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Commentationes Mathematicae Universitatis Carolinae, Vol. 31 (1990), No. 3, 557--565

Persistent URL: http://dml.cz/dmlcz/106890

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On weak solutions to a viscoelasticity model

JAROSLAV MILOTA, JINDŘICH NEČAS, VLADIMÍR ŠVERÁK

Abstract. The existence of global in time weak solutions of a viscoelasticity model is proved. There is no restriction on the dimension but it is supposed that the memory response is linear and a kernel has special properties.

Keywords: Weak solution, viscoelasticity, global existence, Galerkin approximations, a priori estimates, compact imbedding, monotone operators

Classification: 45K05, 73F99

1. Introduction.

The purpose of this paper is to prove the existence of global in time weak solutions of equations of motion for a model of viscoelastic body. We assume that the body occupies a reference configuration $\Omega \subset \mathbb{R}^N$ ($\Omega$ is a bounded domain with smooth boundary) and has unit reference density. We denote by $u(x, t)$ the displacement at the time $t$ of the particle with the reference position $x$. The strain $\epsilon$ is given by

\begin{equation}
\epsilon(x, t) = \nabla_x u(x, t)
\end{equation}

and the equation of balance of linear momentum has the form

\begin{equation}
\dot{u}(x, t) = \text{div}_x \sigma(x, t) + f(x, t),
\end{equation}

where $\sigma$ is the stress and $f$ is a body force. The body is characterized by constitutive assumptions which relate the stress to the motion. General constitutive theories are discussed for example in Coleman & Noll [2], Coleman & Mizel [1] and Saut & Joseph [11]. For the comprehensive account see the recent monograph Renardy & Hrusa & Nohel [10].

We shall limit our attention to constitutive relations of the type

\begin{equation}
\sigma(x, t) = \int_{-\infty}^{t} k(t-s) G(\epsilon(x, t), \epsilon(x, s)) ds.
\end{equation}

Here $k$ is a given nonincreasing positive function which satisfies certain growth conditions at 0 and $\infty$. We suppose that the tensor function $G$ has the special form

\begin{equation}
G(a, b) = g(a) + h(b).
\end{equation}

Moreover, our crucial assumption is that $h$ is linear. Assumptions on $g$ are stated below.
Substitution of the constitutive relations into (1.2) yields

\begin{equation}
\begin{aligned}
\text{(1.5)} \\
u_{tt}(x, t) = \text{div}_x g(\nabla u(x, t)) - \int_{-\infty}^{t} k(t-s)\Delta u(x, s) ds + f(x, t),
\end{aligned}
\end{equation}

\( x \in \Omega, t \geq 0 \). We consider this equation together with the Dirichlet boundary condition

\begin{equation}
\begin{aligned}
\text{(1.6)} \\
u|_{\partial \Omega} = 0.
\end{aligned}
\end{equation}

We shall write (1.5) and (1.6) in the form

\begin{equation}
\begin{aligned}
\text{(1.7)} \\
u_{tt} + \phi(u) + k \ast \Delta u = f.
\end{aligned}
\end{equation}

We seek a vector function \( u \) which satisfies (1.7) in weak sense together with the initial conditions

\begin{equation}
\begin{aligned}
\text{(1.8)} \\
u(0) = u_0, \quad \nu_t(0) = u_1.
\end{aligned}
\end{equation}

We remark that little is known about the existence of weak solutions for viscoelastic models. Recently, Nohel & Rogers & Tzavaras [9] established the global existence of weak solutions to the initial value problem above in the special case \( \Omega = \mathbb{R} \) and \( g = h \) in (1.4).

In Section 2 we introduce appropriate function spaces. Section 3 is devoted to the proof of the existence of weak solutions. The proof is standard: we use the Galerkin method to construct approximate solutions, establish a priori estimates and use compact imbeddings and the theory of monotone operators to prove convergence. The method of monotone operators was used in similar situation in [6].

The authors are indebted to John A. Nohel for the discussion of the preliminary version of this paper.

After this paper was finished we learned about the work of H.Engler [3] dealing with more general equation than (1.5) in one space dimension.

2. Appropriate spaces and operators.

Let \( \Omega \subset \mathbb{R}^N \) be a bounded domain with smooth boundary. The spaces \( V = H_0^1(\Omega), \quad H = L^2(\Omega) \) and \( V' = H^{-1}(\Omega) \) of \( \mathbb{R}^N \)-valued functions are defined in the usual way. We denote by \( (\cdot, \cdot) \) and \( (\cdot, \cdot) \) respectively the scalar product in \( H \) and \( V \). The corresponding norms are denoted by \( | \cdot | \) and \( \| \cdot \| \). The duality between \( V' \) and \( V \) is denoted by \( \langle \cdot, \cdot \rangle \). The Laplace operator \( \Delta : V \rightarrow V' \) is defined by

\[ \langle -\Delta u, v \rangle = \int_{\Omega} \nabla u \cdot \nabla v = (u, v), \]

\( u, v \in V \).

Consider the orthogonal basis of \( H \) consisting of the eigenfunctions \( w_n \in V \) of \( -\Delta \). We assume

\[ -\Delta w_n = \lambda_n w_n, \quad |w_n| = 1, \quad n = 1, 2, \cdots, \]

where \( 0 < \lambda_1 < \lambda_2 \leq \lambda_3 \) is the sequence of eigenvalues of \( -\Delta \).

We denote by \( P_n \) the orthogonal projection (in \( H \)) of \( H \) onto the linear hull \( V_m \) of the first \( m \) eigenfunctions. The following statement is obvious:
Lemma 1. The operators $P_m$ can be extended to the orthogonal projections in $V'$.

The extension of $P_m$ to $V'$ will be denoted also by $P_m$.

Let $u \in V$ and let $c_k = (w_k, u)$. We define

$$[u]_s^2 = \sum_{k=1}^{\infty} \lambda_k^s c_k^2.$$  

We shall assume $-\frac{1}{2} < s < \frac{1}{2}$. We can consider $[\cdot]_s$ as a norm on $V$ and the completion of $V$ in this norm is denoted by $H^s(\Omega)$.

Let $E$ be a Hilbert space and let $a < b \in \mathbb{R}$. The space $L^2(a, b; E)$ is defined in the usual way. For $0 < \nu < \frac{1}{2}$ and $u \in L^2(a, b; E)$ we define

$$\| u \|_\nu^2 = \int_0^1 dt \cdot t^{-(2\nu+1)} \int_{\mathbb{R}} | u(\tau - t) - u(\tau) |^2_E d\tau.$$  

(We extend $u$ by zero outside $(a, b)$.) The space of all $u \in L^2(a, b; E)$ for which $\| u \|_\nu$ is finite is denoted by $\mathcal{H}^{\nu}(a, b; E)$. For $u \in L^2(a, b; E)$ we denote by $\hat{u}$ the Fourier transform of $u$. It is well-known (see e.g. [5]) that

$$[\int_{\mathbb{R}} (1 + \sigma^2)^\nu | \hat{u}(\sigma) |^2_E d\sigma]^\frac{1}{2}$$  

is an equivalent norm on $\mathcal{H}^{\nu}(a, b; E)$. We put $\mathcal{H}^0(a, b; E) = L^2(a, b; E)$ and for $0 \leq \alpha < \frac{1}{2}$ define $\mathcal{H}^{1+\alpha}(a, b; E)$ as the space of all $u \in L^2(a, b; E)$ with distributional derivatives $u'$ belonging to $\mathcal{H}^\alpha(a, b; E)$. The norm on $\mathcal{H}^{1+\alpha}(a, b; E)$ is defined by

$$\| u \|_{1+\alpha} = \| u \|_{L^2} + \| u' \|_\alpha.$$  

We also introduce the spaces

$$\mathcal{H}^{1+\alpha}_{-}(a, b; E) = \{ u \in \mathcal{H}^{1+\alpha}(a, b; E), u(a) = 0 \}$$  

and

$$\mathcal{H}^{1+\alpha}_{0}(a, b; E) = \{ u \in \mathcal{H}^{1+\alpha}(a, b; E), u(a) = u(b) = 0 \}.$$  

Throughout this article we assume $0 < \nu < \frac{1}{2}$.

Lemma 2. The natural imbedding

$$\mathcal{H}^{1+\nu}_{-}(a, b; V') \cap L^\infty(a, b; H) \hookrightarrow \mathcal{H}^1_{-}(a, b; V')$$  

is compact.

PROOF: Let $(u_m)$ be a bounded sequence in

$$\mathcal{H}^{1+\nu}_{-}(a, b; V') \cap L^\infty(a, b; H).$$
We fix a smooth function $\theta$ vanishing on $(b + 1, \infty)$ and $= 1$ in a neighborhood of $b$ and define

$$v_m(t) = \begin{cases} u_m(t), & t \in [a, b] \\ \theta(t)u_m(b), & t \in [b, b + 1]. \end{cases}$$

The sequence $v_m$ is bounded in $C^1(a, b + 1; V') \cap L^\infty(a, b + 1; H)$. Let $\beta \in (0, 1), \nu' \in (0, \nu)$ satisfy $\beta(1 + \nu) > 1 + \nu'$. We notice that

$$\lambda_k^{-\beta}(1 + \sigma^2)^{1+\nu'} \leq \lambda_k^{-1}(1 + \sigma^2)^{1+\nu} + 1$$

and using the definition of the norm $[\cdot]_\nu$ and the expression (2.2) we see that

$$C(a, b + 1; V') \cap L^\infty(a, b + 1; H).$$

is continuously imbedded into $C(a, b + 1; H^{-\beta}(\Omega))$. The last space is compactly imbedded into $H^1(a, b + 1; V')$. (See the proof of Theorem 1.5.2. in [4], for example.)

To construct the operator $\phi$ in (1.7) we fix a convex function $F : \mathbb{R}^{N \times N} \to \mathbb{R}$ of the class $C^2$ satisfying

$$F(0) = 0, \quad \frac{\partial F}{\partial p_{ij}}(0) = 0, \quad i, j = 1, \ldots, N,$$

$$|\frac{\partial^2 F}{\partial p_{ij} \partial p_{kl}}| \leq M, \quad i, j, k, l = 1, \ldots, N,$$

$$\sum_{i, j, k, l = 1}^{N} \frac{\partial^2 F}{\partial p_{ij} \partial p_{kl}} \xi_{ij} \xi_{kl} \geq \mu |\xi|^2$$

for some positive $\mu$ and $M$. We define the operator $\varphi : V \to V'$ by the formula

$$\langle \varphi(u), v \rangle = \int_{\Omega} \frac{\partial F}{\partial p_{ij}}(\nabla u) \frac{\partial v_i}{\partial x_j},$$

$u, v \in V$. Clearly $\varphi$ is Lipschitz continuous and satisfies

$$\mu \| u - v \|^2 \leq \langle \varphi(u) - \varphi(v), u - v \rangle \leq M \| u - v \|^2,$$

$u, v \in V$. For fixed $T \in (0, \infty)$ we introduce the operator

$$\phi : L^2(0, T; V) \to L^2(0, T; V')$$

by

$$\langle \phi(u), v \rangle_T = \int_{0}^{T} \langle \varphi(u(t)), v(t) \rangle dt,$$

$u, v \in L^2(0, T; V)$, where $\langle \cdot, \cdot \rangle_T$ denotes the duality between $L^2(0, T; V)$ and $L^2(0, T; V')$. 

Lemma 3. The operator $\phi$ maps $H^{\nu}(0,T;V)$ into $H^{\nu}(0,T;V')$ and

$$\| \phi(u) \|_{H^{\nu}(0,T;V')} \leq M \| u \|_{H^{\nu}(0,T;V)}$$

for all $u \in H^{\nu}(0,T;V)$.

**Proof:** This follows easily from (2.5).

Let us fix $\alpha, \beta > 0$ and let

$$k(t) = \begin{cases} 0, & \text{for } t \leq 0 \\ \beta t^{-2\nu} e^{-\alpha t}, & \text{for } t > 0. \end{cases}$$

In what follows we could replace $k$ by any function vanishing on $(-\infty, 0)$ and satisfying together with its derivative the same growth conditions at 0 and $\infty$ as the special $k$ above.

We define the operator $K : L^2(0,T;V) \to L^2(0,T;V')$ by

$$Ku(t) = \int_0^t -k(t-s)Au(s)ds.$$ 

It is not difficult to see that

$$\langle Ku, u \rangle \leq \kappa \| u \|_{L^2(0,T;V)}^2,$$

where $\kappa = \int_0^\infty k$.

Lemma 4. The operator $K$ maps $H^{\nu}(0,T;V)$ into $H^{\nu}(0,T;V')$ and

$$\| Ku \|_{H^{\nu}(0,T;V')} \leq \kappa \| u \|_{H^{\nu}(0,T;V)}.$$

**Proof:** This is easy.

Lemma 5. Let $E$ be a Hilbert space and let $v : \mathbb{R} \to E$ satisfy

$$v(t) = 0 \quad \text{for } t \leq 0,$$

$$\lim_{t \to +\infty} v(t) = v_\infty \quad \text{(strong limit)},$$

$$v' \in L^1 \cap L^\infty(\mathbb{R}; E).$$

Then

$$\int_0^\infty (k \ast v(s), v'(s))_E ds = \frac{\kappa}{2} |v_\infty|_E^2 + \frac{1}{2} \int_0^\infty ds k'(s) \int_0^\infty dt |v(t) - v(t-s)|_E^2.$$
PROOF: It is not difficult to see that the following computation is legal
\[
\int_0^\infty (k * v(s), v'(s))_E \, ds - \frac{\kappa}{2} |v_\infty|_E^2 = \int_0^\infty ds \int_0^\infty dt \, k(t)(v(s-t) - v(s), v'(s))_E \\
= \int_0^\infty ds \int_0^\infty dt \, k(s-t)(v(t) - v(s), \frac{d}{ds}(v(s) - v(t)))_E \\
= -\frac{1}{2} \int_0^\infty ds \int_0^\infty dt \, k(s-t) \frac{d}{ds} |v(s) - v(t)|_E^2 \\
= -\frac{1}{2} \int_0^\infty dt \int_0^\infty ds \, k'(s-t)|v(s) - v(t)|_E^2 \\
= -\frac{1}{2} \int_0^\infty \int_0^\infty dt \, k'(t)|v(s) - v(s-t)|_E^2.
\]

The proof is finished.


Our next step consist in the construction of Galerkin approximations for the problem (1.7),(1.8). We assume that the forcing term \( f \) satisfies

\[(3.1) \quad f \in L^2(0, T; H) \cap H^r(0, T; V')\]

for some fixed \( T \in (0, +\infty) \).

Lemma 6. Suppose \( \mu > \kappa \). For each \( m \in \mathbb{N} \) there is a unique function \( u_m \in H^2_T(0, T; V_m) \) satisfying

\[(3.2) \quad \langle (u_m''(t) + \varphi(u_m(t)) + k \ast \Delta u_m(t)), w_j \rangle = \langle f(t), w_j \rangle, \quad j = 1, \ldots, m,\]

for a.e. \( t \in (0, T) \) and

\[u_m(0) = u'_m(0) = 0.\]

The functions \( u_m \) satisfy

\[(3.3) \quad \| u'_m \|_{L^\infty(0, T; H)}^2 + \| u_m \|_{L^\infty(0, T; V)}^2 \leq c,\]

\[(3.4) \quad \| u_m \|_{H^r(0, T; V)}^2 \leq c,\]

where \( c \) does not depend on \( m \).

PROOF: It is standard that (3.2) has a solution on a small interval \( (0, \delta) \). (See e.g. [7]). Let us derive a priori estimates. Let \( t \in (0, T) \) and suppose \( u_m \) is defined on \( (0, t) \). Replacing \( w_j \) by \( u'_m \) in (3.2) and integrating over \( (0, t) \) we obtain

\[
\left(\frac{1}{2} |u'_m|^2 + \int_\Omega F(\nabla u_m))|_{t=\bar{t}}^t \right) = \int_0^t ((k \ast u_m, u'_m)) + \int_0^t (f, u'_m).
\]
From Lemma 5 we see that

\[(3.5) \quad \int_0^t ((k \ast u_m, u'_m)) = \frac{\kappa}{2} \| u_m(t) \|^2 + \frac{1}{2} \int_0^\infty d\tau k'(\tau) \int_0^\infty ds \| u_m^{(t)}(s - \tau) - u_m^{(t)}(s) \|^2,\]

where

\[u_m^{(t)}(s) = \begin{cases} u_m(s), & \text{if } s < t \\ u_m(t), & \text{if } s \geq t. \end{cases}\]

Since the second term on the right-hand side is negative (or nonpositive) and \(u_m(0) = u'_m(0) = 0\), we see that

\[(3.6) \quad \frac{1}{2} |u'_m(t)|^2 + \int_{\Omega} F(\nabla u_m(t)) - \frac{\kappa}{2} \| u_m(t) \|^2 \leq \int_0^t (f, u'_m).\]

Now if \(\mu > \kappa\) then

\[\int_{\Omega} F(\nabla u_m(t)) - \frac{\kappa}{2} \| u_m(t) \|^2 \geq \frac{\mu - \kappa}{2} \| u_m(t) \|^2,\]

and by the standard use of the Gronwall lemma we infer

\[(3.7) \quad |u'_m(t)|^2 + \| u_m(t) \|^2 \leq \frac{1}{2} \int_0^\infty d\tau k'(\tau) \int_0^\infty ds \| u_m^{(t)}(s - \tau) - u_m^{(t)}(s) \|^2 \leq c,\]

where \(c\) is independent of \(m\) and \(t \leq T\).

To estimate the \(H^{\nu}\) norm let us define

\[\tilde{u}_m(s) = \begin{cases} u_m(s), & \text{if } s \leq t \\ 0, & \text{if } s > t. \end{cases}\]

By Lemma 5 the \(H^{\nu}(0,T; V)\) norm of \(u_m\) is estimated by

\[\int_{\mathbb{R}} ((k \ast \tilde{u}_m, \tilde{u}'_m)).\]

But this integral equals to

\[-((k \ast u_m(t), u_m(t))) + \int_0^t ((k \ast u_m, u'_m))\]

since, roughly speaking, \(\tilde{u}'_m\) gives the Dirac measure at \(t\). By (3.5) this amounts to

\[\frac{\kappa}{2} \| u_m(t) \|^2 + \frac{1}{2} \int_0^\infty d\tau k'(\tau) \int_0^\infty ds \| u_m^{(t)}(s - \tau) - u_m^{(t)}(s) \|^2 - ((k \ast u_m(t), u_m(t)))\]

and this expression is bounded by (3.7). Hence

\[\| u_m \|^2_{H^{\nu}(0,T; V)} \leq c,\]

where \(c\) does not depend on \(m\) and \(t \leq T\). The proof is finished. \(\blacksquare\)
Lemma 7. The sequence \( u_m'' \) is compact in \( L^2(0,T;V') \).

PROOF: We notice that (3.2) can be rewritten as

\[
  u_m'' + \tilde{P}_m \phi(u) + \mathcal{K} u_m = \tilde{P}_m f,
\]

where \( \tilde{P}_m : L^2(0,T;V') \rightarrow L^2(0,T;V_m) \) is defined by \( (\tilde{P}_m u)(t) = P_m(u(t)) \) (see Lemma 1). We can now use Lemmas 1-4 together with the estimates (3.3) and (3.4) to infer that the sequence \( u_m' \) is bounded in \( H^{l+\nu}(a,b;V') \cap L^\infty(a,b;H) \) and hence compact in \( H^l(a,b;V') \). This implies that \( u_m'' \) is compact in \( L^2(0,T;V') \).

The proof is finished. ■

Passing to a subsequence, if necessary, we can assume that

\[
  u_m \rightarrow u \quad \text{in} \quad L^2(0,T;V),
\]

\[
  u_m'' \rightarrow u'' \quad \text{in} \quad L^2(0,T;V').
\]

Theorem. Let \( \kappa < \mu \). Then \( u \) is a weak solution of (1.7).

PROOF: The only problem is to show that \( \phi(u_m') \rightarrow \phi(u) \) in \( L^2(0,T;V') \). Since clearly \( \mathcal{K} u_m \rightarrow \mathcal{K} u \) it is enough to show \( B u_m \rightarrow B u \), where \( B = \phi - \mathcal{K} \). By our assumptions, \( B \) is strongly monotone and Lipschitz continuous. We can suppose \( B u_m \rightarrow \chi \) in \( L^2(0,T;V') \). Clearly \( u'' + \chi = f \). For any \( v \in L^2(0,T;V) \) we have

\[
  \langle \chi - Bv, u - v \rangle_T = \langle -u'' + f - Bv, u - v \rangle_T = \lim_{m \rightarrow \infty} \langle u_m'' + f - Bv, u_m - v \rangle_T = \langle Bu_m - Bv, u_m - v \rangle_T \geq 0.
\]

From this we can infer \( \chi = B u \) by "Minty's trick". (See, for example, [8]). The proof is finished. ■

Remark. If \( T = \infty \) and \( f \in L^1(0,\infty;H) \) the procedure above yields a weak solution \( u \) on the interval \( (0,\infty) \) which belongs to the space \( \mathcal{H}^\nu(0,\infty;V) \). This follows easily from the estimate

\[
  \| u_m \|_{\mathcal{H}^\nu(0,\infty;V)} \leq \| f \|_{L^1(0,\infty;H)},
\]

which can be obtained in a similar way as (3.7).

References

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(Received March 27, 1990)