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Abstract. This paper presents some examples of φ-measures of noncompactness and some fixed point theorems for multivalued mappings in general topological vector spaces. An example of a (φ,γ)-condensing mapping in a general topological vector space which has a fixed point is given.

Keywords: Fixed point, measure of noncompactness, condensing mapping, topological vector space

Classification: 47H10

1. Introduction.

In the fixed point theory in locally convex spaces, the invariability of the formation of the convex hull of a measure of noncompactness is of great importance. But the known nontrivial measures of noncompactness ([1], [2], [9], [10]) in locally convex spaces are not measures of noncompactness in nonlocally convex topological vector spaces, since we have $\gamma(\operatorname{co} M) \neq \gamma(M)$, in general. In [3], [4], [5], [6] and [7] Hahn and Hadžić proved that for some mappings $\varphi : [0,\infty) — [0,\infty)$ the inequality $\gamma(\operatorname{co} M) \leq \varphi(\gamma(M))$ on special sets of Zima's type ([5]) in paranormed spaces is true, where γ is the well-known Kuratowski's or the inner Hausdorff's measure of noncompactness. To this, Hadžić introduced in [5] the notion of the φ-measure of noncompactness. We shall give some examples for φ-measures of noncompactness on a set of Zima's type in a general topological vector space.

Using this result, we obtain further fixed point theorems for multivalued mappings ([5], [6], [7]). Finally, we shall give a nontrivial example of a (φ,γ)-condensing mapping in a general topological vector space which has a fixed point. In this paper, every topological vector space will be assumed to be Hausdorff and real.

Let E be a topological vector space and $K \subseteq E$. By $\mathfrak{U}(E)$ we denote a fundamental system of circled, closed neighbourhoods of zero in E, by $\mathfrak{F}_{\mathfrak{U}}$ the set of all nonnegative functions on $\mathfrak{U}(E)$ with the natural order and by p_U the Minkowski functional of $U \in \mathfrak{U}(E)$. Moreover, we denote by $\overline{K}, \operatorname{co} K, \overline{\operatorname{co}} K$ and δK the closed hull, the convex hull, the closed convex hull and the boundary of K. We define $2^K := \{M \subseteq K : M \neq \emptyset\}, b(K) := \{M \in 2^K : M \text{ is bounded }\}, \operatorname{cc}(K) := \{M \in 2^K : M \text{ is closed in } K, M \text{ is convex}\}$ and $\operatorname{fucc}(E) := \{K \subseteq E : K = \bigcup_{i\in I} K_i, I \text{ is finite}, K_i \in \operatorname{cc}(E) \text{ for all } i \in I\}$.

K is said to be <u>admissible</u>, if for every compact subset M of K and every neighbourhood U of zero in E, there exists a continuous mapping $T_U : M \to K$ such that $\dim T_U(M)^{\text{lin}} < \infty$ and $x - T_U(x) \in U (x \in M)$.

The set K will be called <u>locally convex</u> ([8]), iff for any $x \in K$ there exists in K a base of neighbourhoods $U(x)$ of x with $U(x) = W(x) \cap K$ and $W(x)$ is a convex subset of E. Jerofsky proved that every locally convex set $K \in \text{fucc}(E)$ is admissible ([**8**, Satz 1.5.3.]).

We say that K is starshaped, relative to some $u \in K$, iff $tx + (1-t)u \in K$ for all $x \in K$ and all $t \in [0,1]$.

Finally, $K \in \text{fucc}(E)$ is said to be pseudoconvex ([**8**]), if there is a finite dimensional subspace E_0 of E, such that for all finite dimensional subspaces $E' \supseteq E_0$, the set $K \cap E'$ is a retract of E'. Especially, K is pseudoconvex, if K is starshaped, relative to some $u \in K$ and $K \in \text{fucc}(E)$ ([**7**]).

2. φ-measures of noncompactness in topological vector spaces.

Let E be a topological vector space, $K \in 2^E$, $M \in b(\overline{\text{co}}K)$ and $U \in \mathfrak{U}(E)$. Let us define:

$$\alpha(M,U) := \inf\{a > 0 : \text{ There exist } x_1, \ldots, x_n \in E \text{ such that } M \subseteq \bigcup_{i=1}^{n}(x_i + aU)\},$$

$$\beta(M,U) := \inf\{a > 0 : \text{ There exist } x_1, \ldots, x_n \in M \text{ such that } M \subseteq \bigcup_{i=1}^{n}(x_i + aU)\},$$

$$\chi(M,U) := \inf\{a > 0 : \text{ There exist } D_1, \ldots, D_n \subseteq E \text{ such that } M \subseteq \bigcup_{i=1}^{n} D_i \text{ and } D_i - D_i \subseteq aU \, (i \in \{1, \ldots, n\})\}$$

and

$$J(M,U) := \sup\{a \geq 0 : M \text{ contains a countably infinite set } \{x_n : n \in \mathbb{N}\} \text{ with } x_i - x_k \notin aU \text{ for } i \neq k\}$$

($\sup \emptyset = 0$, by definition).
By $[\gamma_{\mathfrak{U}}(M)](U) := \gamma(M,U)$ there is defined a mapping $\gamma_{\mathfrak{U}} : b(\overline{\text{co}}K) \to \mathfrak{F}_{\mathfrak{U}}$ for $\gamma \in \{\alpha, \beta, \chi, J\}$.

If $(E, \|\cdot\|)$ is a normed space, M a bounded subset of E and $U := \{x \in E : \|x\| \leq 1\}$, then the well-known measures of noncompactness — the Hausdorff's, the Kuratowski's and the Istrătescu's measure of noncompactness — of the set M are defined by $\gamma(M,U)$ for $\gamma \in \{\alpha, \beta, \chi, J\}$ ([**1**], [**2**], [**9**], [**10**]). As [**1**, Proposition 1] the following lemma shows that $\alpha_{\mathfrak{U}}, \beta_{\mathfrak{U}}, \chi_{\mathfrak{U}}$ and $J_{\mathfrak{U}}$ are, in a sense, all equivalent.

Lemma 1. *Let E be a topological vector space, $K \in 2^E, U \in \mathfrak{U}(E), V \in \mathfrak{U}(E)$ and $V + V \subseteq U$. Then, for every $M \in b(\overline{\text{co}}K) : \alpha(M,U) \leq \beta(M,U) \leq J(M,U) \leq \chi(M,U) \leq \alpha(M,V)$.*

PROOF : ([**1**, p. 404]) The first inequality is easy. Let $a > J(M,U)$. We choose a maximal family of elements $x_1, \ldots, x_n \in M$ such that $x_i - x_k \notin aU$ for $i \neq k$. Then $M \subseteq \bigcup_{i=1}^{n}(x_i + aU)$ and therefore $\beta(M,U) \leq J(M,U)$.
Let us suppose that $J(M,U) > a > \chi(M,U)$. Then there are $D_1, \ldots, D_m \subseteq E$ such that $M \subseteq \bigcup_{j=1}^{m} D_j$ and $D_j - D_j \subseteq aU \, (j \in \{1, \ldots, m\})$. Moreover, there is

a countably infinite subset $\{x_n : n \in \mathbb{N}\}$ of M with $x_i - x_k \notin aU$ for $i \neq k$. At least one of D_j's contains an infinite number of elements $x_{n_i} \in \{x_n : n \in \mathbb{N}\}$. Hence there are $x_{n_i}, x_{n_k} \in \{x_n : n \in \mathbb{N}\}$ with $x_{n_i} - x_{n_k} \notin aU$ and $x_{n_i} - x_{n_k} \in D_j - D_j \subseteq aU$ for some j. This is a contradiction. Therefore $J(M,U) \leq \chi(M,U)$.
Let $\alpha(M,V) < a$. Then there exist $x_1,\ldots,x_n \in E$, such that $M \subseteq \bigcup_{i=1}^{n}(x_i + aV)$. Put $D_i := x_i + aV$. Then $D_i - D_i \subseteq a(V+V) \subseteq aU\,(i \in \{1,\ldots,n\})$ and $M \subseteq \bigcup_{i=1}^{n} D_i$. This means that $\chi(M,U) \leq \alpha(M,V)$. The proposition is proved. ■

Now we shall state some properties of $J_{\mathfrak{U}}$. Most of them are well-known for the Istrătescu's measure of noncompactness([1]) in normed spaces.

Proposition 1. *Let E be a topological vector space, $U,V \in \mathfrak{U}(E)$, $V+V \subseteq U$, $K \in 2^E, M,N \in b(\overline{\text{co}}K)$ and $s,t > 0$. Then*

(1) $J(M \cup N, U) = \max\{J(M,U), J(N,U)\}$,
(2) $N \subseteq M \Rightarrow J(N,U) \leq J(M,U)$,
(3) $M + N \in b(\overline{\text{co}}K) \Rightarrow J(M+N,U) \leq J(M,V) + J(N,V)$,
(4) $sM \in b(\overline{\text{co}}K) \Rightarrow J(sM,tU) = s \cdot t^{-1} J(M,U)$,
(5) $J(M,U) = J(\overline{M},U)$,
(6) $J_{\mathfrak{U}}(M) \equiv 0$ *iff M is precompact.*

Proof : (1) The inequality $\max\{J(M,U), J(N,U)\} \leq J(M \cup N, U)$ follows from $M,N \subseteq M \cup N$ and from the definition of J. Let $0 < a < J(M \cup N, U)$. Then there is a countably infinite set $\{x_n : n \in \mathbb{N}\} \subseteq M \cup N$ with $x_i - x_k \notin aU$ for $i \neq k$. The set $\{x_n : n \in \mathbb{N}\}$ contains an infinite number of elements of M or of N. Hence there is $J(M,U) \geq a$ or $J(N,U) \geq a$. So we obtain the inequality $\max\{J(M,U), J(N,U)\} \geq J(M \cup N, U)$, too.

(2) follows from $M = N \cup (M \setminus N)$ and (1).

(3) Suppose that $J(M+N,U) > J(M,V) + J(N,V)$. Without loss of generality, we may assume that $J(N,V) \leq J(M,V)$. We choose $a > 0$ with $J(M,V) < a < J(M+N,U)$. Then there is a countably infinite set $\{z_n = x_n + y_n : x_n \in M, y_n \in N, n \in \mathbb{N}\}$ such that $z_i - z_k \notin aU$ for $i \neq k$. Since $a > J(M,V)$, there is an infinite subset $\{x_{n_j} : j \in \mathbb{N}\}$ of $\{x_n : n \in \mathbb{N}\}$ with $x_{n_i} - x_{n_k} \in aV$ for $i \neq k$. Then there must be $y_{n_i} - x_{n_k} \notin aV$ for $i \neq k$, where $y_{n_j} = z_{n_j} - x_{n_j} \in N (j \in \mathbb{N})$. Therefore $J(N,V) \geq a$. It is contradictory to $J(N,V) \leq J(M,V) < a$. So (3) is true.

The properties (4) and (5) can be established easily.

(6) Let $a > 0$ and M precompact. Then there are $x_1,\ldots,x_n \in M$ such that $M \subseteq \bigcup_{i=1}^{n}(x_i + aV)$. Let M_0 be an arbitrary infinite subset of M. At least one of the sets $x_i + aV$ contains an infinite number of elements of M_0. Hence there are $x,y \in M_0, x \neq y$, such that $x - y \in a(V+V) \subseteq aU$. Therefore $J(M,U) = 0$. If $J(M,U) = 0$, then there are $x_1,\ldots,x_n \in M$ such that $M \subseteq \bigcup_{i=1}^{n}(x_i + U)$. From this, the assertion (6) follows. Thus, the proposition is proved. ■

Remark. The mappings $\alpha_{\mathfrak{U}}$ and $\chi_{\mathfrak{U}}$ satisfy also the properties (1) to (6), $\beta_{\mathfrak{U}}$ only (3), (4), (5) and (6), in general ([1, p. 404]). In [5], Hadžić introduced the following notion.

Definition 1. Let E be a topological vector space, $K \in 2^E, A$ a partially ordered set with the partial ordering $\leq, \varphi : A \to A$ and $\mathfrak{M}$ a system of subsets of $\overline{\text{co}}K$ such that:

$$M \in \mathfrak{M} \Rightarrow (\overline{M} \in \mathfrak{M}, \operatorname{co} M \in \mathfrak{M}, M \cup \{u\} \in \mathfrak{M} (u \in K), N \in \mathfrak{M} (N \subseteq M)).$$

Let γ be a mapping of $\mathfrak{M}$ into A. The mapping γ is said to be a φ-measure of noncompactness on K, iff the following conditions are satisfied:

$$\begin{aligned} &(1) \quad \gamma(\overline{M}) = \gamma(M \cup \{u\}) = \gamma(M) \geq \gamma(N), \quad (M \in \mathfrak{M}, N \subseteq M, u \in K), \\ &(2) \quad \gamma(\operatorname{co} M) \leq \varphi(\gamma(M)), \quad (M \in \mathfrak{M}). \end{aligned}$$

Remark. Let c be a real number with $c \geq 1$. If $\varphi(t) = c \cdot t (t \in A)$, then γ is said to be a c-measure of noncompactness on K ([7]). For $c = 1$, the mapping γ will be called a measure of noncompactness on K.

Definition 2 ([5]). Let E be a topological vector space and $K \in 2^E$. The set K is said to be of Zima's type, iff for every $U \in \mathfrak{U}(E)$ there exists $V \in \mathfrak{U}(E)$ such that $\operatorname{co}(V \cap (K - K)) \subseteq U$.

Some examples of the sets of Zima's type in paranormed spaces ([**3**, p. 34]) can be found in [**3**], [**4**], [**5**], [**6**], [**7**]. Let (E, p) be a paranormed space. It is well-known that E is a metrizable topological vector space in which the topology is introduced by the family $\mathfrak{O} = \{v_r : r > 0\}$ of neighbourhoods of zero in E, where $V_r := \{x \in E : p(x) \leq r\}$. Let $K \in 2^E$ and $\varphi : (0, \infty) \to (0, \infty)$. The set K is said to be of Z_φ-type, iff, for every $r > 0$, $\operatorname{co}(V_r \cap (K - K)) \subseteq V_{\varphi(r)}$ ([**5**]). We say that K satisfies the Zima condition (with the constant r), iff there exists $r > 0$ such that $p(tx) \leq rtp(x)$ for all $t \in [0, 1]$ and all $x \in K - K$ ([**3**]).

It is clear that a set is of Z_φ-type, if it satisfies the Zima condition. Moreover, every set which is of Z_φ-type, is of Zima's type also, if the mapping φ is such that $\inf_{r>0} \varphi(r) = 0$ ([**5**]).

Lemma 2. *Let E be a topological vector space and $K \in b(E)$ which is of Zima's type and starshaped, relative to some $u \in K$. Then, for every $U \in \mathfrak{U}(E)$ there is $W \in \mathfrak{U}(E)$ such that*

$$\beta(\operatorname{co} M, U) \leq \beta(M, W) \quad (M \subseteq K).$$

PROOF : If M is precompact, then $\operatorname{co} M$ is also precompact, because K is of Zima's type ([**3**]). It is clear that in this case the assertion is true.

Now, we suppose that M is not precompact. We choose $V \in \mathfrak{U}(E), W \in \mathfrak{U}(E)$ such that $V + V \subseteq U, W \subseteq V, \operatorname{co}(W \cap (K - K) \subseteq V$ and $\beta(M, W) > 0$. This is possible, because K is of Zima's type and M is not precompact.

Without loss of generality, we may assume that $\beta(M, W) \geq 1$. Otherwise, we choose $c > 1$ with $c\beta(M, W) = \beta(M, c^{-1}W) \geq 1$ and replace W by $c^{-1}W$.

Let $a > \beta(M, W)$. Then there exist $x_1, \ldots, x_m \in M$ such that $M \subseteq \bigcup_{i=1}^m (x_i + aW)$.

Let $y \in \operatorname{co} M$. Then there are $y_k \in M, c_k \geq 0 (k \in \{1, \ldots, n\})$ with $\sum_{k=1}^n c_k = 1$, so that $y = \sum_{k=1}^n c_k y_k$. Since $y_k \in M (k \in \{1, \ldots, n\})$, there exists x_{i_k}

($i_k \in \{1,\ldots,m\}$) such that $y_k - x_{i_k} \in aW$. We put $z := \sum_{k=1}^{n} c_k x_{i_k}$. Then $z \in \mathrm{co}\{x_1,\ldots,x_m\}$ and $y - z = \sum_{k=1}^{n} c_k(y_k - x_{i_k}) \in \mathrm{co}(aW \cap (K-K)) \subseteq aV$, because $K - K$ is starshaped and $a \geq 1$.

From the precompactness of the set $\mathrm{co}\{x_1,\ldots,x_m\}$ it follows that there exists $\{z_1,\ldots,z_p\} \subseteq \mathrm{co}\{x_1,\ldots,x_m\}$ such that $\mathrm{co}\{x_1,\ldots,x_m\} \subseteq \bigcup_{j=1}^{p}(z_j + aV)$.

Hence, there is $j \in \{1,\ldots,p\}$ such that $y - z_j = y - z + z - z_j \in a(V+V) \subseteq aU$. Therefore, we have $\beta(\mathrm{co}\, M, U) \leq a$ and finally $\beta(\mathrm{co}\, M, U) \leq \beta(M, W)$. ∎

From Lemma 1 and Lemma 2 we obtain the

Corollary. *Let $\gamma \in \{\alpha, \chi, J\}$ and assume that the hypotheses of Lemma 2 are satisfied. Then, for every $U \in \mathfrak{U}(E)$ there exists $W \in \mathfrak{U}(E)$ such that*

$$\gamma(\mathrm{co}\, M, U) \leq \gamma(M, W) \text{ for all } M \subseteq K.$$

Let K be a nonempty bounded and convex subset of a paranormed space. Hądžić proved in [5] that the inner Hausdorff's and the Kuratowški's measure of noncompactness satisfy the condition (2) from Definition 1, if K is of Z_φ-type and $\varphi : [0,\infty) \to [0,\infty)$ is a right continuous and a continuous mapping, respectively. Special results of this kind can be found in [4], [6] and [7]. Using Lemma 2 and the Corollary, we shall prove an analogous statement in general topological vector spaces.

Proposition 2. *Let E be a topological vector space, $K \in b(E)$ which is of Zima's type and starshaped, relative to some $u \in K$, and $\gamma_{\mathfrak{U}} \in \{\alpha_{\mathfrak{U}}, \chi_{\mathfrak{U}}, J_{\mathfrak{U}}\}$. Then there is a mapping $\varphi^* : \mathfrak{F}_{\mathfrak{U}} \to \mathfrak{F}_{\mathfrak{U}}$ such that $\gamma_{\mathfrak{U}}$ is a φ^*-measure of noncompactness on K.*

PROOF : From Proposition 1 and the remark following it, it follows that $\gamma_{\mathfrak{U}}$ satisfies the condition (1) of Definition 1.

We prove (2) of Definition 1. For every $U \in \mathfrak{U}(E)$, we can choose some $W_U \in \mathfrak{U}(E)$ (fixed) such that $\gamma(\mathrm{co}\, M, U) \leq \gamma(M, W_U)$ for every $M \subseteq K$ (Lemma 2, Corollary). By $\nu(U) := W_U (U \in \mathfrak{U}(E))$, we define a mapping $\nu : \mathfrak{U}(E) \to \mathfrak{U}(E)$. Then we have $[\gamma_{\mathfrak{U}}(M) \circ \nu](U) = [\gamma_{\mathfrak{U}}(M)](W_U) \quad (U \in \mathfrak{U}(E), M \subseteq K)$.
Put $\varphi^*(f) := f \circ \nu$ for every $f \in \mathfrak{F}_{\mathfrak{U}}$. Then φ^* is a mapping of $\mathfrak{F}_{\mathfrak{U}}$ into $\mathfrak{F}_{\mathfrak{U}}$. Since

$$[\gamma_{\mathfrak{U}}(\mathrm{co}\, M)](U) = \gamma(\mathrm{co}\, M, U) \leq \gamma(M, W_U) = [\gamma_{\mathfrak{U}}(M)](W_U) = [\varphi^*(\gamma_{\mathfrak{U}}(M))](U)$$

for every $U \in \mathfrak{U}(E)$ and every $M \subseteq K$, we obtain

$$\gamma_{\mathfrak{U}}(\mathrm{co}\, M) \leq \varphi^*(\gamma_{\mathfrak{U}}(M)) \quad (M \subseteq K).$$

∎

Remark. Of course $\beta_{\mathfrak{U}}$ satisfies (2) of Definition 1 relative to φ^*, too. However $\beta_{\mathfrak{U}}$ is not, in general, a φ^*-measure of noncompactness ([1, p. 404]).

3. Fixed point theorems.

Let E be a topological vector space, $M \subseteq E$ and $K \subseteq E$. We consider multivalued mappings of the kind $F : M \to 2^K$. A point $x \in M$ will be called a fixed point, iff $x \in F(x)$. $F : M \to 2^K$ is said to be upper semicontinuous, iff for every closed subset A of K the set $F^{-1}(A) := \{x \in M : F(x) \cap A \neq \emptyset\}$ is closed in M. We say that $F : M \to 2^K$ is compact, iff F is upper semicontinuous and $\overline{F(M)}$ is compact. Finally, a mapping $G : M \to K$ will be called a generalized contraction ([**9**, Definition 2.3]), iff for every $U \in \mathfrak{U}(E)$ there exists a real function q_U with $0 < \sup\{(q_U/[a,b])(c) : c \in [a,b]\} < 1\, (0 \leq a < b < \infty)$ such that we have

$$p_U(G(x) - G(y)) \leq q_U(p_U(x-y))p_U(x-y)$$

for all $x \in M, y \in M$.

Definition 3 ([**4**, Definition 6]). Let E be a topological vector space, $M \in 2^E, K \in 2^E, M \subseteq K, F : M \to \mathrm{cc}\, K$ an upper semicontinuous mapping, $\varphi : A \to A$ (see Definition 1) and γ a φ-measure of noncompactness on K. We call F a (φ, γ)- condensing mapping, iff for every $N \subseteq M$ the following implication holds:

$$\gamma(N) \leq \varphi(\gamma(F(N))) \Rightarrow \overline{F(N)} \text{ is compact.}$$

Remark. If $\varphi(t) = c \cdot t\, (t \in A)$, where $c \geq 1$, then F is said to be γ-pseudo-condensing ([**6**, Definition 5]). Using a theorem of Jefrosky ([**8**, Folgerung 4.3.5]), the following theorem can be proved in the same way as Theorem 1 from [**5**].

Theorem 1. *Let E be a topological vector space and K an admissible subset of E with $K \in \mathrm{fucc}(E)$ which is starshaped, relative to some $u \in K$. Let $U \subseteq K$ be an in K closed neighbourhood of u, γ a φ-measure of noncompactness on K and $F : U \to \mathrm{cc}(K)$ a (φ, γ)-condensing mapping with*

$$x \notin tF(x) + (1-t)u \quad (x \in \delta_K U, t \in (0.1)).$$

Then F has a fixed point.

Hadžić stated in [**5**] a special variant of Theorem 1 for a convex subset K of a paranormed space, where K is a special set of Zima's type (Corollary to Theorem 1 from [**5**]). The following Corollary is a generalization of this result. Since every set of Zima's type is an admissible set, we obtain (using Proposition 2 to Theorem 1) the following

Corollary. *Let E be a topological vector space and $K \in \mathrm{fucc}(E) \cap b(E)$ which is of Zima's type and starshaped, relative to some $u \in K$. Let $U \subseteq K$ be an in K closed neigbourhood of u, $\gamma \in \{\alpha_{\mathfrak{U}}, \chi_{\mathfrak{U}}, J_{\mathfrak{U}}\}$, φ^* the mapping defined in the proof of Proposition 2 and $F : U \to \mathrm{cc}(K)$ a (φ^*, γ)-condensing mapping with*

$$x \notin tF(x) + (1-t)u \quad (x \in \delta_K U, t \in (0,1)).$$

Then F has a fixed point.

The next statement can be proved in the same way as Theorem 2(5) from [**7**].

Theorem 2. *Let E be a topological vector space, $K \in 2^E$ a pseudoconvex and locally convex set, γ a φ-measure of noncompactness on K and $F : K \to \mathrm{cc}(K)$ a (φ, γ)-condensing mapping. Then there exists $x \in K$ such that $x \in F(x)$.*

Corollary. *Let E be a topological vector space, $K \in b(E)$ of Zima's type and starshaped, relative to some $u \in K$. Let φ^* be the mapping constructed in the proof of Proposition 2, let $\gamma \in \{\alpha_{\mathfrak{U}}, \chi_{\mathfrak{U}}, J_{\mathfrak{U}}\}$ and $F : K \to \mathrm{cc}(K)$ a (φ^*, γ)-condensing mapping. Then F has a fixed point.*

PROOF : Since every set of Zima's type is a locally convex set ([6, Proposition 1]) and every starshaped set is pseudoconvex, the assertion follows from Theorem 2 and Proposition 2. ∎

Hahn gave nontrivial examples of χ-pseudo-condensing mappings in paranormed spaces in [6] and [7], where χ is the in metric spaces well-known Kuratowski's measure of noncompactness. Now we shall give an example of a $(\varphi^*, J_{\mathfrak{U}})$-condensing mapping in a general topological vector space.

Proposition 3. *Let E be a topological vector space, $M \in 2^E, K \in 2^E, M \subseteq K$ and $\mathrm{co}\, K \in b(E)$. Moreover, let $F : M \to 2^K$ be a mapping with the following properties:*

(1) $F = F_1 + F_2$,
(2) $F_1 : M \to K$ *is a generalized contraction,*
(3) $F_2 : M \to 2^K$ *is compact.*

Then for every $U \in \mathfrak{U}(E)$, every $V \in \mathfrak{U}(E)$ with $V + V \subseteq U$ and for every $N \subseteq M$, the inequality

$$J(F(N), U) \leq \sup\{q_V(p_V(x-y)) : x, y \in N\} \cdot J(N, V)$$

holds.

PROOF : Suppose that $J(F_1(N), V) > a > 0$. Then there exists a subset $\{x_n : n \in \mathbb{N}\}$ of N such that

$$p_V(F_1(x_i) - F_1(x_k)) > a \text{ for } i \neq k.$$

Because N is bounded, we have the estimate

$$0 \leq p_V(x_i - x_k) \leq \sup\{p_V(x-y) : x, y \in N\} < \infty \text{ for all } i, k \in \mathbb{N}.$$

Since F_1 is a generalized contraction, there is

$$\begin{aligned} a < p_V(F_1(x_i) - F_1(x_k)) &\leq q_V(p_V(x_i - x_k)) p_V(x_i - x_k) \\ &\leq \sup\{q_V(p_V(x-y)) : x, y \in N\} \cdot p_V(x_i - x_k) \text{ for } i \neq k. \end{aligned}$$

This means that

$$J(F_1(N), V) \leq \sup\{q_V(p_V(x-y)) : x, y \in N\} \cdot J(N, V).$$

Since $F(N) \subseteq F_1(N) + F_2(N)$ and $\overline{F_2(N)}$ is compact, now we obtain from Proposition 1

$$J(F(N), U) \leq J(F_1(N), V) + J(F_2(N), V) \leq \sup\{q_V(p_V(x-y)) : x, y \in N\} \cdot J(N, V).$$

∎

Proposition 4. *Let E be a quasicomplete topological vector space, $K \in b(E)$ which is of Zima's type and starshaped, relative to some $u \in K, M \in 2^K, \varphi^* : \mathfrak{F}_{\mathfrak{U}} \to \mathfrak{F}_{\mathfrak{U}}$ the mapping constructed in the proof of Proposition 2 and $F : M \to \mathrm{cc}(K)$ a mapping with the properties* (1), (2), (3) *from Proposition 3. Moreover, for every $U \in \mathfrak{U}(E)$, every $V \in \mathfrak{U}(E)$ with $V + V \subseteq U$ and every $N \subseteq M$, the following conditions are satisfied:*

(1) $\sup\{q_V(p_V(x-y)) : x, y \in N\} \cdot J(N,V) \leq \sup\{q_U(p_U(x-y)) : x, y \in N\} \cdot J(N,U)$,

(2) $[\varphi^*(J_{\mathfrak{U}}(F(N)))](U) < [\sup\{q_U(p_U(x-y)) : x, y \in N\}]^{-1} \cdot J(F(N),U)$, *if* $J(F(N),U) > 0$.

Then F is a $(\varphi^*, J_{\mathfrak{U}})$-condensing mapping.

Remark. (1) and (2) are conditions on the real functions q_U and q_V which characterize the mapping F_1. Especially, (2) can be compared with the properties of the contractions in Proposition 4 from [**6**] and Theorem 3 from [**7**].

Proof of Proposition 4. The mapping $\mathfrak{J}_{\mathfrak{U}}$ is a φ^*-measure of noncompactness on K (Proposition 2). Let $N \subseteq M$ and

$$\varphi^*(J_{\mathfrak{U}}(F(N))) \geq J_{\mathfrak{U}}(N) \tag{i}$$

We suppose that $\overline{F(N)}$ is not compact. Since E is quasicomplete, there exists $U \in \mathfrak{U}(E)$ such that $J(F(N),U) > 0$. We choose a neighbourhood $V \in \mathfrak{U}(E)$ with $V + V \subseteq U$. From (1), (2) and Proposition 3 we obtain

$$J(N,U) \geq \frac{\sup\{q_V(p_V(x-y)) : x, y \in N\}}{\sup\{q_U(p_U(x-y)) : x, y \in N\}} \cdot J(N,V) \geq$$
$$\geq [\sup\{q_U(p_U(x-y)) : x, y \in N\}]^{-1} \cdot J(F(N),U) > [\varphi^*(J_{\mathfrak{U}}(F(N)))](U),$$

contradictory to (i). Therefore, $\overline{F(N)}$ is compact and F is a $(\varphi^*, J_{\mathfrak{U}})$-condensing mapping. ∎

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