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A new variant for the Meijer’s integral transform

J. RODRÍGUEZ

Abstract. In this paper a new aspect of the Meijer’s integral transform is treated, for which its corresponding inversion formula has been duly achieved. It turns out to exist a relation between this transform and Laplace’s, which opens the way to define different types of convolutions. Furthermore, some operational rules are obtained.

Keywords: $M_{\alpha,\beta}$-integral transform, Meijer, Laplace, Kratzel, Bessel, Bessel—Clifford, convolution, operational rule

Classification: 44A15

1. Introduction.

In this paper a new version of Meijer’s integral transform has been studied, which will be referred to as the $M_{\alpha,\beta}$-integral transform. This variant generalizes those of E. Kratzel’s [6], J. Conlan’s, E.L. Koh’s [3] and J. Rodríguez [9] as well, among others, and it is given as

\begin{align}
F(s) &= \int_0^\infty (st)^{\alpha+\beta-1} L_{\alpha-1}(st)f(t) \, dt \\
f(t) &= \frac{1}{\pi i} \int_{\Gamma_c} (st)^{-\beta} E_{\alpha-1}(st)F(s) \, ds
\end{align}

with $\Gamma_c = \{s/s \in \mathbb{C}, \text{Re}\sqrt{2s} > c > 0\}$. The functions $L_{\alpha-1}(t)$ and $E_{\alpha-1}(t)$ appear in their respective kernels, and are solutions of the differential equation [5]

\begin{equation}
ty'' + \alpha y' - y = 0\end{equation}

$E_{\alpha-1}(t)$ admits the following expansion

\begin{equation}
E_{\alpha-1}(t) = \sum_{n=0}^{\infty} \frac{t^n}{n! \Gamma(\alpha + n)}
\end{equation}

and it is known as the modified (or hyperbolic) Bessel—Clifford function of first kind and order $(\alpha - 1)$. When $(\alpha - 1)$ is a non-integer, then $t^{1-\alpha} E_{1-\alpha}(t)$ constitutes in itself another solution of (1.3), which is non-linearly dependent on $E_{\alpha-1}(t)$. Similarly, $L_{\alpha-1}(t)$ will be referred to as the modified Bessel—Clifford function of third kind and order $(\alpha - 1)$, and it is given as

\begin{equation}
L_{\alpha-1}(t) = \frac{\pi}{2\sin(\alpha - 1)\pi} (E_{\alpha-1}(t)^{1-\alpha} E_{1-\alpha}(t))
\end{equation}
It is of interest to emphasize the fact that \( E_{\alpha-1}(t) \) and \( L_{\alpha-1}(t) \) are linked to their corresponding Bessel functions by the following expressions

\[
E_{\alpha-1}(t) = t^{-\alpha-1/2} I_{\alpha-1}(2\sqrt{t})
\]

(1.6)

\[
L_{\alpha-1}(t) = t^{-\alpha-1/3} K_{\alpha-1}(2\sqrt{t}).
\]

\( L_{\alpha-1}(t) \) admits the generalization given in [7] and [8] as

\[
\eta(\varrho, \alpha; z) = \int_0^\infty \tau^{-\alpha} e^{-\tau - \pi \varrho - \tau^2} \, d\tau \quad (\varrho > 0, |\text{arg } z| < \frac{\pi}{2}),
\]

which for \( \varrho = 1 \), reduces to

\[
\eta(1, \alpha; z) = 2L_{\alpha-1}(z).
\]

(1.7)

The asymptotic behaviour of \( L_{\alpha-1}(t) \) can be interfered from \( \eta(1, \alpha; t) \), as follows

\[
L_{\alpha-1}(t) \sim \begin{cases} 
\frac{\Gamma(\alpha-1)}{2} t^{1-\alpha} & \text{if } \text{Re } \alpha - 1 > 0 \\
\frac{\Gamma(\alpha-1)}{2} t^{1-\alpha} + \frac{\Gamma(1-\alpha)}{2} & \text{if } \text{Re } \alpha - 1 = 0, \alpha - 1 \neq 0 \\
-1 nt & \text{if } \alpha - 1 = 0 \\
\frac{\Gamma(1-\alpha)}{2} & \text{if } \text{Re } \alpha - 1 < 0
\end{cases}
\]

(1.9)

for \( t \to 0^+ \), and

\[
L_{\alpha-1}(t) \sim \frac{\sqrt{\pi}}{2} t^{-\frac{2\alpha-1}{4}} e^{-2\sqrt{t}}
\]

(1.10)

for \( t \to +\infty \).

As for \( E_{\alpha-1}(z) \), it can be referred to from [10] that

\[
E_{\alpha-1}(z) \sim \frac{1}{\Gamma(\alpha)} \quad \text{if } \text{Re } \alpha > 0 \text{ and } z \to 0^+
\]

(1.11)

and also that

\[
z^{\frac{\alpha}{2} - \frac{1}{4}} E_{\alpha-1}(z) \sim \frac{1}{\sqrt{2\pi}} (e^{2\sqrt{z}} \pm i e^{-2\sqrt{z} + i(\alpha-1)\pi})(1 + 0(|z|^{-1/2}))
\]

(1.12)

for \( z \to +\infty \).

Similarly, the following integral representations for \( L_{\alpha-1}(t) \) can be derived from (1.5) through appropriate changes:

\[
L_{\alpha-1}(st) = \frac{1}{2} \int_0^\infty \tau^{-\alpha} e^{-\tau - st/\tau} \, d\tau
\]

(1.13)

\[
L_{\alpha-1}(st) = \frac{1}{2} s^{1-\alpha} \int_0^\infty \tau^{-\alpha} e^{-\tau - st/\tau} \, d\tau
\]

(1.14)

\[
L_{\alpha-1}(st) = \frac{1}{2} t^{1-\alpha} \int_0^\infty \tau^{-\alpha-2} e^{-\tau - st/\tau} \, d\tau
\]

(1.15)

which will be used to express the \( M_{\alpha, \beta} \)-integral transform in terms of the Laplace transform, so as to enable us to obtain convolutions for that transformation.
2. The $M_{\alpha,\beta}$-integral transform.

Its existence is based on the following:

**Proposition 1.** Let $\alpha, \beta$ be complex numbers and $f(t)$ a locally integrable function on $(0, \infty)$, such that

\[
f(t) = \begin{cases} 
0(t^{-\beta}) & \text{if } \Re(\alpha - 1) \geq 0 \\
0(t^{1-\alpha-\beta}) & \text{if } \Re(\alpha - 1) < 0
\end{cases}
\]

for $t \to 0^+$, and

\[f(t) = 0(e^{c\sqrt{2t}})\]

for $t \to +\infty$.

Under these conditions the integral given as

\[
F(s) = M_{\alpha,\beta}\{f(t)\} = \int_0^\infty (st)^{\alpha+\beta-1}L_{\alpha-1}(st)f(t)\,dt
\]

converges for $\Re(\sqrt{2s}) > c$. Besides, $f(s)$ proves to be analytic on the convergence domain.

**PROOF :** Set

\[
F(s) = \int_0^\epsilon (st)^{\alpha+\beta-1}L_{\alpha-1}(st)f(t)\,dt + \int_\epsilon^T (st)^{\alpha+\beta-1}L_{\alpha-1}(st)f(t)\,dt + \int_T^{\infty} (st)^{\alpha+\beta-1}L_{\alpha-1}(st)f(t)\,dt \quad \text{for } 0 < \epsilon < T < +\infty.
\]

It can be noted that the first integral in the right-hand side exists due to (1.9) together with the hypothesis. The second integral exists because of $f(t)$ being locally integrable and $(st)^{\alpha+\beta-1}L_{\alpha-1}(st)$ a continuous function. Finally, existence for the third integral is guaranteed by (1.10) provided that $\Re(\sqrt{2s}) > c$.

Analyticity proves obviously.

Now, the following inversion formula can be established.

**Proposition 2.** Let $\alpha, \beta$ be complex numbers with $\Re(\alpha) > 0$. Assume that $F(s)$ is analytic over the domain $\Omega = \{s/s \in \mathbb{C} \text{ and } \Re(\sqrt{2s}) > B \geq 0\}$ and also that $|F(s)| \leq M|s|^{-q}$ holds, $M$ and $q$ being real constants non-depending on $s$ and such that $q > -\Re(\beta) + \frac{5}{4}$. Then, for any fixed real $c > B$, the following expression

\[
F(s) = \int_0^\infty (st)^{\alpha+\beta-1}L_{\alpha-1}(st)f(t)\,dt
\]

is valid for $\Re(\sqrt{2s}) > c$. Here $f(t)$ is given by

\[
f(t) = \frac{1}{\pi i} \int_{\Gamma_c} (zt)^{-\beta}E_{\alpha-1}(zt)F(z)\,dz
\]
with \( \Gamma_c = \{ z/w \in \mathbb{C} \text{ and } \text{Re}\sqrt{2z} = c \} \).

**Proof:** Assume \( s \) to be fixed and that \( 1 < R < \infty \). Set:

\[
I(s, T) = \int_0^T (st)^{\alpha+\beta-1} L_{\alpha-1}(st) f(t) \, dt = \frac{1}{\pi i} \int_0^T (st)^{\alpha+\beta-1} L_{\alpha-1}(st) \int_{\Gamma_c} (zt)^{-\beta} E_{\alpha-1}(zt) F(z) \, dz
\]

where

\[
\Gamma_c = \{ w/w \in \mathbb{C} \text{ and } \text{Re}\sqrt{2w} = c \} = \left\{ w = a + bi/a = \frac{1}{2}(c^2 - t^2), b = ct, t \in (-\infty, +\infty) \right\}.
\]

Consider, on the other hand, the domain defined as

\[
\Lambda = \{ (t, z)/t \in [0, T], z \in \Gamma_c \}.
\]

To make feasible in (2.3) inversion of the order of integration it suffices to apply Fubini's theorem, previously verifying that

\[
((st)^{\alpha+\beta-1} L_{\alpha-1}(st)(zt)^{-\beta} E_{\alpha-1}(zt) F(z))
\]

proves an absolutely integrable function on \( \Lambda \), provided that

\[
\text{Re} \alpha > 0, \text{ and } q > -\text{Re} \beta + \frac{5}{4}.
\]

Therefore, the following holds true

\[
I(s, T) = \frac{s^{\alpha+\beta-1}}{\pi i} \int_{\Gamma_c} z^{-\beta} F(z) \int_0^T t^{\alpha-1} E_{\alpha-1}(zt) L_{\alpha-1}(st) \, dtdz.
\]

Now, by invoking equality [11]

\[
\int_0^T t^{\alpha-1} E_{\alpha-1}(zt) L_{\alpha-1}(st) \, dt = \frac{T^\alpha}{z-s} (z E_{\alpha}(zT)L_{\alpha-1}(sT)+ sE_{\alpha-1}(zT)L_{\alpha}(sT)) - \frac{s^{1-\alpha}}{2(z-s)}
\]

and by substituting its right-hand side for the second part of (2.4), we obtain

\[
I(s, T) = \frac{s^{\alpha+\beta-1}}{\pi i} \int_{\Gamma_c} z^{-\beta} F(z) \left[ \frac{T^\alpha}{z-s} (z E_{\alpha}(zT)L_{\alpha-1}(sT)+ sE_{\alpha-1}(zT)L_{\alpha}(sT)) - \frac{s^{1-\alpha}}{2(z-s)} \right] \, dz.
\]
Now, by virtue of the asymptotic behaviour of \( L_{\alpha-1}(t) \) and \( E_{\alpha-1}(t) \), we can have the following inequality:

\[
\left| \frac{T^\alpha}{z-s} (zE_\alpha(zT)L_{\alpha-1}(sT) + sE_{\alpha-1}(zT)L_\alpha(sT)) \right| \leq N \cdot \frac{|z|^{-\operatorname{Re} \frac{\beta}{2} - \frac{1}{4}|s|^{-\operatorname{Re} \frac{\alpha}{2} + \frac{1}{4}} |z|^{1/2} (|z|^{1/2} + |s|^{1/2})}{||z| - |s||} \cdot e^{-\sqrt{2T} (\operatorname{Re} \sqrt{z} - c)}
\]

and, as a consequence,

\[
\left| \frac{s^{\alpha+\beta-1}}{\pi i} \int_{\gamma_c} z^{-\beta} F(z) \frac{T^\alpha}{z-s} (zE_\alpha(zT)L_{\alpha-1}(sT) + sE_{\alpha-1}(zT)L_\alpha(sT)) \right| < M_1 |s|^{\operatorname{Re} \frac{\alpha}{2} + \operatorname{Re} \beta + \frac{3}{4}} e^{-\sqrt{2T} (\operatorname{Re} \sqrt{z} - c)} \int_{\Gamma_c} |z|^{-q} \operatorname{Re} \frac{\beta}{2} - \frac{1}{4} \, dz
\]

is true for \( \operatorname{Re} \sqrt{2s} > c \), due to \( |z|^{1/2} (|z|^{1/2} + |s|^{1/2}) / ||z| - |s|| \) being a bounded function.

On the other hand, the last integral converges because

\[
q > - \operatorname{Re} \beta + 1 > - \operatorname{Re} \frac{\alpha}{2} - \operatorname{Re} \beta + \frac{3}{4}.
\]

Thus, for every fixed \( s \), with \( \operatorname{Re} \sqrt{2s} > c > 0 \), this integral proves uniformly convergent on \( 1 < T < \infty \) and then it is valid to take up the limit for \( T \to \infty \):

\[
\lim_{T \to \infty} I(s, T) = \int_0^\infty (st)^{\alpha+\beta-1} L_{\alpha-1}(t)f(t) \, dt = \frac{s^\beta}{2\pi i} \int_{\Gamma_c} \frac{z^{-\beta} F(z)}{s - z} \, dz
\]

To finish the proof, it only remains to perform the evaluation of the integral

\[
\int_{\Gamma_c} \frac{z^{-\beta} F(z)}{s - z} \, dz
\]

which can be achieved by considering the closed domain drawn in this figure:
whose contour (considered by J. Betancor [1]) admits the following parametric representation

\[ R_{1,y} = \begin{cases} 
  a(t) = \frac{1}{2}(y^2 - t^2) & t \in [-y, y] \\
  b(t) = yt & t \in [c, y] 
\end{cases} \]

\[ R_{2,y} = \begin{cases} 
  a(t) = \frac{1}{2}(t^2 - y^2) & t \in [c, y] \\
  b(t) = -yt & t \in [c, y] 
\end{cases} \]

\[ R_{3,y} = \begin{cases} 
  a(t) = \frac{1}{2}(t^2 - y^2) & t \in [c, y] \\
  b(t) = ty & t \in [c, y] 
\end{cases} \]

\[ \Gamma_{c,y} = \begin{cases} 
  a(t) = \frac{1}{2}(c^2 - t^2) & t \in [-y, y]. \\
  b(t) = ct & t \in [-y, y]. 
\end{cases} \]

If \( F(z) \) is holomorphic on \( \Omega = \{z/z \in \mathbb{C} \text{ and } \text{Re} \sqrt{2z} > B > 0\} \), then it follows from Cauchy's theorem that

\[ \int_{R_y} \frac{z^{-\beta}F(z)}{s - z} \, dz = 2\pi is^{-\beta}F(s). \]

But according to the previously established bounds we can write

\[ |\int_{R_{1,y}} \frac{z^{-\beta}F(z)}{s - z} \, dz| \leq \frac{M}{d(s)} \cdot y^{-2(q+\text{Re } \beta - 1)} \]

which tends to zero for \( y \to +\infty \) in view that \( q > -\text{Re } \beta + 1 \). Here \( d(s) \) denotes the distance from \( s \) to \( R_{1,y} \).

The same procedure and conditions lead to

\[ |\int_{R_{2,y}} \frac{z^{-\beta}F(z)}{s - z} \, dz| \to 0, \quad |\int_{R_{3,y}} \frac{z^{-\beta}F(z)}{s - z} \, dz| \to 0, \]

for \( y \to \infty \).

Hence

\[ \int_{R_y} \frac{z^{-\beta}F(z)}{s - z} \, dz = \int_{-\Gamma_{c,y}} \frac{z^{-\beta}F(z)}{s - z} \, dz \]

and, as a consequence,

\[ \lim_{T \to \infty} I(s, T) = F(s) \]

can be easily inferred.

In the following, several propositions will be given in order to express the \( M_{\alpha,\beta} \)-integral transform in terms of Laplace's. We always take the assumption that every integral is absolutely convergent.
A new variant for the Meijer’s integral transform

Proposition 3. The integral transform

\[ F(s) = M_{\alpha, \beta} \{ f(t) \} = \int_0^\infty (st)^{\alpha+\beta-1} L_{\alpha-1}(st)f(t) \, dt \]

can be re-written for \( \text{Re}\alpha > \frac{1}{2} \) as:

(2.5) \[ F(s) = M_{\alpha, \beta} \{ f(t) \} = \frac{\sqrt{\pi}}{\Gamma(\alpha - 1/2)} s^{\alpha+\beta-1} \mathcal{L} \left\{ (1 - \tau)^{\alpha-\frac{3}{2}} \tau^\beta f(\tau^2) \, d\tau; 2\sqrt{s} \right\} \]

To justify this we will invoke the well-known connection existing between the K-integral transform and Laplace’s [4], given as

\[ \int_0^\infty (xy)^{1/2} K_{\alpha-1}(xy)g(x) \, dx = \frac{\sqrt{\pi} \cdot 2^{1-\alpha}}{\Gamma(\alpha - \frac{1}{2})} y^{\alpha-\frac{1}{2}} \int_0^\infty e^{-yz} \int_0^z (x^2 - r^2)^{\alpha-\frac{3}{2}} (y^2 - r^2)^{\frac{1}{2} - \alpha} g(r) \, dr \, dx \]

Now, by performing the changes of variable \( x = \sqrt{t} \), \( y = 2\sqrt{s} \) and \( r = \sqrt{st} \) and also by using the relation

\[ L_{\alpha-1}(x) = x^{-\frac{\alpha-1}{2}} K_{\alpha-1}(2\sqrt{x}) \]

we obtain

(2.6) \[ \int_0^\infty (st)^{\alpha+\beta-1} L_{\alpha-1}(st)f(t) \, dt = \frac{\sqrt{\pi} s^{\alpha+\beta-1}}{2\Gamma(\alpha - 1/2)} \int_0^\infty e^{-2\sqrt{st}} t^{\alpha+\beta-1} \int_0^1 (1 - \tau)^{\alpha-\frac{3}{2}} \tau^\beta f(t\tau) \, d\tau \, dt \]

where \( f(t) = t^{-\frac{\alpha-1}{2}} g(\sqrt{t}) \).

Finally, the new change \( t = \xi^2 \) in the right-hand side of (2.6) leads to the result stated in (2.5).

Proposition 4. The \( M_{\alpha, \beta} \)-integral transform can be expressed as

(2.7) \[ F(s) = M_{\alpha, \beta} \{ f(t) \} = \frac{s^\beta}{2} \mathcal{L} \{ f_{-\alpha, \alpha+\beta-1}(t) \} \]

provided that \( \text{Re}s > 0 \), which proves equivalent to stating that

(2.8) \[ F(s) = M_{\alpha, \beta} \{ f(t) \} = \frac{s^{\alpha+\beta-1}}{2} \mathcal{L} \{ f_{\alpha-2, \beta}(t) \} \]
with

$$f_{\lambda, \gamma}(t) = \int_0^\infty t^{\lambda \gamma - 1} e^{-t} f(\tau) d\tau.$$  \hfill (2.9)

In fact, by substituting the integral representation of (1.14) for the last part of (2.1), we have

$$F(s) = M_{\alpha, \beta} \{ f(t) \} = \frac{1}{2} \int_0^\infty (st)^{\alpha + \beta - 1} f(t) s^{1 - \alpha} \int_0^\infty x^{-\alpha} e^{-x s - \frac{1}{2}} dx dt =$$

$$= \frac{s^\beta}{2} \int_0^\infty e^{-sx} dx \int_0^\infty x^{-\alpha} t^{\alpha + \beta - 1} f(t) e^{-\frac{1}{2} t} dt = \frac{s^\beta}{2} \mathcal{L} \{ f_{-\alpha, \alpha + \beta - 1}(x); s \}$$

once the integration order has been inverted.

Now, to obtain (2.8) substitute (1.15) for (2.1) and invert the order of integration.

**Proposition 5.** The $M_{\alpha, \beta}$-integral transform can be given for $Re s > 0$ as:

$$F(s) = M_{\alpha, \beta} \{ f(t) \} = \frac{s^\beta}{2} \mathcal{L} \{ x^{-\alpha} \mathcal{L} \{ t^{\alpha + \beta - 1} f(t); x^{-1} \} ; s \}$$  \hfill (2.10)

or else

$$F(s) = M_{\alpha, \beta} \{ f(t) \} = \frac{s^{\alpha + \beta - 1}}{2} \mathcal{L} \{ x^{\alpha - 2} \mathcal{L} \{ t^\beta f(t); x^{-1} \} ; s \}$$  \hfill (2.11)

3. *Convolutions for the $M_{\alpha, \beta}$-integral transform.*

In this section several convolutions for the $M_{\alpha, \beta}$-integral transform are given.

a). Define convolution $*$ of two functions $f(t)$ and $g(t)$, as:

$$f(t) * g(t) = \frac{1}{2} t^{-\beta} I^{\alpha - 1} \int_0^t (t - \xi)^{\beta} d\xi \int_0^1 \eta^{\alpha + \beta - 1} (1 - \eta)^{\alpha + \beta - 1} d\eta,$$

$$= \mathcal{F}(f(t) g((1 - \eta)(t - \xi))) d\eta,$$

where $I^{\alpha - 1}$ stands for the Riemann—Liouville fractional integral [10].

**Proposition 6.** If we define convolution $f(t) * g(t)$ as in (3.1); $f(t), g(t), f(t) * g(t)$ being $M_{\alpha, \beta}$-transformable functions for $Re \sqrt{2s} > c > 0$, then

$$M_{\alpha, \beta} \{ f(t) * g(t) \} = s^{1 - \alpha - \beta} M_{\alpha, \beta} \{ f(t) \} \cdot M_{\alpha, \beta} \{ g(t) \}$$

is true.

**Proof:** In fact, from (2.11) it follows that

$$F(s) = M_{\alpha, \beta} \{ f(t) \} = \frac{s^{\alpha + \beta - 1}}{2} \int_0^\infty e^{-s \tau^{\alpha - 2}} \int_0^\infty e^{-\tau^{-1} t^\beta f(t)} dt d\tau =$$

$$= \frac{s^{\alpha + \beta - 1}}{2} \mathcal{L} \{ \tau^{\alpha - 2} f_0(\tau); s \}$$
where \( f_0(\tau) = \int_0^\infty e^{-\tau^{-1}t} f(t) \, dt \).

Similarly,

\[
G(s) = M_{\alpha,\beta} \{ g(t) \} = \frac{s^{\alpha+\beta-1}}{2} \mathcal{L} \{ \tau^{\alpha-2} g_0(\tau); s \}.
\]

Hence,

\[
M_{\alpha,\beta} \{ f(t) \} \cdot M_{\alpha,\beta} \{ g(t) \} = \frac{s^{2\alpha+2\beta-2}}{4} \mathcal{L} \{ \tau^{\alpha-2} f_0(\tau) \} \cdot \mathcal{L} \{ \tau^{\alpha-2} g_0(\tau) \} = \frac{s^{2\alpha+2\beta-2}}{4} \mathcal{L} \left\{ \int_0^t \xi^{\alpha-2} f_0(\xi)(t - \xi)^{\alpha-2} g_0(t - \xi) \, d\xi \right\}.
\]

Now, the change \( \xi = tu \) leads to

\[
\frac{s^{2\alpha+2\beta-2}}{4} \mathcal{L} \left\{ t^{2\alpha-3} \int_0^1 u^{\alpha-2} f_0(tu) g_0[(1 - u)t] \, du \right\} = \frac{s^{2\alpha+2\beta-2}}{4},
\]

\[
\mathcal{L} \left\{ t^{2\alpha-3} \int_0^\infty \int_0^\infty \int_0^1 u^{\alpha-2}(1-u)^{\alpha-2} e^{-t^{-1}[u^{-1}\tau+(1-u)^{-1}]} \tau^\beta y^\beta f(\tau) g(y) \, du \, d\tau \, dy \right\}
\]

which combined with

\[
x = u^{-1}\tau + (1 - u)^{-1}y, \quad \xi = u^{-1}\tau
\]
yields

\[
\frac{s^{2\alpha+2\beta-2}}{4} \mathcal{L} \left\{ t^{2\alpha-3} \int_0^\infty e^{-t^{-1}x} \, dx \int_0^x (x - \xi)^\beta \xi^\beta \, d\xi \cdot \int_0^1 u^{\alpha+\beta-1}(1-u)^{\alpha+\beta-1} f(u\xi) g[(x - \xi)(1 - u)] \, du \right\} = \frac{s^{2\alpha+2\beta-2}}{4} \mathcal{L} \left\{ t^{\alpha-2} t^{-\alpha-1} \int_0^\infty e^{-t^{-1}x} H(f, g; x) \, dx \right\}
\]

where

\[
H(f, g; x) = \int_0^x (x - \xi)^\beta \xi^\beta \, d\xi \int_0^1 u^{\alpha+\beta-1}(1-u)^{\alpha+\beta-1} f(u\xi) g[(x - \xi)(1 - u)] \, du
\]

and by taking into account that

\[
t^{\alpha-1} \int_0^\infty e^{-t^{-1}x} H(f, g; x) \, dx = \int_0^\infty e^{-t^{-1}x} t^{\alpha-1} H(f, g; x) \, dx
\]

holds, then it can be easily inferred that (3.2) can be re-written as:

\[
s^{\alpha+\beta-1} \left[ \frac{s^{\alpha+\beta-1}}{2} \mathcal{L} \left\{ t^{\alpha-2} \mathcal{L} \left\{ x^\beta x^{-\beta} t^{-\alpha-1} H(f, g; x); t^{-1} \right\}; s \right\} \right] = s^{\alpha+\beta-1} M_{\alpha,\beta} \{ f(t) \ast g(t) \}.
\]
b). If we define convolution $\ast$ of two functions $f(t), g(t)$ as

$$f(t) \ast g(t) = \frac{t^{1-\alpha-\beta}}{2^\alpha \Gamma(\alpha - 1)} \int_0^1 (t - \xi)^{\alpha + \beta - 1} \xi^{\alpha + \beta - 1} d\xi \cdot \int_0^1 (1 - \eta)^{\beta} f(\xi\eta) g((1 - \eta)(t - \xi)) d\eta,$$

the following holds true:

**Proposition 7.** If convolution $f(t) \ast g(t)$ is defined as in (3.3) and if $f(t), g(t)$ and $f(t) \ast g(t)$ are $M_{\alpha, \beta}$-transformable functions for $\text{Re} \sqrt{2s} > c > 0$, then

$$M_{\alpha, \beta} \{f(t) \ast g(t)\} = s^{-\beta} M_{\alpha, \beta} \{f(t)\} \cdot M_{\alpha, \beta} \{g(t)\}$$

holds.

By using of (2.10), proof follows a similar procedure as in the previous proposition.

c). Let $\alpha, \beta$ be real numbers with $\alpha > 1$. It is feasible to define a convolution for the $M_{\alpha, \beta}$-integral transform in the space $C(\beta)$, which is made up of all complex functions of the form $f(t) = t^{\gamma-\beta} f_1(t)$, where $\gamma > -1$ and $f_1(t)$ being a continuous function on $[0, \infty)$.

Define in $C(\beta)$ the following operation:

$$(3.4) \quad f(t) \circ g(t) = \frac{t^{\alpha+\beta}}{2^\gamma \Gamma(\alpha - 1)} \int_0^1 \int_0^1 \int_0^1 t_3^{1+2\beta}(1 - t_3)^{\alpha-2}(t_2(1 - t_2))^{\beta} \cdot (t_1(1 - t_1))^{\alpha+\beta-1} f(tt_1t_2t_3) g(tt_3(1 - t_1)(1 - t_2)) dt_1 dt_2 dt_3$$

By virtue of Weierstrass’ approximation theorem the operation $(\circ)$ is completely defined by invoking

$$t^{\gamma-\beta+p} \circ t^{\gamma-\beta+q} = \frac{\Gamma(\gamma + \alpha + p)\Gamma(\gamma + \alpha + q)\Gamma(\gamma + p + 1)\Gamma(\gamma + q + 1)}{\Gamma(2\gamma + 2\alpha + p + q)\Gamma(2\gamma + \alpha + p + q + 1)} t^{\alpha+2\gamma-\beta+p+q}$$

for each $p, q \in \mathbb{N}$ with $\gamma > -1$.

Let us now consider the integral transform

$$T_{\alpha, \beta} \{f(t)\} = \int_0^\infty t^{\alpha+\beta-1} L_{\alpha-1}(st) f(t) dt$$

which is closely related to $M_{\alpha, \beta}$.

It is proved in [2] that the $(\circ)$-operation proves a convolution for the transformation $T_{\alpha, \beta}$ in the subset of $C(\beta)$ denoted as $C(\beta, c)$, with $c > 0$, and defined as follows:

$$C(\beta, c) = \left\{ f(t) \middle| f(t) \in C(\beta) \quad \text{and} \quad f(t) = 0(e^{c\sqrt{2t}}) \quad \text{for} \quad t \to \infty \right\}.$$
Note that \( M_{\alpha,\beta} \{ f(t) \} = s^{\alpha+\beta-1} T_{\alpha,\beta} \{ f(t) \} \) and also that
\[
\frac{2}{\Gamma(-\alpha - \beta + 1) \Gamma(-2\alpha - \beta + 2)} T_{\alpha,\beta} \{ t^{-2\alpha-2\beta+1} \} = s^{\alpha+\beta-1}
\]
hold provided that \(-\alpha - \beta + 1 > 0\) and \(-2\alpha - \beta + 2 > 0\).

Under these conditions we define the operation
\[
f(t) \circ g(t) = \frac{2}{\Gamma(-\alpha - \beta + 1) \Gamma(-2\alpha - \beta + 2)} t^{-2\alpha-2\beta+1} \circ (f(t) \circ g(t))
\]
and then the following can be established:

**Proposition 8.** If \( \alpha > 1, -\alpha - \beta + 1 > 0 \) and \(-2\alpha - \beta + 2 > 0\) for each \( f(t), g(t) \in C(\beta, c) \) in such a way that the expressions \( t^{-2\alpha-2\beta+1} \circ (f(t) \circ g(t)) \) and \( f(t) \circ g(t) \in C(\beta, c) \) belong to \( C(\beta, c) \), then the following holds:

\[
M_{\alpha,\beta} \{ f(t) \circ g(t) \} = M_{\alpha,\beta} \{ f(t) \} \cdot M_{\alpha,\beta} \{ g(t) \}
\]
for \( \text{Re} \sqrt{2s} > c \).

**Proof:** It suffices to note that
\[
M_{\alpha,\beta} \{ f(t) \circ g(t) \} = s^{\alpha+\beta-1} T_{\alpha,\beta} \{ f(t) \circ g(t) \} =
\]
\[
2s^{\alpha+\beta-1} \Gamma(-\alpha - \beta + 1) \Gamma(-2\alpha - \beta + 2) T_{\alpha,\beta} \{ t^{-2\alpha-2\beta+1} \circ (f(t) \circ g(t)) \} =
\]
\[
\frac{2s^{\alpha+\beta-1}}{\Gamma(-\alpha - \beta + 1) \Gamma(-2\alpha - \beta + 2)} T_{\alpha,\beta} \{ t^{-2\alpha-2\beta+1} \} \cdot T_{\alpha,\beta} \{ (f(t) \circ g(t)) \} =
\]
\[
2^{\alpha+2\beta-2} T_{\alpha,\beta} \{ f(t) \} \cdot T_{\alpha,\beta} \{ g(t) \} = M_{\alpha,\beta} \{ f(t) \} \cdot M_{\alpha,\beta} \{ g(t) \}.
\]

4. **Operational rules.**

The following operational rule, which relates the operator \( A_{\alpha,\beta} = t^{1-\alpha-\beta} Dt^\alpha Dt^\beta \) to the \( M_{\alpha,\beta} \)-integral transform, comes in very useful in numerous applications.

**Proposition 9.** Let \( f(t) \in C^2((0, \infty)) \), with
\[
f(t) = 0(t^m) \quad \text{if } m > \max(-\text{Re} \beta, -\text{Re}(\alpha + \beta))
\]
\[
Dt^\beta f(t) = 0(t^n) \quad \text{if } n > \max(-1, -\text{Re} \alpha)
\]
for \( t \to 0^+ \) and
\[
f(t) = 0(e^{c\sqrt{2t}})
\]
for \( t \to +\infty \).

Then
\[
M_{\alpha,\beta} \{ A_{\alpha,\beta} f(t) \} = s M_{\alpha,\beta} \{ f(t) \}
\]
holds.
In fact
\[ M_{\alpha,\beta} \{ A_{\alpha,\beta} f(t) \} = \int_0^\infty (st)^{\alpha+\beta-1} L_{\alpha-1}(st) t^{1-\alpha-\beta} Dt^\alpha Dt^\beta f(t) \, dt = \]
\[ = s^{\alpha+\beta-1}(a_1 - A_2 + \int_0^t t^\beta Dt^\alpha DL_{\alpha-1}(st) f(t) \, dt = sM_{\alpha,\beta} \{ f(t) \} , \]
can be stated after performing two integrations by parts and verifying that
\[ A_1 = t^\alpha Dt^\beta f(t)L_{\alpha-1}(st)]_0^\infty = 0 \]
\[ A_2 = t^\beta f(t)DL_{\alpha-1}(st)]_0^\infty = 0 \]
in view of the behaviour of \( f(t) \) and \( L_{\alpha-1}(st) \).
This result can be extended by induction as it is shown in the following:

**Proposition 10.** Let \( k \) be a positive integer and \( f(t) \in C^{2k}((0, \infty)) \), with
\[ A_{\alpha,\beta}^{k-1} f(t) = 0(t^p), \quad \text{if } p > \max(-\text{Re} \, \beta, -\text{Re}(\alpha + \beta)) \]
\[ Dt^\beta A_{\alpha,\beta}^{k-1} f(t) = 0(t^q) \quad \text{if } q > \max(-1, -\text{Re} \, \alpha) \]
for \( t \to 0^+ \), and
\[ f(t) = 0(e^{c\sqrt{2t}}) \]
for \( t \to +\infty \).
Then, the following holds
\[ M_{\alpha,\beta} \{ A_{\alpha,\beta}^k f(t) \} = s^k M_{\alpha,\beta} \{ f(t) \} . \]

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