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Abstract. We construct a compactification, which we call a Fan–Gottesman type compactification, of a regular frame having a normal base. It is shown that the Stone–Čech compactification of a normal regular frame and the least compactification of a regular continuous frame are examples of compactifications of such type. We also characterize those precompact uniformities on a frame whose Samuel compactification is of Fan–Gottesman type.

Keywords: frame, compactification, strong inclusion, uniform frame

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In [4] Fan and Gottesman construct a compactification of a regular topological space having a normal base. It is shown there that the Stone–Čech compactification of a normal Hausdorff space can be obtained using this general construction if one takes as a normal base the collection of all open subsets of the topological space. It is also shown that the Alexandroff one–point compactification of a locally compact, non–compact Hausdorff space X can be so obtained if one takes as a normal base the family of all those open subsets U such that either cl U or $X - U$ is compact.

This classical construction for topological spaces provided the motivation to construct compactifications of regular frames having a base satisfying properties analogous to that for normal bases as defined in [4]. Such compactification we shall call compactifications of Fan–Gottesman type. We construct this compactification for a regular frame with a so–called normal base in Section 1.

In Sections 2 and 3 we show that just as for the classical case the Stone–Čech compactification of a normal regular frame and the least compactification of a regular continuous frame are examples of compactifications of Fan–Gottesman type. We also give in Section 2 an alternative proof (which avoids the use of Joyal's lemma [6, p.91]) of P.T. Johnstone's [7] result that the Wallman compactification of a normal regular frame is the same as its Stone–Čech compactification. The Wallman compactification for such a frame, we may deduce then, is of Fan–Gottesman type.

In Section 4 we discuss uniform frames with a view to characterizing those precompact uniformities on a frame whose Samuel compactification is of Fan–Gottesman type.

0. Preliminaries. Recall that a frame (locale) is a complete lattice satisfying the infinite distributive law $x \wedge \bigvee A = \bigvee_{a \in A}(x \wedge a)$ for any $x \in L$, $A \subset L$. These are the objects of the category **Frm** whose morphisms are those functions which preserve finite meets and arbitrary joins. We denote the top of L by e and the bottom by 0.

A frame L is compact if $e = \bigvee A$ implies that there exists a finite $S \subset A$ such that $e = \bigvee S$. A frame L is regular if for each $a \in L$, $a = \bigvee_{x \prec a} x$. Here $x \prec a$ is read as x is "rather below" a and is defined by $x \wedge y = 0$ and $y \vee a = e$ for some $y \in L$, or equivalently $x^* \vee a = e$, where x^* is the pseudocomplement of x. L is normal if given a and b in L with $a \vee b = e$ there exists c and d with $c \wedge d = 0$, $c \vee b = e$ and $a \vee d = e$. A frame map $h : M \longrightarrow L$ is called dense if $h(x) = 0$ implies that $x = 0$. A compactification of L is a compact regular frame M together with a dense onto map $h : M \longrightarrow L$. A strong inclusion on L is a binary relation $\lhd$ on L such that

(i) $x \leq a \lhd b \leq y \Longrightarrow x \lhd y$
(ii) $\lhd \subset L \times L$ is a sublattice, i.e. $0 \lhd 0$, $e \lhd e$,
$x \lhd a, b \Longrightarrow x \lhd a \wedge b$, $x, y \lhd a \Longrightarrow x \vee y \lhd b$
(iii) $x \lhd a \Longrightarrow x \prec a$
(iv) $\lhd$ interpolates, that is $x \lhd z \Longrightarrow x \lhd y \lhd z$ for some $y \in L$
(v) $x \lhd a \Longrightarrow a^* \lhd x^*$
(vi) $a = \bigvee_{x \lhd a} x$

If $\lhd$ is a strong inclusion on L, then this determines a compactification of L defined as follows: An ideal $J \subset L$ is called strongly regular (with respect to $\lhd$) if $x \in J$ implies that $x \lhd y$ for some $y \in J$. Let $\gamma L = \{J | J$ is a strongly regular ideal of $L\}$. Then γL is a compact regular subframe of Idl(L), the frame of ideals of L. The join map $\bigvee : \gamma L \longrightarrow L$ is dense and onto so that $(\gamma L, \bigvee)$ is a compactification of L.

The concept of strong inclusion and the construction of the compactification determined by it is due to Banaschewski [2]. We are unaware of any published reference of this fact. For a general reference on frames see [6].

1. Fan–Gottesman compactification. In [4] Fan and Gottesman constructed a compactification of a regular toplogical space having a so called normal base, which includes Wallman's compactification for normal Hausdorff spaces. As a direct frame translation of the conditions for this base we may formulate

1.1 Definition. A base $B \subset L$ for a regular frame L is said to be a *normal base* if it satisfies

(i) $a, b \in B \Longrightarrow a \wedge b \in B$
(ii) $a \in B \Longrightarrow a^* \in B$
(iii) for any $c \in L$, $a \in B$ with $a \prec c$ there exists $b \in B$ such that $a \prec b \prec c$.

1.2 Proposition. *Let L be regular and B a normal base for L. Define $\lhd$ on L by: $x \lhd y$ if there exists $b \in B$ with $x \prec b \prec y$. Then $\lhd$ is a strong inclusion on L.*

PROOF : We check that the six conditions for a strong inclusion are satisfied:
(i) $x \leq a \lhd b \leq y \Longrightarrow x \leq a \prec c \prec b \leq y$ for some $c \in B$. Thus $x \prec c \prec y$ and hence $x \lhd y$.
(ii) We have $0 \lhd 0$ since $0 \prec 0 \prec 0$ and $0 \in B$. Also $e \lhd e$ since $e \prec e \prec e$ and $e \in B$. Now suppose $x \lhd a, b$. Find $c, d \in B$ such that $x \prec c \prec a$, $x \prec d \prec b$. Then $x \prec c \wedge d \prec a \vee b$. Since $c \wedge d \in B$ we have $x \lhd a \wedge b$. If $x, y \lhd b$, then there exist $a, c \in B$ such that $x \prec a \prec b$, $y \prec c \prec b$. Thus $a \vee c \prec b$ and hence $(a \vee c)^{**} \prec b$. Thus $x \vee y \prec (a \vee c)^{**} \prec b$. Since $(a \vee c)^{**} = (a^* \wedge c^*)^* \in B$ we have $x \vee y \lhd b$.
(iii) If $x \lhd y$, then $x \prec y$ follows from the definition.

(iv) Suppose $x \lhd z$. Then there exists $a \in B$ such that $x \prec a \prec z$. By the third condition of the definition of the base B, there exist $b, c \in B$ such that $x \prec a \prec b \prec c \prec z$. Hence $x \lhd b \lhd z$.
(v) If $x \lhd a$ then there exists $b \in B$ such that $x \prec b \prec a$. Then $a^* \prec b^* \prec x^*$, and since $b^* \in B$ we have $a^* \lhd x^*$.
(vi) Let $a \in L$. By regularity and the fact that B is a base for L, we have $a = \bigvee_{z \prec a, z \in B} z$. Now if $z \prec a$ and $z \in B$, then there exists $c \in B$ such that $z \prec c \prec a$. Hence $z \lhd a$, and thus $a = \bigvee_{x \lhd a} x$. ■

The compactification γL associated with the above $\lhd$ (or $\gamma_B L$, to emphasize that this is with respect to a normal base B for L) we shall call the *Fan–Gottesman compactification* of L. Any compactification of L isomorphic with $\gamma_B L$ for some normal base B for L will be called a Fan–Gottesman type compactification.

Let $S(L)$ be the set of all strong inclusions on L partially ordered by inclusion and let $K(L)$ be the set of all compactifications (M, h) of L partially ordered by: $(M, h) \leq (K, f)$ if and only if there exists a frame homomorphism $g : M \to K$ such that $fg = h$. It is known that $S(L) \cong K(L)$ (Banaschewski [2]). As we are unaware of this result appearing in the published literature, we sketch a proof below from Banaschewski [2].

1.3 Proposition. $S(L) \cong K(L)$.

Proof (sketch):

Consider the maps $S(L) \to K(L)$ given by $\lhd \rightsquigarrow (\gamma L, \bigvee)$ (as defined above) and $K(L) \to S(L)$ given by $(M, h) \rightsquigarrow \lhd$. Here $\lhd$ is defined by $x \lhd y$ if and only if $l(x) \prec l(y)$ where $l : L \to M$ is the right adjoint of h given by $l(a) = \bigvee_{h(x)=a} x$. That these maps are order-preserving can be easily shown. We show these maps are inverses of each other.

Consider $S(L) \to K(L) \to S(L)$ where $\lhd \rightsquigarrow (\gamma L, \bigvee) \rightsquigarrow \lhd_0$. For $a \in L$, $k(a) = \{x \in L | x \lhd a\} \in \gamma L$. Furthermore $\bigvee J \leq a$ if and only if $J \subset k(a)$ so that k is the right adjoint of $\bigvee : \gamma L \to L$. Thus for $\lhd_0$ determined by $(\gamma L, \bigvee)$, $x \lhd_0 a$ if and only if $k(x) \prec k(a)$.

Now $x \lhd a \implies k(x) \prec k(a)$ (with a little calculation) $\implies x \lhd_0 a$. Conversely $x \lhd_0 a \implies k(x) \prec k(a) \implies$ there exists $J \in \gamma L$ such that $k(x) \cap J = 0$ and $k(a) \vee J = L$ from which we may obtain $x \lhd a$. Thus $S(L) \to K(L) \to S(L)$ is the identity.

To show $K(L) \to S(L) \to K(L)$ is the identity, where $(M, h) \rightsquigarrow \lhd \rightsquigarrow (\gamma L, \bigvee)$, we must show $(M, h) \cong (\gamma L, \bigvee)$. Consider

$$\begin{array}{ccccc} M & \xrightarrow{\;k_M\;} & \gamma M & \xrightarrow{\;\gamma h\;} & \gamma L \\ & {\scriptstyle h}\searrow & & \swarrow{\scriptstyle \bigvee} & \\ & & L & & \end{array}$$

where $k_M(a) = \{x \in M | x \prec a\}$ and $(\gamma h)(I) = \bigcup\{\downarrow h(x) | x \in I\}$.

The map k_m is a frame map since M is compact regular. Furtermore γh is a frame map so that $(\gamma h)k_m : M \longrightarrow \gamma L$ is a frame map. Also the above diagram commutes and $(\gamma h)k_m$ is dense since both h and $\bigvee$ are. As one can verify $(\gamma h)k_m$ is also onto. Thus $(\gamma h)k_m$ is an isomorphism since M and γL are compact regular. ■

1.4 Proposition. *Let L be a regular frame, B a normal base for L and R the set of regular elements of B, that is $R = \{b \in B | b = b^{**}\}$. Then R is a normal base for L and $\gamma_R L$ is isomorphic to $\gamma_B L$.*

PROOF : That R is a normal base follows from the following:
(i) If $a, b \in R$ then $(a \wedge b)^{**} = a^{**} \wedge b^{**} = a \wedge b$. Hence $a \wedge b \in R$.
(ii) If $a \in R$, then $(a^*)^{**} = a^*$. Hence $a^* \in R$.
(iii) If $a \in R$, $a \prec c$ then there exists $b \in B$ such that $a \prec b \prec c$. Thus $a \prec b^{**} \prec c$ with $b^{**} \in R$.
(iv) If $a \in L$, then $a = \bigvee_{x \in B, x \prec a} x$. Since $x \in B$ and $x \prec a$ implies that $x^{**} \in R$ and $x^{**} \prec a$ we have $a = \bigvee_{z \prec a, z \in R} z$.

To complete the proof we show that $\lhd_B = \lhd_R$, where $\lhd_B$ and $\lhd_R$ are the strong inclusions with respect to the bases B and R respectively. Obviously if $x \lhd_R y$ then $x \lhd_B y$ since $R \subset B$. If $x \lhd_B$, then $x \prec a \prec y$ for some $a \in B$. But then $x \lhd_R y$ since we have $x \prec a \leq a^{**} \prec y$ and $a^{**} \in R$. Hence $\lhd_B = \lhd_R$. ■

It might be thought that if B and B' are normal bases for L such that $\gamma_B L$ and $\gamma_{B'} L$ are isomorphic then they contain the same regular elements. The following example shows this is not the case.

1.5 Example: Let $X = [0,1]$ with the usual topology. Then $\mathcal{O}X$ is compact, regular and normal, where $\mathcal{O}X$ is the frame of open sets of X. Since, as is well known, every dense frame map between compact regular frames is an embedding, any compactification of $\mathcal{O}X$ is isomorphic with $\mathcal{O}X$. Now $\mathcal{O}X$ is a normal base for $\mathcal{O}X$ and thus the set R of all the regular elements of $\mathcal{O}X$ (i.e. the regular open subsets of X) is a normal base as well by Proposition 1.4. Now $R' = \{g \in \mathcal{O}X | G$ is a finite union of open intervals in $X\}$ is evidently a normal base for $\mathcal{O}X$. We have $\gamma_R \mathcal{O}X \cong \gamma_{R'} \mathcal{O}X (\cong \mathcal{O}X)$, but $R' \subsetneq R$.

2. Normal regular frames. If L is normal regular, recall that the rather below relation $\prec$ interpolates and that the Stone–Čech compactification can be described as $(RL, \bigvee)$ where RL consists of all the regular ideals of L and $\bigvee : RL \longrightarrow L$ is the join map. (See e.g. [1],[2],[6]). An ideal $J \subset L$ is said to be regular if $x \in J$ implies that there exists $y \in J$ such that $x \prec y$. Since $\prec$ interpolates it is clear that L itself is a normal base and that $\lhd_L = \prec$. Thus $(\gamma_L L, \bigvee)$ is isomorphic to $(RL, \bigvee)$. We have shown:

2.1 Proposition. *For normal regular L, L itself is a normal base and the Fan-Gottesman compactification $(\gamma_L L, \bigvee)$ is the Stone-Čech compactification of L.*

The remainder of this section is devoted to an alternative proof (which avoids the use of Joyal's lemma ([6]) of Johnstone's result ([7]) that the Wallman compactification of a normal regular frame is the Stone–Čech compactification of L. Hence by

Proposition 2.1, the Wallman compactification of such a frame is a compactification of Fan–Gottesman type.

Let us firstly recall Johnstone's ([7]) construction of the Wallman compactification of a subfit frame L, i.e. a frame satisfying $\nabla(a) \subset \nabla(b) \Longrightarrow a \leq b$, where $\nabla(a) = \{c \in L | a \vee c = e\}$: Let j be the nucleus on the frame $\mathrm{Idl}(L)$ of ideals of a subfit frame L given by

$$j(I) = \{a \in L | (\forall b \in L)(a \vee b = e) \Longrightarrow (\exists c \in I)(c \vee b = e)\}.$$

The Wallman compactification of L is defined to be the frame $\mathrm{Idl}(L)_j$ of j-fixed ideals of L.

2.2 Lemma. *If L is regular, then for any ideal I of L, $\bigvee I = \bigvee j(I)$.*

PROOF : $I \subset j(I)$ so that $\bigvee I \leq \bigvee j(I)$.
Now let $a \in j(I)$ be arbitrary. Take any $x \prec a$. Then $x^* \vee a = e$. Since $a \in j(I)$, there exists $c \in I$ such that $x^* \vee c = e$, i.e. $x \leq c \leq \bigvee I$. By regularity $a = \bigvee_{y \prec a} y$ so that we have $a \leq \bigvee I$. Hence $\bigvee j(I) \leq \bigvee I$.

■

2.3 Lemma. *If L is normal regular then* $(\mathrm{Idl}(L)_j, \bigvee)$ *is a compactification of L.*

PROOF : That $\mathrm{Idl}(L)_j$ is compact regular is proved in [7]. We need to show that $\bigvee : \mathrm{Idl}(L)_j \longrightarrow L$ is a frame homomorphism which is dense and onto.
That $\bigvee$ is dense is clear; also L subfit $\Longrightarrow$ every principal ideal of L is j-fixed ([7]) so that $\bigvee$ is onto. That $\bigvee$ preserves finite meets is clear. Now take $I, J \in \mathrm{Idl}(L)_j$. Then $\bigvee(I \vee_j J) = \bigvee(j(I \vee J) = \bigvee(I \vee J)$ (from Lemma 2.2) $= \bigvee I \vee \bigvee J$. Now take any collection of updirected ideals $\{I_i\}$ in $\mathrm{Idl}(L)_j$. Then $\bigvee(\bigvee_j I_i) = \bigvee j(\bigcup I_i) = \bigvee(\bigcup I_i) = \bigvee \bigvee I_i$. Thus $\bigvee$ preserves arbitrary joins and hence is a frame homomorphism. ■

2.4 Proposition ([7]). *If L is normal regular then* $(\mathrm{Idl}(L)_j, \bigvee)$ *is the Stone–Čech compactification of L.*

PROOF : Let M be compact regular, $h : M \longrightarrow L$ a frame map. Define

$$g : M \longrightarrow \mathrm{Idl}(L)_j \text{ by}$$

$$g(b) = j\left(\bigvee_{\mathrm{Idl}(L)} {\downarrow} h(c)\,(c \prec b)\right)$$

Then

$$\begin{aligned}
\bigvee g(b) &= \bigvee \bigvee_{\mathrm{Idl}(L)} {\downarrow} h(c)(c \prec b) \qquad \text{(from Lemma 2.2)}\\
&= \bigvee \bigvee {\downarrow} h(c)(c \prec b)\\
&= \bigvee h(c)(c \prec b)\\
&= h\left(\bigvee c(c \prec b)\right)\\
&= h(b)
\end{aligned}$$

We need to show only that g is a frame map:

$$g(0) = j(0),\ g(e) = L \text{ is clear;}$$

$$\begin{aligned} g(b \wedge d) &= j\left(\bigvee_{\mathrm{Idl}(L)} \downarrow h(s)(s \prec b)\right) \wedge j\left(\bigvee_{\mathrm{Idl}(L)} \downarrow h(t)(t \prec d)\right) \\ &= j\left(\bigvee_{\mathrm{Idl}(L)} (\downarrow h(s) \wedge \downarrow h(t))\,(s \prec b,\ t \prec d)\right) \\ &= j\left(\bigvee_{\mathrm{Idl}(L)} \downarrow h(s \wedge t)(s \prec b,\ t \prec d)\right) \\ &\subseteq g(b \wedge d) \end{aligned}$$

Since $g(b \wedge d) \subseteq g(b) \wedge g(d)$ is clear, we have $g(b \wedge d) = g(b) \wedge g(d)$. To show $g(b) \vee_j g(d) = g(b \vee d)$: Obviously $g(b) \vee_j g(d) \subseteq g(b \vee d)$. Now $g(b \vee d) = \bigvee_j \downarrow h(c)(c \prec b \vee d)$.

Take any $c \prec b \vee d$. By regularity and compactness of B we can find $s \prec b$, $t \prec d$ such that $c \prec s \vee t$. Then $c = (c \wedge s) \vee (c \wedge t)$ which implies

$$h(c) = h(c \wedge s) \vee h(c \wedge t) \in g(b) \vee g(d) \subseteq g(b) \vee_j g(d)$$

Thus $\downarrow h(c) \subseteq g(b) \vee_j g(d)$ and hence $g(b \vee d) = g(b) \vee_j g(d)$. Now suppose $\{b_i\}$ is and updirected subset of B. To show $g(\bigvee b_i) \subseteq j\,(\bigvee g(b_i)) = j\,(\bigcup g(b_i))$. Let $x \in g(\bigvee b_i)$ and $x \vee y = e$. Then there exists $c \in \bigvee \downarrow h(s)(s \prec \bigvee b_i)$ such that $c \vee y = e$. Now $c \in \bigcup \downarrow h(s)(s \prec \bigvee b_i)$ so that there exists $x \prec \bigvee b_i$, $c \leq h(s)$. By compactness of B, $s \prec b_i$ for some i. Thus $c \in \downarrow h(s) \subseteq j(\bigvee \downarrow h(w)(w \prec b_i)) = g(b_i)$. Hence $x \in j\,(\bigcup g(b_i))$ as required. Thus g is a frame homomorphism.

■

3. Regular continuous frames. Recall that in any complete lattice L, $x << y$ (x is "way below" y) if $y \leq \bigvee x_i$ implies that $x \leq x_{i_1} \vee x_{i_2} \vee \cdots \vee x_{i_n}$ for some $i_1, i_2, \ldots, i_n$. A complete lattice L is said to be continuous if for each $a \in L$, $a = \bigvee x\ (x \prec a)$. A continuous frame is a distributive continuous lattice, also called a locally compact frame (see e.g. [2],[6]). For regular continuous L, $x << y$ if and only if $x \prec y$ and $\uparrow x^*$ is compact, where $\uparrow x^* = \{z \in L | z \geq x^*\}$. Furthermore such a frame has a smallest strong inclusion given by: $x \triangleleft y$ if and only if $x \prec y$ and $\uparrow x^*$ is compact. This means then that L has a least compactification which is the frame conterpart to the Alexandroff one–point compactification of a locally compact non–compact Hausdorff space. We then have the following result which is just the localic version of Exercise iv 2.7 in [6].

3.1 Proposition. *Let L be a regular continuous frame. Let $B = \{a \in L|$ either $\uparrow a$ or $\uparrow a^*$ is compact$\}$. Then B is a normal base for L and $(\gamma_B L, \bigvee)$ is the least compactification of L.*

PROOF : That B is a normal base follows from:
(i) Let $a \in B,\ b \in B$. If either $\uparrow a^*$ or $\uparrow b^*$ is compact, then $\uparrow (a \wedge b)^*$ is compact and hence $a \wedge b \in B$. If $\uparrow a^*$ and $\uparrow b^*$ are not compact, then $\uparrow a$ and $\uparrow b$ are compact. Hence $\uparrow (a \wedge b)$ is compact and thus $a \wedge b \in B$.
(ii) Let $a \in B$. If $\uparrow a$ is compact, then since $a \leq a^{**}$ we have $\uparrow a^{**}$ is compact. Thus $a^* \in B$.
(iii) Let $a \in B,\ a \prec c$. If $\uparrow a^*$ is compact then $a << c$. Since the "way below" relation interpolates there exists $b \in L$ such that $a << b << c$. Now since $b << c$ we have $b \prec c$ and $\uparrow b^*$ is compact. This says $b \in B$. Thus $a \prec b \prec c$ with $b \in B$. If, on the other hand, $\uparrow a$ is compact, then $\uparrow c$ is compact also. Now $a \prec c$ implies that $a^* \vee c = e$ and hence $\bigvee c \vee x\,(x << a^*) = e$. Since $\uparrow c$ is compact we can find $x << a^*$ such that $c \vee x = e$. Now $x << a^*$ implies that $x \prec a^*$ and $\uparrow x^*$ is compact. Thus $a \leq a^{**} \prec x^* \prec c$ with $x^* \in B$ as required.
(iv) That B is indeed a base follows from the fact that $x << a$ implies that $x \in B$. ■

To show that $\gamma_B L$ is the least compactification, we show that $\triangleleft_B = \triangleleft$. Suppose $x \triangleleft_B y$. Find $c \in B$ such that $x \prec c \prec y$. If $\uparrow c$ is compact, then $\uparrow y$ is compact and hence $x \triangleleft y$. If $\uparrow c^*$ is compact, then $c \triangleleft y$ and hence $x \triangleleft y$. Suppose now that $x \triangleleft y$. Find $a, b \in L$ such that $x \triangleleft a \triangleleft b \triangleleft y$. If $\uparrow a^*$ is compact, then $a \in B$ and we have $x \prec a \prec y$. Thus $x \triangleleft_B y$. If $\uparrow b$ is compact, then $b \in B$ and $x \prec b \prec y$, so again $x \triangleleft_B y$.

4. Precompact uniform frames. It is well known that every Hausdorff compactification of a Tychonoff space is the Samuel compactification of a uniform space with respect to a precompact uniformity. The same is true for compactifications of frames as well. In this section we characterize those precompact uniformities on a frame whose Samuel compactification is of Fan–Gottesman type. Let us recall some preliminaries on uniform frames which we shall need. Uniform frames were introduced in [**8**], called uniform locales therein; also see [**5**], [**9**]. A cover of a frame L is a subset $A \subseteq L$ such that $\bigvee A = e$. Denote by $\mathrm{Cov}(L)$, the set of all covers of L. For $A, B \in \mathrm{Cov}(L)$, we write $A \leq B$ if for each $a \in A$ there is a $b \in B$ such that $a \leq b$. For $A, B \in \mathrm{Cov}(L)$, set $A \wedge B = \{a \wedge b | a \in A,\ b \in B\}$. Clearly $A \wedge B \in \mathrm{Cov}(L)$. For $A \in \mathrm{Cov}(L),\ x \in L$, let $\mathrm{st}(x, A) = \bigvee\{a \in A | a \wedge x \neq 0\}$. For $A, B \in \mathrm{Cov}(L)$, A is said to star-refine B, written $A^* \leq B$ if $\{\mathrm{st}(a, A) | a \in A\} \leq B$.

4.1 Definition ([**9**]). Let L be a frame. A non-empty set of covers μ of L is said to be a *uniformity* on L if

(i) $A \in \mu$ and $A \leq B \Longrightarrow B \in \mu$
(ii) $A \in \mu$ and $B \in \mu \Longrightarrow A \wedge B \in \mu$
(iii) For each $A \in \mu$ there exists $B \in \mu$ such that $B^* \leq A$
(iv) For each $a \in L,\ a = \bigvee x$ (for some $A \in \mu,\ \mathrm{st}(x, A) \leq a$)

(L, μ) is called a *uniform frame* .

As in the classical theory of uniform spaces we say that a non–empty subfamily $\mu' \subseteq \mu$ is a uniformity basis for μ if each member of μ is refined by some member of μ'. A non–empty subfamily $\mu'' \subseteq \mu$ is a uniformity subbasis for μ if the set of alll finite meets of members of μ'', is a basis for μ. Clearly μ' is a basis for some uniformity on L if and only if it is a filter basis satisfying (iii) and (iv) above. A uniform frame (L, μ) is said to be *precompact* (or totally bounded) if the finite uniform covers form a base for μ.

We recall the Samuel compactification of a uniform frame as defined by Banaschewski ([3]):

For a uniform frame (L, μ) define $x \triangleleft y$ if there is an $A \in \mu$ such that $\text{st}(x, A) \leq y$. Then $\triangleleft$ is a strong inclusion on L, as one may verify. The *Samuel compactification* of (L, μ) is defined to be $(RL, \bigvee)$, where RL consists of all the strongly regular ideals (with respect to $\triangleleft$) and $\bigvee : RL \to L$ is the join map.

For a frame L let $P(L)$ be the set of all precompact uniformities on L partially ordered by inclusion, and as earlier let $S(L)$ and $K(L)$ be the set of all strong inclusions and compactifications of L respectively. In [5] it is shown that every strong inclusion on L is induced by a unique precompact uniformity: Given $\triangleleft, \mu_0 = \{C_a^b | a, b \in L,\ a \triangleleft b\}$ where $C_a^b = \{a^*, b\}$ forms a subbasis for a precompact uniformity $\mu(\triangleleft)$ on L. Any uniformity μ on L induces a strong inclusion $\triangleleft(\mu)$ given by: $x \triangleleft(\mu) y$ if and only if there is an $A \in \mu$ such that $\text{st}(x, A) \leq y$. The maps $S(L) \to P(L)$ given by $\triangleleft \rightsquigarrow \mu(\triangleleft)$, and $P(L) \to S(L)$ given by $\mu \rightsquigarrow \triangleleft(\mu)$ are clearly order preserving, and by the result in [5] stated above are inverses of each other. Thus we have the proposition, the second statement of which follows from the first.

4.2 Proposition.
(a) $S(L) \cong P(L) \cong K(L)$.
(b) Every compactification of L is the Samuel compactification of L with respect to a precompact uniformity.

4.3 Definition. Let μ be a precompact uniformity on L. A base B for L is said to *generate* μ if the family of all finite covers of L from B is base for μ.

We may now prove

4.4 Proposition. *Let (L, μ) be a precompact uniform frame. Then the Samuel compactification $(RL, \bigvee)$ of (L, μ) is of Fan–Gottesman type if and only if μ possesses a generating base B which is normal.*

PROOF : ($\Leftarrow$): Assume μ possesses a generating base B which is normal. Let $\triangleleft$ be the strong inclusion induced by μ and $\triangleleft_B$ the strong inclusion associated with B. It suffices to show $\triangleleft = \triangleleft_B$. Suppose $x \triangleleft y$. Then there exists z, $x \triangleleft z \triangleleft y$. Find $A \in \mu$ such that $\text{st}(x, A) \leq z$. Find finite $C \subseteq B$ such that $\bigvee C = e$ and $C \leq A$. Let $C = \{b_1, b_2, \ldots, b_n\}$, say. By relabelling, if necessary, let $b_1, b_2, \ldots b_k$ be those elements of C for which $b_i \wedge x = 0$ and $b_{k+1}, \ldots, b_n$ be those for which $b_i \wedge x \neq 0$. Then

$$x \leq b_{k+1} \vee \cdots \vee b_n \leq (b_{k+1} \vee \cdots \vee b_n)^{**}$$

Now $(b_{k+1} \vee \cdots \vee b_n)^{**} = (b_{k+1}^* \wedge \cdots \wedge b_n^*)^* \in B$.
Further $x \prec b_{k+1} \vee \cdots \vee b_n$ (separating element being $b_1 \vee \cdots \vee b_k$) so that $x \prec$

$(b_{k+1} \vee \cdots \vee b_n)^{**}$.

We have $\mathrm{st}(x, C) \leq z \triangleleft y$ so that $b_{k+1} \vee \cdots \vee b_n \leq z \prec y$ and hence $(b_{k+1} \vee \cdots \vee b_n)^{**} \prec y$. We have found an element $b \in B$ such that $x \prec b \prec y$, i.e. $x \triangleleft_B y$. If on the other hand $x \triangleleft_B y$, then there exists z, $x \triangleleft_B z \triangleleft_B y$. Thus there exists $b, c \in B$ such that $x \prec b \prec z \prec c \prec y$. Then $\{b^*, c\} \subseteq B$, $b^* \vee c = e$ so that $\{b^*, c\} \in \mu$. Further $\mathrm{st}\,(x, \{b^*, c\}) = c \leq y$ so that $x \triangleleft y$.

($\Rightarrow$): Now assume the Samuel compactification of (L, μ) is a Fan–Gottesman type compactification. Then there exists a normal base B of L such that $(\gamma_B L, \bigvee) = (RL, \bigvee)$. We show B is a generatingg base for μ. Since $\gamma_B L = RL$, the coresponding strong inclusions on L are the same, i.e. $\triangleleft_B = \triangleleft$. Since μ is precompact and μ induces $\triangleleft$ it is evident from Proposition 4.2 and the remarks preceding it that μ has a subbasis $\{C_a^b = \{a^*, b\} | a \triangleleft b\}$. Take any C_a^b, $a \triangleleft b$. Then $a \triangleleft_B b$ and hence there exists c such that $a \triangleleft_B c \triangleleft_B b$, i.e. there exists $b_1, b_2 \in B$ such that $a \prec b_1 \prec c \prec b_2 \prec b$. Then $\{b_1^*, b_2\}$ is a cover of L from B which refines C_a^b. This implies every basic member, and hence every member of μ is refined by a cover of L from B. ∎

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