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Commentationes Mathematicae Universitatis Carolinae, Vol. 31 (1990), No. 3, 489--499

Persistent URL: http://dml.cz/dmlcz/106884

Lucid operators on Banach spaces

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Abstract. We consider an ideal of operators which have a pointwise unconditional representation, and we investigate the relationship between them and some other operator ideals.

Keywords: Operator ideals, unconditional convergence

Classification: Primary 47D30, Secondary 40A30

In this note we will study a class of operators on Banach spaces, which is closely related to the notion of unconditional convergence.

An operator $T : X \to Y$ is said to be lucid (i.e. it is a linear operator with unconditionally converging image's decomposition), if there exist sequences $(a_n) \subset X^*, (y_n) \subset Y$, such that for all $x \in X$

$$Tx = \sum_{n=1}^{\infty} a_n(x) y_n \tag{$*$}$$

and the series $(*)$ converges unconditionally.

It is not hard to see that these operators together with a canonically defined norm form a complete normed ideal (in the sense of Pietsch) which we denote by Λ. We will characterize the lucid operators by factorization properties through spaces with an unconditional basis. We also investigate the relationship between Λ and certain other operator ideals and we study some hull procedures of operator ideals applied to Λ. Furthermore we consider the behaviour of Λ in relation to Banach spaces with certain special properties (for example local unconditional structure in the sense of Gordon and Lewis).

We use the usual terminology of Banach space theory. X^* denotes the topological dual space of the Banach space X, T^t the dual operator of the operator T.

1. Definition and simple properties.

In the sequel let X, Y be (real or complex) Banach spaces. A linear operator $T : X \to Y$ is said to be *lucid*, if there exist sequences $(a_n) \subset X^*, (y_n) \subset Y$, such that for all $x \in X$

$$Tx = \sum_{n=1}^{\infty} a_n(x) y_n \tag{1.1}$$

The authors would like to thank Prof. J.R. Retherford, Baton Rouge, for many stimulating discussions.

where the series (1.1) converges unconditionally. Let $\Lambda(X,Y)$ denote the set of all lucid operators between X and Y and let for $T \in \Lambda(X,Y)$

$$(1.2) \qquad \lambda(T) = \inf \sup_{\varepsilon_n=\pm 1} \sup_{\|x\|\leq 1} \| \sum_{n=1}^{\infty} \varepsilon_n a_n(x) y_n \|$$

where the infinitum is taken over all representations of T of the form (1.1). We omit the proof of the following general fact.

Theorem 1.1. *(Λ, λ) is a complete normed ideal.*

If X is a Banach space with an unconditional basis (x_n) with coordinate functionals (e_n), then for any $x \in X$ the series

$$x = \sum e_n(x) x_n$$

converges unconditionally, hence the identity operator I_X in X is lucid. This shows that every operator which factors through a space with an unconditional basis is lucid, moreover we have the following theorem.

Theorem 1.2. *An operator $T : X \to Y$ is lucid, if and only if there exist a space U with an unconditional basis and operators $P : U \to Y, Q : X \to U$, such that $T = PQ$. Then*

$$\lambda(T) = \inf \|P\| \cdot \|Q\| \cdot \chi(U)$$

where the infimum is taken over all possible factorizations and $\chi(U)$ is the unconditional basis constant.

PROOF : Let T be lucid with a lucid representation

$$Tx = \sum a_n(x) y_n,$$

let U be the Banach space of all sequences (ξ_n) such that $\sum \xi_n y_n$ converges unconditionally and let the norm on U be given by

$$\|(\xi_n)\| = \sup_{\varepsilon_n=\pm 1} \|\varepsilon_n \xi_n y_n\|_Y.$$

Then the unit vectors e_n form an unconditional basis in U with the unconditional basis constant $\chi(U) = 1$. Thus we have a factorization $T = PQ$, where $Qx = (a_n(x)), P(\xi_n) = \sum \xi_n y_n$ and

$$\lambda(T) \leq \|P\| \lambda(I_U) \|Q\| = \|P\| \chi(U) \|Q\|.$$

The proof can be completed by standard arguments. ∎

This shows that Λ is quite large. Especially we mention the operators between the $\mathcal{L}_\infty$-space $C[0,1]$ and the $\mathcal{L}_1$-space $L_1[0,1]$. These operators factor through a Hilbert space and thus they are lucid, although neither $C[0,1]$ nor $L_1[0,1]$ possess an unconditional basis.

If the identity operator I_X in a Banach space X is lucid, then in the factorization $I_X = PQ$ the operator Q is injective and the operator P is surjective. From $QP(U) = Q(X)$ and $QPQP = QP$ follows that QP is a continuous projection of U onto $Q(X)$, hence X is isomorphic to a complemented subspace of a space with an unconditional basis. Since every space with an unconditional basis is a complemented subspace of Pelczynski's universal space [**6**], we have shown:

Theorem 1.3.

(a) *I_X is lucid, iff X is isomorphic to a complemented subspace of Pelczynski's universal space.*

(b) *A linear operator T is lucid if and only if it factors through Pelczynski's universal space.*

Later we will characterize these operators which factor through a not necessarily complemented subspace or through a quotient space of a space with an unconditional basis. Obviously the problem, whether a Banach space with a lucid identity operator possesses an unconditional basis, is equivalent to Lindenstrauss's problem if any complemented subspace of a space with an unconditional basis has an unconditional basis.

2. Comparison with other ideals.

The main result of this section will be that the ideal of lucid operators is not comparable with the most of the common operator ideals. We start with a simple observation (we adapt the terminology of Persson—Pietsch [**8**]).

Proposition 2.1.

(a) *Every p-nuclear operator* $(1 \leq p < \infty)$ *is lucid with* $\lambda(T) \leq \nu_p(T)$.

(b) *Every absolutely-p-summing operator* $(1 \leq p \leq 2)$ *with a separable range is lucid with* $\lambda(T) \leq \pi_p(T)$.

(c) *Every p-integral operator* $(1 \leq p \leq 2)$ *with a separable range is lucid with* $\lambda(T) \leq i_p(T)$.

PROOF : Since every p-nuclear operator factors through l_p, statement (a) is clear. Since every absolutely-p-summing or p-integral operator $(1 \leq p \leq 2)$ factors through $L_2(U^0, \mu)$, it remains to show by standard arguments that it factors through a separable subspace of $L_2(U^0, \mu)$, which possesses an unconditional Schauder basis. This proves (b) and (c). On the other hand, if (K, μ) is a non-separable compact measure space, then $C(K) \hookrightarrow L_p(K, \mu)$ is p-integral but not lucid, since $C(K)$ is dense in $L_p(K, \mu)$, but the range of a lucid operator is necessarily separable. ∎

Remark. Proposition 2.1(b) is not true in case $p > 2$: O. Reinov has shown in [**10**] that for any $p > 2$ there are separable Banach spaces X, Y and an operator $T : X \to Y$ such that T is absolutely-p-summing but not even the pointwise limit of a sequence of finite rank operators and so of course T cannot be lucid. (A. Pelczynski has pointed out that the construction of Kwapien [**3**] yields similar examples.)

To study the connection between the approximable and lucid operators we begin with the following lemma. (An operator T is said to be approximable iff it is the norm-limit of a sequence of finite rank operators.)

Lemma 2.2. *Let X be a Banach space and (X_n) a sequence of finite dimensional spaces with the property: There exist sequences of operators $S_n : X \to X_n, T_n : X_n \to X$ such that $S_n T_n = I_{X_n}$ and $\gamma = \sup \|S_n\| \cdot \|T_n\| < \infty$. Then*

$$\lambda(I_{X_n}) \leq \gamma \lambda(T_n I_{X_n} S_n).$$

PROOF : Let

$$T_n S_n x = \sum_k a_k(x) y_k, \quad x \in X \tag{2.1}$$

be a lucid representation of $T_n S_n$. From

$$y = \sum_k a_k(T_n y) S_n y_k = S_n(T_n S_n) T_n y, \; y \in X_n$$

we obtain a lucid representation of I_{X_n}. Then

$$\lambda(I_{X_n}) = \lambda(S_n(T_n S_n) T_n) \leq \|S_n\| \cdot \|T_n\| \cdot \lambda(T_n S_n).$$

■

Theorem 2.3. *Let X be a Banach space, $(X_n), (S_n), (T_n), (I_{X_n})$, γ be the same as in Lemma 2.2. If $\{\lambda(I_{X_n}), n \in \mathbb{N}\}$ is unbounded, then in X there exists an approximable (hence compact) operator which fails top be lucid.*

PROOF : We assume that every approximable operator in X is lucid, then the space $(\mathcal{A}(X), \lambda)$ of all approximable operators endowed with the norm λ is a Banach space. By the Open Mapping Theorem the norms $\| \quad \|$ and λ are equivalent on $(\mathcal{A}(X), \lambda)$, i.e. there exists an $\eta \geq 1$, such that for all $T \in \mathcal{A}(X)$

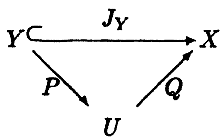
$$\|T\| \leq \lambda(T) \leq \eta \|T\|.$$

Since $\lambda(I_{X_n})$ is unbounded, so is $\lambda(T_n S_n)$. But this contradicts

$$\lambda(T_n S_n) \leq \eta \|T_n\| \cdot \|S_n\| \leq \gamma \cdot \eta.$$

■

Gordon and Lewis [2] introduced the following notion: A Banach space X is said to have a local unconditional structure (LUST) iff there exists a real $\mu > 0$, such that for each finite dimensional subspace $Y \subset X$ there exists a factorization



of the canonical inclusion $J_Y = QP$ through a space U with an unconditional basis, such that $\|P\| \, \|Q\| \chi(U) \leq \mu$. The infimum, $\chi_u(X)$, of all such μ is called the LUST-constant of X.

It can be easily verified that

$$\chi_u(X) \leq \lambda(I_X) \leq \chi(X)$$

and, if $\dim X < \infty$

$$\chi_u(X) \leq \lambda(I_X).$$

If X does not have LUST, (i.e. $\chi_u(X) = \infty$), then there exists a sequence of finite dimensional subspaces X_n with $\lambda(I_{X_n}) \nearrow \infty$. To find concrete examples of such spaces X we make use of the ideas of Gordon and Lewis [2] concerning sufficiently euclidean spaces and tensor products of Banach spaces. A Banach space X is said to be sufficiently euclidean, [2], iff there exists a real $\beta > 0$, sequences of operators $S_n : X \to l_2^n, T_n : l_2^n \to X$, such that $S_n T_n = I_{l_2^n}$ and $\|S_n\| \cdot \|T_n\| \leq \beta$.

Examples of sufficiently euclidean spaces are the $\mathcal{L}_p$- spaces $(1 < p < \infty)$.

Now we are able to show the existence of non-lucid approximable operators.

Corollary 2.4. *Let X, Y be sufficiently euclidean Banach spaces. Then there exists an approximable operator T on $X\tilde{\otimes}_\alpha Y, \alpha \in \{\varepsilon, \pi\}$, which fails to be lucid.*

PROOF : Let S_n^X, T_n^X, resp. S_n^Y, T_n^Y be operators with $S_n^X : X \to l_2^n, T_n^X : l_2^n \to X, S_n^Y : Y \to l_2^n, T_n^Y : l_2^n \to Y$, such that $S_n^X T_n^X = S_n^Y T_n^Y = I_{l_2^n}$ and $\sqrt{\gamma} = \sup_n(\|S_n^X\| \cdot \|T_n^X\|, \|S_n^Y\| \cdot \|T_n^Y\|) < \infty$.

Let

$$A_n = (S_n^X \otimes S_n^Y) : X\tilde{\otimes}_\alpha Y \to l_2^n \tilde{\otimes}_\alpha l_2^n$$
$$B_n = (T_n^X \otimes T_n^Y) : l_2^n \tilde{\otimes}_\alpha l_2^n \to X\tilde{\otimes}_\alpha Y$$

be the canonical tensor products, then $A_n, B_n = I_{l_2^n \tilde{\otimes}_\alpha l_2^n}$ and we have $\|A_n B_n\| \leq \gamma$. Gordon and Lewis have shown that for $n \in \mathbb{N}$

$$\chi_u(l_2^n \tilde{\otimes}_\alpha l_2^n) \geq \frac{\sqrt{n}}{9}$$

hence by Theorem 2.3 there exist approximable non-lucid operators on $X\tilde{\otimes}_\alpha Y$. ∎

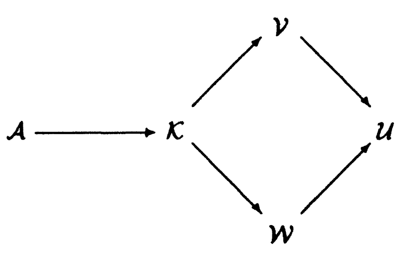
Now it is easy to show that the ideal Λ is not comparable with most of the common ideals.

Theorem 2.5. *The ideals*

$\mathcal{A}$ of approximable
$\mathcal{K}$ of compact
$\mathcal{V}$ of completely continuous
$\mathcal{W}$ of weakly compact
$\mathcal{U}$ of unconditionally converging

operators are not comparable with Λ, i.e. Λ is not contained in one of them, and none of them is contained in Λ.

PROOF : From



follows that we have to give an example of an operator $T \in \mathcal{A}$ which is not in Λ (done in Corollary 2.4) and an operator T in Λ which is not unconditionally converging (for instance is the identity on c_0). The latter is lucid, but not unconditionally converging, since in c_0 there exist σ-summable sequences which are not norm-summable. ∎

3. Hulls of the ideal of lucid operators.

In Theorem 1.3 we have shown that every lucid operator factors through a complemented subspace of Pelczynski's universal space U, hence through U itself. Here we will focus our interest on those operators which factor through an arbitrary subspace of U.

In order to characterize these operators we need the notion of injective or surjective hull of an operator ideal (see e.g. Pietsch [8]):

The *injective hull* $\Lambda^{\mathrm{inj}}(X,Y)$ of the ideal Λ is the set of all operators $T : X \to Y$, such that there exists an injection J into a larger Banach space Y_∞, such that $JT \in \Lambda(X, Y_\infty)$ and $J(Y)$ is closed in Y_∞. The *surjective hull* $\Lambda^{\mathrm{surj}}(X,Y)$ is the set of all $T : X \to Y$, such that there exists a surjection Q of X_1 onto X, such that $TQ \in \Lambda(X_1, Y)$. Obviously, $\Lambda \subset \Lambda^{\mathrm{inj}}, \Lambda \subset \Lambda^{\mathrm{surj}}$, and Y_∞ resp. Y_1 can be chosen of type $l_\infty(\Gamma)$ resp. $l_1(\Gamma)$.

Theorem 3.1.

(a) Λ^{inj} *is the class of all operators which factor through a subspace of a Banach space with an unconditional basis.*

(b) Λ^{surj} *is the class of all operators which factor through a quotient space of a space with an unconditional basis.*

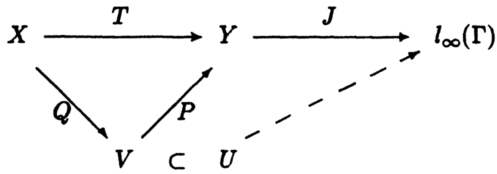
PROOF : (a) Let $T \in \Lambda^{\mathrm{inj}}(X,Y)$, then we have the factorization

$$\begin{array}{ccccc} X & \xrightarrow{T} & Y & \xrightarrow{J} & Y_\infty \\ & {\scriptstyle Q}\searrow & & \nearrow{\scriptstyle P} & \\ & & U & & \end{array}$$

This implies

$$\begin{array}{ccc} X & \xrightarrow{\quad T \quad} & Y \\ \downarrow & & \uparrow {\scriptstyle J^{-1}} \\ \overline{Q(X)} & \xrightarrow[P \mid \overline{Q(X)}]{} & J_Y \end{array}$$

This shows that T factors through the subspace $\overline{Q(X)}$ of a space U with an unconditional basis. On the other hand, if V is a subspace of a space with an unconditional basis, then the diagram



shows that the mapping JP can be extended to a mapping $\tilde{P} : U \to l_\infty(\Gamma)$, such that $JT = \tilde{P}Q$, and we see that JT is lucid by Theorem 1.2.

(b) The proof of the second part of Theorem 3.1 uses similar arguments. ∎

Since every separable Banach space is isomorphic to a quotient of l_1, Λ^{surj} consists of all operators with a separable range. Also it is true that any separable Banach space X is the range of a lucid mapping: trivially take the canonical surjection $Q : l_1 \to X$ if X is isomorphic to l_1/N, N a closed subspace of l_1. X is isomorphic to a subspace of a space with an unconditional basis iff $I_X \in \Lambda^{\mathrm{inj}}$. Since every compact operator factors through a subspace of c_0, we have $K \subset \Lambda^{\mathrm{inj}}$.

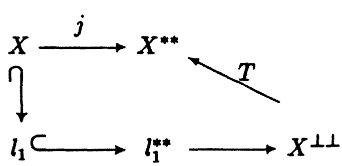
A third interesting example of a hull ideal of Λ is the so-called regular hull Λ^{reg} (see [9]). The ideal Λ^{reg} consists of the operators $T : X \to Y$ such that $j \circ T \in \Lambda(X, Y^{**})$ where $j : Y \to Y^{**}$ is the canonical embedding.

Proposition 3.2. *The operator ideal Λ is not regular, that is, Λ is a proper subclass of Λ^{reg}.*

Proof : There is a Banach space X such that I_X is not lucid but the canonical embedding $j : X \to X^{**}$ is:
Let X be the Lindenstrauss space (see [4]) which is defined to be the kernel of any surjection from l_1 onto $L_1[0,1]$. In [6] it is shown that X is not a complemented subspace of a Banach space with an unconditional basis, and so I_X is not lucid. On the other hand in [4] it is shown that

$$X^{\perp\perp} = \{z \in l_1^{**} | z(y) = 0 \text{ for each } y \in l_1^* \text{ s.t. } \quad y(x) = 0 \text{ for each } x \in X\},$$

the biannihilator of X in l_1^{**}, is complemented in l_1^{**} and there is an isometry $T : X^{\perp\perp} \to X^{**}$such that $T \mid X$ is the identity of X. So we obtain the following factorization of j with canonical mappings:



Since the diagram commutes, j is lucid and we have $I_X \in \Lambda^{\mathrm{reg}}$ but $I_X \notin \Lambda$. ∎

Since X is isomorphic to a subspace of a space with an unconditional basis iff $I_X \in \Lambda^{\mathrm{inj}}$, and every compact operator factors through a subspace of c_0, we have $K \subset \Lambda^{\mathrm{inj}}$.

Another procedure for forming a new ideal from a given one is the construction of Λ^{dual}:
$\Lambda^{\text{dual}}(X,Y)$ is the set of all $T : X \to Y$ such that $T^t : Y^* \to X^*$ belongs to Λ. An ideal $\mathcal{I}$ is said to be symmetric iff $\mathcal{I} \subset \mathcal{I}^{\text{dual}}$. Without additional assumptions Λ is far from being symmetric: the identity on l_1 is in Λ, but not its adjoint, hence $\Lambda \not\subset \Lambda^{\text{dual}}$; on the other hand, the identity on $C(\omega^\omega)$ does not belong to Λ [5], but its adjoint does, since $C(\omega^\omega)$ is isometrically isomorphic l_1. Hence $\Lambda^{\text{dual}} \not\subset \Lambda$.

Theorem 3.3. *Let X be a Banach space. Then X^* does not contain a subspace isomorphic to c_0 if and only if for every Banach space Y*

$$\Lambda(X,Y) \subset \Lambda^{\text{dual}}(X,Y).$$

PROOF : If X^* contains a subspace isomorphic to c_0, then X contains a complemented subspace X_1 isomorphic to l_1 [4, p. 41]. Let $P : X \to X_1$ be a projection, $J : X_1 \to l_1$ an isomorphism and $Y = l_1$, and let T be defined by $T = JP$. Then $T^t : l_\infty \to X^*$ is lucid by assumption. We will construct a contradiction by showing that the range of T^t is not separable, e.g. $\exists \gamma > 0 \forall (\xi_n) \in l_\infty \, \|T^t(\xi_n)\|_{X^*} \to \gamma \|(\xi_n)\|_{l_\infty}$. This follows from the inequalities

$$\begin{aligned}\|T^t(\xi_n)\| &= \sup_{\substack{x\in X\\ \|x\|\le 1}} \left|\sum \xi_n (Tx)_n\right| \\ &= \sup_{\substack{x\in X\\ \|x\|\le 1}} \left|\sum \xi_n (JP(x))_n\right| \ge \sup_{\substack{x\in X_1\\ \|x\|\le 1}} \left|\sum \xi_n (Jx)_n\right| \\ &\ge \sup_{\substack{y\in l_1\\ \|y\|\le \|J^{-1}\|^{-1}}} \left|\sum \xi_n y_n\right| = \|J^{-1}\|^{-1}\|(\xi_n)\|.\end{aligned}$$

On the other hand, if X does not contain a subspace isomorphic to c_0, and if $T : X \to Y$ is lucid with a lucid representation $Tx = \sum a_n(x) y_n$, then for all $b \in Y^*$ we obtain $(T^t b(x) = \sum a_n(x) b(y_n)$, i.e. $\sum b(y_n) a_n$ is σ^*-convergent to $T^t b \in X^*$. By a known lemma (see e.g. [13, p. 423]) $\sum b_n(y_n) a_n$ is a σ-unconditionally Cauchy sequence on X. Since X has no subspace isomorphic to c_0, by a classical result of Pelczynski, $\sum b(y_n) a_n$ is norm-convergent with limit $T^t b$ hence $T^t b = \sum y_n(b) a_n$, and this series converges unconditionally. Thus T^t is lucid, i.e. $T \in \Lambda^{\text{dual}}(X,Y)$. ■

Corollary 3.4. *If Y is a Banach space such that I_Y is lucid and Y^* does not contain a subspace isomorphic to c_0 then I_{Y^*} is lucid.*

PROOF : Letting $X = Y$ in Theorem 3.3 yields $\Lambda(Y) \subseteq \Lambda^{\text{dual}}(Y)$. Since $I_Y \in \Lambda$ we have $I_Y \in \Lambda^{\text{dual}}(Y)$, that is $I_{Y^*} \in \Lambda$. ■

Since a reflexive space does not contain a subspace isomorphic to c_0, it follows from Corollary 3.4 that the following is true:

Corollary 3.5. *If Y is reflexive then I_Y is lucid if and only if I_{Y^*} is lucid.*

Corollary 3.6. *If Y is a Banach space with the properties: Y^* does not contain a subspace isomorphic to c_0 and*

Y^* is not separable (for example $Y = C[0,1]$)
then any embedding $j : Y \to Z$, where Z is another Banach space, j injective and $j(Y)$ closed in Z, is not lucid.

PROOF : Assume there is a Banach space Z and a lucid embedding $j : Y \to Z$ as claimed. Theorem 3.3 yields $\Lambda(Y,Z) \subseteq \Lambda^{\text{dual}}(Y,Z)$. Hence j is an element of $\Lambda^{\text{dual}}(Y,Z)$, that is $j^t : Z^* \to Y^*$ is lucid, but this is impossible since Y^* is assumed not to be separable. ∎

Thus Theorem 3.3 together with Corollary 3.6 is a way to show that $C[0,1]$ is not embeddable into a Banach space with an unconditional basis (the embedding operator j would be lucid) in terms of lucid mappings.

4. Weakly nuclear operators, lucid operators and LUST.

A class of operators which are closely related to approximable lucid operators is the ideal of weakly nuclear operators due to Pietsch [9, 23.2].

An operator $T : X \to Y$ is said to be *weakly nuclear*, if there exist sequences $(a_n) \subset X^*, (y_n) \subset Y$, such that

$$T = \sum a_n \otimes y_n$$

where this series is unconditionally convergent in the operator norm. Let $\mathcal{N}_\sigma$ be the ideal of weakly nuclear operators endowed with the norm-topology given by

$$\nu_\sigma(T) = \inf \sup_{\substack{\|x\|\leq 1 \\ \|b\|\leq 1}} \sum |a_n(x)b(y_n)|$$

where the infimum is taken over all weakly nuclear representations of T.

Proposition 4.1.

(a) $(\mathcal{N}_\sigma, \nu_\sigma) \subset (\Lambda, \lambda)$
(b) *If $T \in \mathcal{L}(X,Y), \dim X < \infty$, then $\nu_\sigma(T) = \lambda(T)$*
(c) *If X^* and Y possess the metric approximation property then every degenerate $T \in \mathcal{L}(X,Y)$ fulfils*
$$\nu_\sigma(T) = \lambda(T).$$

PROOF : (a) and (b) are elementary facts; (c) follows from an observation of H.U. Schwarz [12] because of the easy proved fact that $\nu_\sigma(T) = \lambda(T)$ for each $T \in \mathcal{L}(X,Y)$ when X and Y are finite dimensional spaces. ∎

Problems.

Is it true that for every finite rank operator $\lambda(T) = \nu_\sigma(T)$?
Is $\mathcal{N}_\sigma$ equal to $\mathcal{A} \cap \Lambda$?

There are two further procedures for forming ideals which are of interest in this context:
Let $(\mathcal{B}, \beta)$ be an operator ideal. Then the *maximal hull* $(\mathcal{B}^{\max}, \beta^{\max})$ of $(\mathcal{B}, \beta)$ is the

class of all operators $T \in \mathcal{L}(X,Y)$, such that for all operators $R \in \mathcal{A}(X_0,X), S \in \mathcal{A}(Y,Y_0)$ hold $STR \in \mathcal{B}(X_0,Y_0)$ with

$$\beta^{\max}(T) = \sup\{\beta(STR) : \|S\| \leq 1, \|T\| \leq 1\}.$$

The *minimal kernel* $(\mathcal{B}^{\min}, \beta^{\min})$ of $(\mathcal{B}, \beta)$ is defined as the class of all $T \in \mathcal{L}(X,Y)$, such that there exist operators $S \in \mathcal{A}(X,X_0), R \in \mathcal{B}(X_0,Y_0), Q \in \mathcal{A}(Y_0,Y)$ with $T = QRS$ and

$$\beta^{\min}(T) = \inf\{\|Q\|\beta(R)\|S\|, T = QRS\}.$$

Pietsch [**6**, 23.3.1] has shown that

$$(\mathcal{N}_\sigma^{\max}, \nu_\sigma^{\max}) = (I_\sigma, i_\sigma)$$

where I_σ is the class of all weakly integral operators, defined by the existence of a factorization through a Banach lattice.

Thus $I_X \in I_\sigma$ if and only if X has LUST, since in this case X^{**} is a complemented subspace of a Banach lattice (see [**1**]), we obtain the following facts.

Proposition 4.2.

(a) $(\Lambda^{\max}, \lambda^{\max}) = (\mathcal{N}_\sigma^{\max}, \nu_\sigma^{\max}) = (I_\sigma^{\max}, i_\sigma^{\max})$
(b) $(\Lambda^{\min}, \lambda^{\min}) = (\mathcal{N}_\sigma, \nu_\sigma)$
(c) *Y has LUST if and only if for any Banach space X $\mathcal{A}(X,Y) \subseteq \Lambda(X,Y)$.*

((c) *has been observed by Pietsch* [**9**] *in a similar manner for $\mathcal{N}_\sigma$ instead of Λ.*)

PROOF : The proof of (a) uses the property, that the basis constructed in the proof of Theorem 1.2 is hyperorthogonal, thus the statement follows from [**9**, 23.3.4], (b) follows from [**9**, 23.3.2].
If Y has LUST, then for every finite dimensional subspace $Y_0 \subset Y$ with its canonical embedding J_{Y_0}

$$\lambda(J_{Y_0}) = \inf\{\|P\| \cdot \|Q\|\chi(U)\} \leq \chi_u(X)$$

where $J_{Y_0} = PQ$ is a factorization through a space U with an unconditional basis. Let $T \in \mathcal{A}(X,Y), (T_n)$ an approximating sequence of finite rank operators. Since (T_n) is a $\|\ \ \|$-Cauchy sequence, for $\varepsilon > 0$ there exists n_ε, such that for $n \geq n_\varepsilon$ $\|T_n - T_m\| \leq \varepsilon/\chi_u(Y)$. Let $Y_0 = \mathrm{span}[T_nY, T_mY] \subset Y$, then

$$\lambda(T_n - T_m) = \lambda(J_{Y_0}(T_n - T_m)) \leq \lambda(J_{Y_0}) \cdot \|T_n - T_m\| < \varepsilon.$$

Thus (T_n) is a λ-Cauchy sequence with

$$\lambda - \lim T_n = T \in \Lambda(X,Y).$$

If for all Banach spaces X

$$\mathcal{A}(x,Y) \subset \Lambda(X,Y),$$

then $I_Y \in \Lambda^{\max} = J_\sigma^{\max}$. Then Y has a LUST. ■

Since $C[0,1]$ is a space with LUST, we have e.g. $I_{C[0,1]} \in \Lambda^{\max}$. Hence $\Lambda^{\max} \neq \Lambda^{\mathrm{inj}}$.

REFERENCES

[1] Figiel T., Johnson W.B., Tzafriri L., *On Banach lattices and spaces having LUST with applications to Lorentz function spaces*, J. Approximation Theory **13** (1975), 393–412.

[2] Gordon Y., Lewis D.R., *Absolutely summing operators and local unconditional structures*, Acta Math. **133** (1974), 27–48.

[3] Kwapien S., *Comments to Enflo's construction of a Banach space without the approximation property*, Seminaire Goulaouic—Schwartz, 1972–1973.

[4] Lindenstrauss J., *On a certain subspace of l_1*, Bull. de l'Académie Polonaise des Sciences **12** (1964), 539–542.

[5] Lindenstrauss J., Tzafriri L., *Classical Banach Spaces*, Springer, Heidelberg, 1973/79.

[6] Pelczynski A., *Universal bases*, Studia Math. **32** (1969), 247–268.

[7] Pelczynski A., Wojtaszczyk P., *Banach spaces with finite dimensional expansions of identity and universal bases of finite dimensional subspaces*, Studia Math. **40** (1971), 91–108.

[8] Persson A., Pietsch A., *p-nukleare und p-integrale Abbildungen auf Banachräumen*, Studia Math. **33** (1969), 16–62.

[9] Pietsch A., *Operator Ideals*, Deutscher Verlag der Wissenschaften, Berlin, 1978.

[10] Reinov O., *Approximation properties of order p and the existence of non-p-nuclear operators with p-nuclear second adjoints*, Math. Nachr. **109** (1982), 135–144.

[11] Retherford J.R., *Applications of Banach ideals of operators*, Bull. AMS **81** (1979), 763–781.

[12] Schwarz H.U., *Dualität und Approximation von Normidealen*, Math. Nachr. **66** (1975), 305–317.

[13] Singer I., *Bases in Banach Spaces I*, Springer Verlag, Heidelberg.

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(*Received* January 3, 1990)