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Remarks on delta-convex functions

EVA KOPECKÁ AND JAN MALÝ

Abstract. We construct a delta-convex function on $\mathbb{R}^2$ which is strictly differentiable at 0, but this property possesses none of its control functions. Further, we prove that if a function $H$ on an open convex subset $A$ of a normed linear space $X$ is controlled by a bounded continuous function $h_E$ on each bounded closed set $E$ contained in $A$, then $A$ is delta-convex on $A$. We construct various counterexamples showing that this result is the best possible generalization of its finite-dimensional counterpart, which is due to P. Hartman.

Keywords: Delta-convex functions, differentiability, normed linear spaces

Classification: 26B25, 46A55

Introduction.

Let $A$ be a convex subset of a normed linear space $X$. A function $H: A \rightarrow \mathbb{R}$ is termed delta-convex on $A$ if $H$ can be expressed as a difference of two continuous convex functions on $A$. Let $H$ and $h$ be functions on $A$. We say that $h$ is a control function to $H$ on $A$, or, that $h$ controls $H$ on $A$, if both the functions $h - H$ and $h + H$ are continuous and convex. Then, of course, $h$ is also continuous and convex and $H$ is continuous. We may define equivalently delta-convex functions on $A$ as those functions, which have control functions. The family of all delta-convex functions on $A$ is the linear span of the set of all continuous convex functions on $A$.

A difficulty of the concept of delta-convexity consists in the fact that in general we cannot found a "canonical" control function to $H$, which would be controlled by all control functions to $H$.

The notion of delta-convex function was introduced by A.D. Aleksandrov [1] in $n$-dimensional case. It was observed by L. Zajíček [4], that the infinite-dimensional version of this notion allows to characterize the sets of Gâteaux nondifferentiability of continuous convex functions on separable Banach spaces.

A survey of results in the theory of delta-convex functions and mappings (see Remark 17 below) can be found in an important article by L. Veselý and L. Zajíček [3].

In this paper we solve Problem 3 and Problem 5 from [3]. In Section 1 we show an example of a delta-convex function $H$ on $\mathbb{R}^2$ such that the function $H$ is strictly differentiable at 0, but none of its control functions is (Fréchet) differentiable at 0.

In Section 2 we compare local and global delta-convexity. A function $H$ is said to be locally delta-convex on an open set $A$, if every point $z \in A$ has an open convex neighborhood $V$ in $A$ such that $H$ is delta-convex on $V$. P. Hartman [2] proved that if $A \subset \mathbb{R}^n$ is an open convex set and $H$ is a locally delta-convex function on $A$, we thank Luděk Zajíček and Libor Veselý for valuable remarks.
then it is delta-convex. We show that analogous statement is not true in infinite-dimensional spaces even if $A$ is bounded and $H$ is locally delta-convex on the whole space. A further example shows that there is a function $H$ on $l^2$ such that $H$ is delta-convex on each bounded convex subset of $l^2$ but $H$ is not delta-convex on $l^2$. The only positive result remains: If a function $H$ on a normed linear space $X$ is controlled by a bounded continuous convex function on each bounded convex set, then $H$ is delta-convex on $X$. The same result holds for mappings (cf. Remark 17).

Differentiability.

Let $X$ be a normed linear space. We denote $B(a, r) = \{ x \in X : |x - a| < r \}$. Let $F$ be a function defined on an open set $A \subset X$. A continuous linear functional $L$ is said to be a strict derivative of $F$ at a point $a$ if for every $\epsilon > 0$ there is $\delta > 0$ such that for each $x, y \in A \cap B(a, \delta)$ we have

$$|F(y) - F(x) - L(y - x)| \leq \epsilon|y - x|.$$ 

If a convex function is differentiable (= Fréchet differentiable) at a point $a$, then it is strictly differentiable at $a$ ([3, Prop. 3.8]). This cannot be said about delta-convex functions. An example of a delta-convex function on $\mathbb{R}^2$ which is differentiable at 0 but not in the strict sense is given in [3, Note 6.4]. Such a function cannot be controlled by a function differentiable at 0.

In this section we construct a delta-convex function $H$ on $\mathbb{R}^2$ such that the function $H$ is strictly differentiable at 0, but none of its control functions is differentiable at 0.

**Example 1.** Find a sequence $\{k_j\}$ of positive integers such that $\cos(2\pi/k_j) \geq 1 - 2^{-j-3}$ and denote

$$M = \{ (2^{-j}\cos(2\pi k/k_j), 2^{-j}\sin(2\pi k/k_j)) : j \in \mathbb{N}, k \in \{1, \ldots, k_j\} \}.$$ 

Set

$$F(x) = |x| + 4|x|^2.$$ 

For each $z \in \mathbb{R}^2 \setminus \{0\}$ we define

$$G_z(x) = F(z) + \langle F'(z), x - z \rangle = (8|z| + 1)\frac{x \cdot z}{|z|} - 4|x|^2.$$ 

Since $F$ is convex and $G_z$ is a tangent function to $F$ at $z$, we have $G_z \leq F$ on $\mathbb{R}^2$. Set

$$G(x) = \sup\{G_z(x) : z \in M\}$$

and

$$H(x) = G(x) - |x|.$$ 

Obviously $G$ is a convex functions on $\mathbb{R}^2$. It follows that the function $H$ is delta-convex. In what follows we will derive further properties of the functions $G$ and $H$. 
Lemma 2. Let $M$ be as in Example 1 and $x \in \mathbb{R}^2$. Suppose $0 < |x| < 1$. Then there is $z \in M$ such that
\[ |z| \leq |x| \leq 2|z| \]
and
\[ \frac{\langle x, z \rangle}{|x||z|} > 1 - \frac{1}{8}|z|. \]

**Proof:** Find $j \in \mathbb{N}$ such that
\[ 2^{-j} \leq |x| < 2^{-j+1}. \]
Further find $z \in M$ such that $|z| = 2^{-j}$ and the angle between the radius vectors of $z$ and $x$ is less than $2\pi/k_j$, i.e.
\[ \frac{\langle x, z \rangle}{|x||z|} > \cos(\frac{2\pi}{k_j}) \]
Then we have
\[ |z| \leq |x| \leq 2|z| \]
and
\[ \frac{\langle x, z \rangle}{|x||z|} > 1 - 2^{-j-3} = 1 - \frac{1}{8}|z|. \]

Lemma 3. The function $G$ from Example 1 satisfies
\[ |x| + |x|^2 \leq G(x) \leq |x| + 4|x|^2 = F(x) \]
for all $|x| < 1$.

**Proof:** Fix $x \in \mathbb{R}^2$ with $0 < |x| < 1$. The inequality $G(x) \leq F(x)$ is obvious. Find $z \in M$ as in Lemma 2. Then we have
\[
G(x) \geq G_z(x) = (8|z| + 1)\frac{\langle x, z \rangle}{|z|} - 4|z|^2 \geq (8|z| + 1)(1 - \frac{1}{8}|z|)|x| - 4|z|^2 \\
= |x| + |z|(8|x| - \frac{1}{8}|x| - |x||z| - 4|z|) \geq |x| + 2|z||x| \geq |x| + |x|^2.
\]

Lemma 4. Let $G, G_z$ be the functions from Example 1. Let $r \in (0,1)$. If
\[ 0 < |z| \leq \frac{r}{9} \quad \text{and} \quad |x| \geq r, \]
then
\[ G_z(x) \leq G(x) - \frac{r^2}{9}. \]

**Proof:** Under the assumptions, using Lemma 3 we obtain
\[
G_z(x) = (8|z| + 1)\frac{\langle x, z \rangle}{|z|} - 4|z|^2 \\
\leq |x| + 8|x||z| \leq G(x) - \frac{r^2}{9}.
\]
Lemma 5. Let \( G, G_z \) be the functions from Example 1 and \( w \in \mathbb{R}^2, 0 < |w| < \frac{1}{16} \). Then

\[
G(w) = \sup \{ G_z(w) : z \in M_w \},
\]

where

\[
M_w = \{ z \in M : |z| \leq 2|w|, \langle w, z \rangle \geq |z||w|(1 - 8|z|) \}.
\]

**Proof:** Choose \( z \in M \setminus M_w \). We will distinguish two cases.

(a) Assume that \(|z| \geq 2|w|\). Then

\[
G_z(w) = (8|z| + 1)\frac{\langle w, z \rangle}{|z|} - 4|z|^2
\]

\[
\leq |w| + 8|w||z| - 4|z|^2 \leq |w|.
\]

(b) Assume that \(|z| \leq 2|w|\) and

\[
\langle w, z \rangle < (1 - 8|z|)|w||z|.
\]

We obtain

\[
G_z(w) = (1 + 8|z|)\frac{\langle w, z \rangle}{|z|} - 4|z|^2
\]

\[
\leq |w| - 64|z|^2|w| - 4|z|^2 \leq |w|.
\]

In both cases (a) and (b), using Lemma 3 we conclude that \( G_z(w) \leq G(w) - |w|^2 \). The assertion easily follows.

Lemma 6. Let \( H \) be the function from Example 1. Then the zero functional is a strict derivative of \( H \) at the origin.

**Proof:** Choose \( \varepsilon \in (0, 1/4) \). Let \( x, y \) be points of \( B(0, \varepsilon^2) \). We will estimate the quantity

\[
|H(y) - H(x)|.
\]

We will distinguish two cases.

(a) Let us assume that \(|y - x| \geq \varepsilon(|x| + |y|)\). Then we obtain

\[
|H(y) - H(x)| \leq |H(y)| + |H(x)| \leq 4(|x|^2 + |y|^2) \leq 4(|x| + |y|)^2
\]

\[
\leq 4(|x| + |y|) \frac{|y - x|}{\varepsilon} \leq 8 \varepsilon |y - x|.
\]

(b) Let us assume that \(|y - x| \leq \varepsilon(|x| + |y|)\). Fix \( z \in M \). Denote

\[
x^* = \frac{|x|}{|z|} z, \quad y^* = \frac{|y|}{|z|} z.
\]

Assume \( z \in M_x \) (for the notation see Lemma 5). Then

\[
|x^* - x|^2 = \frac{1}{|z|^2} ||x||z|| - |z||x|^2 = \frac{1}{|z|^2} (2|x|^2|z|^2 - 2|x||z||(x, z))
\]

\[
\leq 2|x|^2(1 - (1 - 8|z|)) = 16 |z||x|^2 \leq 32 \varepsilon^2 (|x| + |y|)^2.
\]
and
\[ |y^* - y| \leq |y^* - x^*| + |x^* - x| + |x - y| \leq (2\varepsilon + \sqrt{32}) \varepsilon(|x| + |y|) \leq 8\varepsilon(|x| + |y|). \]

Similarly we have
\[ |y^* - y| \leq 8\varepsilon(|x| + |y|) \quad \text{and} \quad |x^* - x| \leq 8\varepsilon(|x| + |y|) \]
assuming that \( z \in M_y \). Now, let \( z \in M_x \cup M_y \). Then
\[
|(G_x(y) - |y|) - (G_x(x) - |x|)| = \left| (8|z| + 1) \frac{\langle y - x, z \rangle}{|z|} - \frac{\langle y, y - x \rangle + \langle x, y - x \rangle}{|x| + |y|} \right| \\
= 8|z| \frac{\langle y - x, z \rangle}{|z|} + \frac{\langle y - x, x^* - x \rangle + \langle y - x, y^* - y \rangle}{|x| + |y|} \\
\leq 8|z| \frac{|x^* - x|}{|x| + |y|} + \frac{|y^* - y|}{|x| + |y|} |y - x| \leq 24\varepsilon|y - x|.
\]

It easily follows
\[ |H(y) - H(x)| \leq \sup \{|(G_x(y) - |y|) - (G_x(x) - |x|)| : z \in M_y \cup M_x \} \leq 24\varepsilon|y - x|. \]

The estimates in (a) and (b) show that the zero functional is a strict derivative of \( H \) at the origin. \( \blacksquare \)

**Theorem 7.** The function \( H \) from Example 1 is delta-convex on \( \mathbb{R}^2 \). The function \( H \) is strictly differentiable at 0, but none of its control functions is (Fréchet) differentiable at 0.

**Proof:** Most of the required properties is proved in Lemma 2 - Lemma 6. It only remains to show that none of the control functions to \( H \) is differentiable at 0. Assume that \( h \) is a control function to \( H \) which is differentiable at 0. We may assume that \( h(0) = h'(0) = 0 \). Find \( r, 0 < r < 1 \), such that
\[ |h(x)| \leq \frac{1}{8}|x| \quad \text{if} \quad |x| \leq 3r. \]

Denote
\[
\xi(t) = (r,t), \quad t \in [-2r, 2r], \\
\varphi(t) = |\xi(t)|, \\
\gamma(t) = G(\xi(t)), \\
\kappa(t) = h(\xi(t)).
\]

The function \( \gamma \) is piecewise linear, as by Lemma 4
\[ G(\xi(t)) = \max \{ G_z(\xi(t)) : z \in M, |z| > \frac{r}{9} \} . \]
We find points $t_i$, $-r = t_0 < t_1 < \cdots < t_m = r$, such that $\gamma$ is linear on each interval $[t_{i-1}, t_i]$. Then

\[ \gamma'_-(t_i) = \gamma'_+(t_{i-1}) \]

for each $i = 1, \ldots, m$. Since $h$ is a control function to $H$, the function $\kappa + \gamma - \varphi$ is convex on $[-2r, 2r]$. Hence for each $i = 1, \ldots, m$ we have

\[ \kappa'_-(t_i) - \kappa'_+(t_{i-1}) + \gamma'_-(t_i) - \gamma'_+(t_{i-1}) - \varphi'(t_i) + \varphi'(t_{i-1}) \geq 0. \]

Using convexity of $\kappa$ we obtain

\[ \kappa'_-(r) \leq \frac{1}{r} (\kappa(2r) - \kappa(r)), \]

\[ -\kappa'_+(r) \leq \frac{1}{r} (\kappa(-2r) - \kappa(-r)). \]

From (1), (2) and (3) it follows

\[ \sqrt{2} = \varphi'(r) - \varphi'(-r) = \sum_{i=1}^{m} (\varphi'(t_i) - \varphi'(t_{i-1})) \]

\[ \leq \sum_{i=1}^{m} (\kappa'_-(t_i) - \kappa'_+(t_{i-1})) \leq \kappa'_-(r) - \kappa'_+((-r)) \]

\[ \leq \frac{1}{r} (\kappa(2r) - \kappa(r) + \kappa(-2r) - \kappa(-r)) \leq 1, \]

which is a contradiction. 

**Local and global delta-convexity.**

Is every locally delta-convex function delta-convex? The answer is positive in the finite-dimensional case ([2]). In this section we will study various related questions in case of infinite dimension. We present several "negative" results, which show why the final result cannot be stronger.

Let us introduce a notation: if $x \in l^2$, then $x^j$ stands for the $j$-th coordinate of $x$. We denote by $e_i$ the element of $l^2$ with $i$-th coordinate 1 and remaining coordinates 0.

**Lemma 8.** Let $X$ be a normed linear space and $R > 0$. Let $H$ be a function on $B(0, R)$. Suppose that there exists a bounded control function $h$ to $H$ on $B(0, R)$. Then $H$, $h - H$ and $h + H$ are bounded on $B(0, R)$.

**Proof:** Denote $F = h - H$, $G = h + H$. From the convexity of $F$ and $G$ we obtain existence of continuous linear functionals $f$, $g$ on $X$ such that for each $x \in A$ we have $\langle f, x \rangle \leq F(x) - F(0)$ and $\langle g, x \rangle \leq G(x) - G(0)$. Then, of course, $f$ and $g$ are bounded on $B(0, R)$ and for each $x \in B(0, R)$ we estimate

\[ F(0) + \langle f, x \rangle \leq F(x) = 2h(x) - G(x) \leq 2h(x) - G(0) - \langle g, x \rangle. \]

Similarly we conclude that $G$ and $H = \frac{1}{2} (G - F)$ are bounded on $B(0, R)$. 

Lemma 9. Let $X$ be a normed linear space and $R > 0$. Let $H$ be a function on $B(0, 2R)$. Suppose that there exists a bounded control function $h$ to $H$ on $B(0, 2R)$. Then $H$ is Lipschitz-continuous on $B(0, R)$.

**Proof:** It is well known (see e.g. [3, Lemma 1.9]) that any bounded convex function on $B(0, 2R)$ is Lipschitz-continuous on $B(0, R)$. If we apply this result to the functions $h + H$ and $h - H$ (which are convex by the definition of a control function and bounded on $B(0, 2R)$ by the preceding lemma), we deduce that $H = \frac{1}{2}((h + H) - (h - H))$ is Lipschitz-continuous on $B(0, R)$.

Lemma 10. There exists a bounded nonnegative delta-convex function $H$ on $l^2$ such that $H(x) = 0$ if $|x| \geq 1$ and none of the control functions to $H$ is bounded on $B(0, 1)$.

**Proof:** If $x \in l^2$, we define

$$F(x) = \sup \{|x|, x^1 - \frac{1}{4}, 2(x^2 - \frac{1}{4}), 3(x^3 - \frac{1}{4}), \ldots \}$$

and

$$G(x) = \max \{1, F(x)\}.$$ 

The functions $F, G$ are convex on $l^2$. If $y \in l^2$, then

$$F(x) = \max \{|x|, x^1 - \frac{1}{4}, 2(x^2 - \frac{1}{4}), 3(x^3 - \frac{1}{4}), \ldots, m(x^m - \frac{1}{4})\}$$

holds for all $x \in B(y, \frac{1}{8})$ and $m = \max \{j : y^j \geq \frac{1}{8}\}$. This proves the continuity of $F$ and $G$. Hence the function $H = G - F$ is delta-convex. Obviously $H$ is bounded and $H = 0$ outside $B(0, 1)$. Denote

$$u_k = \frac{1}{4} e_k, \quad v_k = \left(\frac{1}{4} + \frac{1}{k}\right) e_k, \quad k = 5, 6, 7, \ldots.$$ 

Then

$$F(v_k) = 1, \quad F(u_k) = \frac{1}{4},$$

and thus

$$|H(v_k) - H(u_k)| = \frac{3}{4} \geq \frac{k}{2}|v_k - u_k|.$$ 

It follows that $H$ is not Lipschitz-continuous on $B(0, \frac{1}{2})$, and thus, by Lemma 9, none of the control functions to $H$ is bounded on $B(0, 1)$.

Example 11. Let $\Omega \subset l^2$ be an open convex set. We will construct a function $H$ on $l^2$ such that $H$ is locally delta-convex on $l^2$, but it is not delta-convex on $\Omega$.

Without loss of generality we may assume that $0 \in \Omega$. Denote $U = \{\frac{1}{2} x : x \in \Omega\}$. Let us find a sequence \{x_k\} of points of $U$ and $\delta > 0$ such that the balls $B(x_k, 2\delta)$ are contained in $U$ and pairwise disjoint. By a slight modification Lemma 10 we obtain
for every $k \in \mathbb{N}$ a delta-convex function $H_k$ such that $H_k = 0$ outside $B(x_k, \frac{1}{k}\delta)$ and none of the control functions to $H_k$ is bounded on $B(x_k, \frac{1}{k}\delta)$. Set

$$H = \sum_{k=2}^{\infty} H_k.$$  

Obviously $H$ is locally delta-convex on $l^2$. Assume that there is a control function $h$ to $H$ on $\Omega$. Using the unboundedness property, we find $u_k \in U$ such that $|u_k| \leq \frac{1}{k}\delta$ and $h(x_k + u_k) \geq \max\{h(2x_k), k\}$. From convexity of $h$ on the line connecting $2u_k$, $u_k + x_k$ and $2x_k$ we get $h(2u_k) \geq k$. Since $u_k \to 0$, the function $h$ is not continuous at 0, which is a contradiction.

**Theorem 12.** There exists a function $H$ on $l^2$ which has the following properties: With every point $z \in l^2$ we can associate a continuous convex function $h_z$ on $l^2$ such that $h_z$ is bounded on each bounded subset of $l^2$ and controls $H$ on a neighborhood of $z$. Nevertheless, $H$ is not delta-convex on $B(0,3)$.

**Proof:** For $i, j = 1, 2, \ldots$ we denote

$$z_{ij} = e_i + 2^{-i-1}e_j,$$

$$B_i = B(e_i, 2^{-i-1}),$$

$$B_{ij} = B(z_{ij}, 2^{-i-2}).$$

Let

$$H(x) = \begin{cases} 
  j(1 - \frac{|x - z_{ij}|}{2^{-i-2}}) & x \in B_{ij}, \\
  0 & \text{otherwise.}
\end{cases}$$

It is easy to see that there are control functions $h_z$ to $H$ with the required properties. (Notice that the function $\max(|x| - 1,2|x| - 2)$ controls $\max(1 - |x|,0).$) We will show that $H$ is not delta-convex on $B(0,3)$. Let us assume that there is a control function $h$ to $H$ on $B(0,3)$. By Lemma 8, $h$ is unbounded on each ball $B_i$. Now, for every $i \in \mathbb{N}$ we find a point $x_i \in B_i$, so that $h(x_i) \geq \max\{h(2e_i), i\}$. Set $y_i = 2(x_i - e_i)$. From the convexity of the function $h$ we obtain $h(y_i) \geq i$ (the point $x_i$ belongs to the segment connecting $y_i$ and $2e_i$). Since $|y_i| \leq 2^{-i}$, we have

$$\lim_{i \to \infty} y_i = 0 \text{ and } \lim_{i \to \infty} h(y_i) = \infty,$$

which contradicts the continuity of $h$ at the point 0.

**Remarks 13.** 1. If we do not require $h_z$ to be bounded on bounded sets, then the assertion easily follows from Example 11.

2. A similar example as in Theorem 12 can be constructed in each infinite-dimensional normed linear space.
Theorem 14. There exists a function $H$ on $l^2$ which is delta-convex on each bounded convex subset of $l^2$, but it is not delta-convex on $l^2$.

PROOF: Let us specify Example 11 so that $\Omega = l^2$ and $\lim |x_j| = \infty$. We obtain a function $H$, which is not delta-convex on $l^2$. Nevertheless, $H$ is delta-convex on an arbitrary bounded convex set $M \subset l^2$, as $H$ coincides on $M$ with a sum of a finite family of delta-convex functions.

Lemma 15. Let $H$, $h$ be functions on an open convex subset $A$ of a normed linear space $X$. Suppose that $h$ is convex and continuous. If every point of $A$ has a neighborhood $U$ such that $h$ controls $H$ on $U$, then $h$ controls $H$ on $A$.

PROOF: It is an obvious consequence of the fact that every locally convex function is convex.

The following theorem is an infinite-dimensional generalization of Hartman's result on locally delta-convex functions.

Theorem 16. Let $H_\alpha$ be a family of functions on an open convex subset $A$ of a normed linear space $X$. Suppose that for every bounded closed convex set $E \subset A$ there is a bounded continuous convex function $h_E$ on $E$ which controls each $H_\alpha$ on $E$. Then there is a continuous convex function $h$ on $X$ which controls each $H_\alpha$ on $A$.

PROOF: We may suppose that $0 \in A$. Let $p$ be the Minkowski's functional of $A$, defined by

$$p(x) = \inf\{ \lambda \in (0, +\infty) : \lambda^{-1} x \in A \}.$$ 

and

$$q(x) = |x| + \frac{p(x)}{1 - p(x)}.$$ 

Then $q$ is a continuous convex function on $A$, as the function $y \mapsto \frac{y}{1 - y}$ is increasing and convex on $[0, 1)$. Set

$$E_k = \{ x \in A : q(x) \leq k \}.$$ 

The sets $E_k$ are obviously bounded and closed subsets of $X$ and

$$A = \bigcup E_k.$$ 

Fix $k \in \{0, 1, 2, \ldots \}$. Denote

$$h_k = h_{E_k},$$

$$M_k = \sup_{E_{k+3}} h_{k+3},$$

$$m_k = \inf_{E_{k+3}} h_{k+3},$$

$$c_k = M_k + (k - 2)(M_k - m_k) = m_k + (k - 1)(M_k - m_k),$$

$$f_k(x) = h_{k+3}(x) + (M_k - m_k)q(x) - c_k,$$

$$w_k(x) = 5(M_k - m_k)(q(x) - (k + 1)).$$
Then $f_k$ is a convex function on $E_{k+3}$. We will estimate $f_k(x)$ for some positions of $x$: If $x \in E_{k+3} \setminus E_{k+2}$, then
\[ f_k(x) \leq M_k + (k + 3)(M_k - m_k) - c_k \leq 5(M_k - m_k) \leq w_k(x), \]
if $x \in E_{k-2}$, then
\[ f_k(x) \leq M_k + (k - 2)(M_k - m_k) - c_k = 0 \]
and if $x \in E_{k+1} \setminus E_{k-1}$, then
\[ f_k(x) \geq m_k + (k - 1)(M_k - m_k) - c_k = 0 \geq w_k(x). \]
Set
\[ g_k(x) = \begin{cases} \max\{0, f_k(x), w_k(x)\} & \text{if } x \in E_{k+3}, \\ w_k(x) & \text{if } x \in A \setminus E_{k+3}. \end{cases} \]
Then $g_k = 0$ on $E_{k-2}$, $g_k = f_k$ on $E_{k+1} \setminus E_{k-1}$ and $g_k = w_k$ on $A \setminus E_{k+2}$. It follows that $g_k$ is a continuous convex function on $A$ which controls each $H_\alpha$ on each convex subset of $E_{k+1} \setminus E_{k-1}$. Set
\[ h = \sum_{k=0}^{\infty} g_k. \]
Since for every bounded set $K$ we can find $n \in \mathbb{N}$ such that $g_k = 0$ on $K$ if $k \geq n$, we deduce that $h$ is a continuous convex function on $A$. By Lemma 15, $h$ controls each $H_\alpha$ on $A$.

Remark 17. Let $X$ and $Y$ be normed linear spaces. Let $H$ be a mapping of an open convex set $A \subset X$ into $Y$. Following [3], we say that $H$ is delta-convex, if there is a convex continuous function $h$ which controls $H$, this means that $h$ controls every function $H_\alpha : x \mapsto (f_\alpha, H(x))$, where $\{f_\alpha\}$ is the collection of all linear functionals on $Y$ with $\|f_\alpha\| \leq 1$. From Theorem 16 we immediately see that the following result is true:

Corollary 18. Let $X$ and $Y$ be normed linear spaces. Let $A$ be an open convex subset of $X$ and $H : A \to Y$ be a mapping. Suppose that for every bounded closed convex set $E \subset A$ there is a bounded continuous convex function $h_E$ on $E$ which controls $H$ on $E$. Then $H$ is delta-convex on $X$.

**REFERENCES**


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