 + \Psi(\bar{x}, R) < \varepsilon c_{17}^* c_3(M)$ for $0 < R < R_1$. This fact implies: there exists $R_2$ such that for $0 < R < R_2$

$$
(3.21) \quad \chi(\bar{x}, R) < \tau^n.
$$

from (3.19), (3.20), (3.21) we have

$$
(3.22) \quad \Psi(\bar{x}, \tau R) \leq \tau^{2\mu} \Psi(\bar{x}, R).
$$
By induction we get for every $k$:

$$\Psi(\overline{x}, \tau^k R) \leq \tau^{2\mu k} \Psi(\overline{x}, R)$$

and hence for every $0 < \rho < \overline{R}$, $(\overline{R} < R_2)$:

$$\Psi(\overline{x}, \rho) \leq \tau^{2-n-2\mu} \left(\frac{\rho}{R_x^0}\right)^{2\mu} \Psi(\overline{x}, \overline{R}). \tag{3.23}$$

It is clear ($\Psi$ is continuous in $\overline{x}$) that there exists $0 < \delta < \overline{R}$ such that for every $x^0 \in Q'(\overline{x}, \delta)$, $R_x^0 + \Psi(x^0, R_x^0) < \frac{\epsilon}{c_1 c_3(M)}$, $R_x^0 = \text{dist}(x^0, \partial Q(\overline{x}, \overline{R}) \setminus \Gamma)$. Let $\delta < \frac{\overline{R}}{4}$.

We shall investigate the following cases:

(i) $x^0 \in Q'(\overline{x}, \delta) \cap \Gamma$,
(ii) $x^0 \in Q'(\overline{x}, \delta) \cap \Omega$.

(i) For $(x^0, R_x^0)$ the inequality

$$\Psi(x^0, \rho) \leq \tau^{2-n-2\mu} \left(\frac{\rho}{R_x^0}\right)^{2\mu} \Psi(x^0, R_x^0), \quad 0 < \rho < R_x^0 \tag{3.24}$$

holds.

(ii) Let $x^1$ be the projection of $x^0$ on $\Gamma$ and $d_x^0 = \text{dist}(x^0, \Gamma) = \text{dist}(x^0, x^1)$. If $d_x^0 < \rho < \frac{R_x^0}{2}$, then $Q'(x^0, \rho) \subset Q'(x^1, 2\rho)$ and $\Psi(x^0, \rho) \leq 2^{n-2} \Psi(x^1, 2\rho)$. Using the case (i) ($x^1 \in \Gamma$, $2\rho < R_x^1$), we get

$$\Psi(x^0, \rho) \leq 2^{n-2} \tau^{2-n-2\mu} \left(\frac{2\rho}{R_x^1}\right)^{2\mu} \Psi(x^1, R_x^1). \tag{3.25}$$

In the case when $0 < \rho < d_x^0$, we shall prove

$$d_x^0 \leq \frac{\Psi(x^0, d_x^0)}{c_1 c_3(M)} \tag{3.26}$$

$$[P_{x^0}, d_x^0] \leq \frac{M}{2}. \tag{3.27}$$

Because $d_x^0 < \delta < \frac{\overline{R}}{4}$, (3.25) implies:

$$\Psi(x^0, d_x^0) \leq 2^{n-2+2\mu} \cdot \tau^{2-n-2\mu} \Psi(x^1, R_x^1). \tag{3.28}$$

Let us choose $\overline{R}$ in such a way that

$$\overline{R}^2 + \Psi(\overline{x}, \overline{R}) < \min \left\{ 2^{2-n-2\mu} \tau^{n-2+2\mu} \frac{\epsilon}{c_1 c_3(M)} + 2^{2n-2}(c_{11}^*)^{-1} \tau^{n-2+2\mu} \frac{M}{4} \right\}.$$ 

Then (3.28) implies (3.26). From $Q(x^0, d_x^0) \subset Q'(x^1, 2d_x^0)$ it follows that

$$[P_{x^0}, d_x^0] \leq 2^{2n} c_{11}^* \Psi(x^1, 2d_x^0) + 2.$$
Using the case (i) \((2d_{x_0} < R_{x_1}, x^1 \in \Gamma)\) we get:

\[ [P_{x_0}, d_{x_0}] \leq 2^{2n} c_{11}^* r^{2-n-2\mu} \Psi(x^1, R_{x_1}) + 2 \leq \frac{M}{4} + \frac{M}{4} = \frac{M}{2}. \]

(3.26), (3.27) imply (by the arguments from the proof of interior regularity)

\[ \Psi(x^0, \rho) \leq \text{const} \left( \frac{\rho}{R_{x^0}} \right)^{2\mu} \]  

(3.29)

Now (3.24), (3.25), (3.29) imply

\[ \Psi(x^0, \rho) \leq \text{const}\rho^{2\mu}, \quad 0 < \rho \leq \frac{R - \delta}{2}, \quad x^0 \in Q'(\bar{x}, \delta). \]

(3.30)

Inequality (3.30) and the properties of Campanato spaces (see [8]) imply: \(u \in C^{m-1,\mu}(Q'(\bar{x}, \delta)), \mu \in (0, 1)\).

\[ \sum = \left\{ x \in \Gamma : \limsup_{R \to 0^+} R^{2-n} \int_{Q'(x, R)} |D^m u|^2 \, dy > 0 \right\}. \]

Let now \(\bar{u}\) be the extension of \(u\) from \(Q'(x, R)\) into \(Q(x, R)\) by zero. Then

\[ R^{2-n} \int_{Q'(x, R)} |D^m u|^2 \, dy = R^{2-n} \int_{Q(x, R)} |D^m \bar{u}|^2 \, dy. \]

Using Hölder inequality and the fact that \(u \in H^{m,p}_{\text{loc}}(\Omega), p > 2\) we get: \(\Gamma \setminus \Gamma_0 \subset \sum \subset \sum_1\) where

\[ \sum_1 = \left\{ x \in \Gamma : \limsup_{R \to 0^+} R^{p-n} \int_{Q(x, R)} |D^m \bar{u}|^p \, dy > 0 \right\}. \]

Then Theorem 1 from [6] implies that \(H_{n-p}(\Gamma \setminus \Gamma_0) = 0\).

**Theorem 3.3.** Let the assumptions of Theorem 3.2 be satisfied. Then there exists \(\Gamma_0 \subset \Gamma (\Gamma_0 - \text{open in } \Gamma)\) such that for every \(x \in \Gamma_0\) there exists \(\delta > 0\) such that \(u \in C^{m-1,\mu}(Q'(x, \delta)), \mu \in (0, 1)\) and \(H_{n-p}(\Gamma \setminus \Gamma_0) = 0, p > 2\).

**Proof:** The existence of \(\Gamma_0 \subset \Gamma, \Gamma_0 - \text{open},\) follows from Theorem 3.2. It is clear that \(\Gamma \setminus \Gamma_0 \subset \sum\), where

\[ \sum = \left\{ x \in \Gamma : \limsup_{R \to 0^+} R^{2-n} \int_{Q'(x, R)} |D^m u|^2 \, dy > 0 \right\}. \]

Let now \(\bar{u}\) be the extension of \(u\) from \(Q'(x, R)\) into \(Q(x, R)\) by zero. Then

\[ R^{2-n} \int_{Q'(x, R)} |D^m u|^2 \, dy = R^{2-n} \int_{Q(x, R)} |D^m \bar{u}|^2 \, dy. \]

Using Hölder inequality and the fact that \(u \in H^{m,p}_{\text{loc}}(\Omega), p > 2\) we get: \(\Gamma \setminus \Gamma_0 \subset \sum \subset \sum_1\) where

\[ \sum_1 = \left\{ x \in \Gamma : \limsup_{R \to 0^+} R^{p-n} \int_{Q(x, R)} |D^m \bar{u}|^p \, dy > 0 \right\}. \]

Then Theorem 1 from [6] implies that \(H_{n-p}(\Gamma \setminus \Gamma_0) = 0\).

**Remark 3.2.** The proof of Theorem 1.1 follows directly from Theorem 2.2 and Theorem 3.3.
REFERENCES


Matematicko fyzikálna fakulta, Univerzita Komenského, Mlynská dolina, 842 15 Bratislava, Československo

(Received December 14, 1989)