Commentationes Mathematicae Universitatis Carolinae

Thomas N. Vougiouklis

On rings with zero divisors. Strong V-groups

Commentationes Mathematicae Universitatis Carolinae, Vol. 31 (1990), No. 3, 431--433

Persistent URL: http://dml.cz/dmlcz/106878

On rings with zero divisors. Strong V-groups

THOMAS VOUGIOUKLIS

Abstract. The strong V-groups are groups with elements zero divisors of a ring. Using the above groups on matrices a more refinement inequality than a known one is proved. Moreover, a construction of hyperrings is given.

Keywords: Strong V-group, hyperring

Classification: 16A42, 16A78, 20N99

1. A large class of rings with zero divisors contains strong V-groups which are defined in [5] as follows:

Definition. An additive subgroup $M \neq \{0\}$ of a ring R, with zero divisors, is called a strong V-group (sV-group) if:

$$rm = mr = 0, \quad \forall m \in M \quad \text{iff} \quad r \in M.$$

One can see that in a ring R with the property (Z) (see [2]), the set of all zero divisors is an sV-group. In this case R is a completely primary ring [1]. Of course, we have rings containing an sV-group which are not rings with the property (Z).

In this paper, we firstly find all sV-groups in $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$. Secondly, we introduce an sV-group in the ring of square matrices, a special case of which can be used to obtain a more refinement inequality than the one appeared in [3]. Finally, we use the V-groups to construct a class of hyperrings.

2. In this paragraph, we fix a non-zero natural number m, we consider the set $\mathbb{Z}_m$ and denote the mod m class of the integer n by $\bar{n} = n + m\mathbb{Z}$.

Theorem 1. *The ring $\mathbb{Z}_m$ has an sV-group iff $m = m_e^2, m_e \in \mathbb{Z}$. In this case there exists only one sV-group which is the ideal generated by the element $\bar{m}_e$ of $\mathbb{Z}_m$.*

PROOF : We write the integer m in the form

$$m = p_1^{a_1} p_2^{a_2} \dots p_n^{a_n}, \quad \text{where} \quad p_1 < p_2 < \dots < p_n \quad \text{are primes and} \quad a_i > 0.$$

According to the Chinese remainder theorem the ring $\mathbb{Z}_m$ is isomorphic to the direct product of $\mathbb{Z}_{m_e^2}$ and $\mathbb{Z}_{m_0}$, where m_e^2 is the product of those $p_i^{a_i}$'s, where a_i is an even number and $m_0 = p_{v_1}^{a_{v_1}} \dots p_{v_s}^{a_{v_s}}$, where $a_{v_j} = 2k_j + 1, \quad j = 1, \dots, s$ are all odd exponents. Thus

$$m = m_e^2 p_{v_1}^{2k_1+1} \dots p_{v_s}^{2k_s+1}.$$

We shall prove that the component $\mathbb{Z}_{m_0}$ does not appear iff $\mathbb{Z}_m$ has an sV-group. An element a mod m, in order to be an element of some sV-group, must contain the factor $n = m_e p_{v_1}^{k_1+1} \dots p_{v_s}^{k_s+1}$ because we must have $a^2 \equiv 0 \bmod m$. Therefore, from the definition, any sV-group must contain the ideal J generated by the element $\bar{n}$. Therefore, any sV-group must be equal to the ideal J. Now we observe that if $m = m_e^2$ then J is an sV-group. If one of the factors $p_{v_j}^{2k_j+1}$ appears, then the element $x = m_e p_{v_1}^{k_1} \dots p_{v_s}^{k_s}$ has the property $xJ \equiv 0 \bmod m$, but $x^2 \not\equiv 0 \bmod m$, thus J is not an sV-group. Therefore $\mathbb{Z}_m$ has an sV-group, unique, iff $J = [\bar{m}_e]$. ■

3. Let us denote by F_n the ring of $n \times n$ matrices over the finite field F, with characteristic $\neq 2$, with q elements, i.e. $|F| = q$. Let S_n^{xy} be the set of xy-symmetric $n \times n$ matrices [5] $A = (a_{ij})$, where

$$\left.\begin{array}{ll}(\mathrm{I}) & a_{ij} = a_{n+1-i,j} \\ (\mathrm{II}) & a_{ij} = -a_{i,n+1-j}\end{array}\right\} \quad \text{for all} \quad i, j = 1, \dots, n.$$

The set S_n^{xy} is an sV-group in F_n. One can notice that

$$|S_n^{xy}| = q^{n^2/4} \quad \text{when } n \text{ is even number, and}$$
$$|S_n^{xy}| = q^{(n^2-1)/4} \quad \text{when } n \text{ is odd number.}$$

Theorem 2. *The following relation is valid for $n \geqslant 2$:*

$$|F_n| < s(F_n)|s_n^{xy}|^{4/(n^2-1)} < s(F_n)^{1+1/(n(n-1))},$$

where $s(F_n)$ is the number of singular matrices in F_n.

PROOF : For n even or odd number, we have respectively

$$|s_n^{xy}|^{4n/(n+1)} = q^{n^3/(n+1)} < q^{n^2-1} \quad \text{or} \quad |s_n^{xy}|^{4n/(n+1)} = q^{n^2-n} < q^{n^2-1}.$$

Therefore for every n we have

$$|s_n^{xy}|^{4n/(n+1)} < q^{n^2-1}.$$

But according to the lemma in [3], we have $q^{n^2-1} < s(F_n)$ so $|s_n^{xy}|^{4n/(n+1)} < s(F_n)$ and

$$s(F_n)|s_n^{xy}|^{4/(n^2-1)} = s(F_n)(|s_n^{xy}|^{4n/(n+1)})^{1/(n(n-1))} < s(F_n)^{1+1/(n(n-1))}.$$

On the other hand, using the same lemma, we have for n even or odd respectively

$$s(F_n)|S_n^{xy}|^{4/(n^2-1)} > q^{n^2-1} \cdot q^{(n^2/4)\cdot(4/(n^2-1))} > q^{n^2} = |F_n|$$

or

$$s(F_n)|S_n^{xy}|^{4/(n^2-1)} > q^{n^2-1} \cdot q^{((n^2-1)/4)\cdot(4/(n^2-1))} = q^{n^2} = |F_n|.$$

■

4. The P-hypergroups, introduced in [6] and generalized in [7], also cf. [8], is a large class of hypergroups of Marty defined on semigroups with a given subset P. One can also define P-hyperoperations whenever there are structures with more than one associative operation, see [4]. In the following, we give such a construction on rings with sV-groups.

Theorem 3. *Let M be an sV-group of the ring R and $P \subset M$. We consider the following two P-hyperoperations:*

$$M^* : xM^*y = x + M + y \qquad \textit{addition},$$
$$P^* : xP^*y = xPy \qquad \textit{multiplication}.$$

Then $\langle R, M^*, P^* \rangle$ is a hyperring.

PROOF : Both hyperoperations M^*, P^* are associative. Moreover, for every x, y, z of R we have, since $P \subset M$ and M is an sV-group,

$$xP^*(yM^*z) = xP(y + M + z) \subset xPy + xPM + xPz = xPy + xPz.$$

On the other hand, we have

$$(xP^*y)M^*(xP^*z) = xPy + M + xPz.$$

Therefore, since $0 \in M$, the hyperoperation P^* is distributive, not strong, with respect to M. So $\langle R, M^*, P^* \rangle$ is a hyperring. ■

Remark. If M is an ideal of R, then for every $P \subset R$ the hyperstructure $\langle R, M^*, P^* \rangle$ is a hyperring. This remark can be applied to $\mathbf{Z}_m, m = m_e^2$, see Theorem 1, but not in the general case of xy-symmetric matrices, since in this case s_n^{xy} is not an ideal. We notice that here M is not necessarily an sV-group but an ideal of R. For an analogous construction see also [4].

REFERENCES

[1] Al-Khames Y., *Finite rings in which the multiplication of any two zero-divisors is zero*, Archiv der Mathematik **37** (1981), 144–149.

[2] Corbas B., *Finite rings in which the product of any two zero divisors is zero*, Archiv der Mathematik **21** (1970), 466–469.

[3] Koh K., *On the matrix ring over a finite field*, Linear Algebra and Appl. **66** (1985), 195–197.

[4] Konguetsof L., Vougiouklis T., *Constructions d'hyperanneaux à partir d'anneau*, Acta Univ. Carolinae - Math. et Ph. **28** (1987), 9–13.

[5] Vougiouklis T., *A classification of rings with zero divisors*, Math. Balkanica, N.S.3 **3–4** (1989), 241–243.

[6] Vougiouklis T., *Cyclicity in a special class of hypergroups*, Acta Un. Carolinae-Math. et Ph. **22** (1981), 3–6.

[7] Vougiouklis T., *Generalization of P-hypergroups*, Rend. Circolo Mat. Palermo, S.II **36** (1987), 114–121.

[8] Vougiouklis T., Konguetsof L., *P-hypergroups*, Acta Univ. Carolinae – Math. et Ph. **28** (1987), 15–20.

Democritus University of Thrace, 67100 Xanthi, Greece

(*Received* February 13, 1990)