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# A non-special $\omega_2$-tree with special $\omega_1$-subtrees [1]

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*Abstract.* Answering a question of F. Tall it is shown that if ZF is consistent then so is ZFC + GCH + "there exists a non-special $\omega_2$-Aronszajn tree having only special $\omega_1$-subtrees".

*Keywords:* Tree, Aronszajn, special, consistency proof, forcing, non-reflecting

*Classification*: 03E35

## 1. Basic notions and terminology.

In this paper we follow the standard terminology of the set theory, cf [**2**]. A tree $\mathcal{T} = \langle T, \prec_{\mathcal{T}} \rangle$ is called $\kappa$-tree iff both the cardinality and the height of $\mathcal{T}$ are $\kappa$. We say that a $\kappa$-tree $\mathcal{T}$ is *$\kappa$-Aronszajn* iff $\mathcal{T}$ does not have $\kappa$-branches and the levels of $\mathcal{T}$ have cardinalities $< \kappa$. Given $x, y \in T$ we write "$x \not\parallel_{\mathcal{T}} y$" for "$x$ and $y$ are incomparable in $\mathcal{T}$". Take

$$\mathrm{V}(\mathcal{T}) = \{\langle x, y, z \rangle \in T^3 : x \prec_{\mathcal{T}} y, x \prec_{\mathcal{T}} z \text{ and } y \not\parallel_{\mathcal{T}} z\}.$$

A $\kappa$-tree $\mathcal{T} = \langle T, \prec_{\mathcal{T}} \rangle$ is *special* iff there is a function $f$ on $T$ with $|\mathrm{ran}(f)| < \kappa$ such that there is no $\langle x, y, z \rangle \in \mathrm{V}(\mathcal{T})$ with $f(x) = f(y) = f(z)$. The height of an $x$ element in $\mathcal{T}$ will be denoted by $h_{\mathcal{T}}(x)$ or by $h(x)$. Take $b(x) = b_{\mathcal{T}}(x) = \{y \in T : y \prec_{\mathcal{T}} x\}$.

The set of all finite sequences of elements of a given set $I$ will be denoted by $I^*$. For $x, y \in I^*$ let $x^\frown y$ be the concatenation of them. For $n \in \omega$ and $c \in I$ take $c^n = \langle c \rangle^\frown c^{n-1}$. By an abuse of notation we write $a^\frown x$ instead of $\langle a \rangle^\frown x$ whenever $a \in I$ and $x \in I^*$.

Denote by $<_{\mathrm{On}}$ the usual ordering of ordinals. Given $X, Y \subset On$ we write "$X <_{\mathrm{On}} Y$" to mean that $\max_{<_{\mathrm{On}}} X < \min_{<_{\mathrm{On}}} Y$.

## 2. The result.

In [3] F. Tall investigated some downwards reflection principles. Beside other results he proved that Con(ZFC + $\exists$ huge cardinal) $\to$ Con(ZFC + CH + "every non-special $\omega_2$-tree contains a non-special $\omega_1$-subtree") and raised the following problem: ***Is ZFC + GCH consistent with the existence of a non-special $\omega_2$-tree having only special $\omega_1$-subtrees?*** In this paper we give an affirmative answer proving the following theorem.

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**Theorem 2.1.** *Assume CH. Then there is a $\sigma$-complete poset $\mathcal{P}$ with $|\mathcal{P}| = \omega_2$ such that*

$$V^{\mathcal{P}} \models \text{“There is a non-special } \omega_2\text{-Aronszajn tree having only special } \omega_1\text{-subtrees”.}$$

**Remark.** S. Todorčevič proved that if $\omega_2$ is not weakly compact in L, then there exists a tree as in Theorem 2.1. His result and Theorem 2.1 were proved approximately at the same time.

PROOF : First we define our poset $\mathcal{P} = \langle P, \leq \rangle$. The underlying set of $\mathcal{P}$ consists of triples $\langle T, \prec, \langle f_x : x \in 2^* \rangle \rangle$ satisfying (A)–(E) below:

(A) $T \in [\omega_2]^{\leq\omega}$, $\prec \subset <_{\mathrm{On}} \cap (T \times T)$, $\langle T, \prec \rangle$ is a tree.

(B) $f_x$ is a function, $f_x : T \times T \to [\omega]^{\omega}$, for each $x \in 2^*$.

(C) $f_x(\alpha,\delta) \cap f_y(\alpha,\delta) = \emptyset$ for each $x \neq y \in 2^*$ and $\langle \alpha, \delta \rangle \in T \times T$.

(D) If $\langle \alpha, \beta, \gamma \rangle \in V(\langle T, \prec \rangle)$, $\alpha <_{\mathrm{On}} \delta \in T$, $k, m, n \in \omega$, then

$$f_{0^k}(\alpha,\delta) \cap f_{0^m}(\beta,\delta) \cap f_{0^n}(\gamma,\delta) = \emptyset.$$

(E) If $\alpha, \beta \in T$, $\alpha \prec \beta$, $\alpha <_{\mathrm{On}} \delta \in T$, $n \in \omega$, $x \in 2^* \setminus 1^*$, then

$$f_{0^n}(\alpha,\delta) \cap f_x(\beta,\delta) = \emptyset.$$

We write $p = \langle T_p, \prec_p, \langle f^p_x : x \in 2^* \rangle \rangle$ for $p \in \mathcal{P}$.
The ordering on $P$ is defined as expected:

$$\begin{array}{lll} p \leq q \quad \text{iff} & T_q \subseteq T_p, & \\ & \prec_q = \prec_p \cap\, T_q \times T_q & \text{and} \\ & f^q_x \subset f^p_x & \text{for each } x \in 2^*. \end{array}$$

It is easily seen that $\mathcal{P}$ is a $\sigma$-complete poset with cardinality $\omega_2$.
The following lemma is straightforward.

**Lemma 2.2.** *For each $\alpha \in \omega_2$ the set $D_\alpha = \{p \in \mathcal{P} : \alpha \in T_p\}$ is dense in $\mathcal{P}$.*

**Definition 2.3.** Assume that $\dot{f}$ is a $\mathcal{P}$-name. A condition $p \in \mathcal{P}$ is called *strong for $\dot{f}$* iff $p \Vdash$ “$\dot{f} : \hat{\omega}_2 \to \hat{\omega}_1$ is a function” and $\forall \alpha \in T_p\ \exists \xi < \omega_1\ p \Vdash$ “$\dot{f}(\hat{\alpha}) = \hat{\xi}$”.

**Lemma 2.4.** *Assume that $p \in \mathcal{P}$, $\dot{f}$ is a $\mathcal{P}$-name and $p \Vdash$ “$\dot{f} : \hat{\omega}_2 \to \hat{\omega}_1$ is a function”. Then the conditions which are strong for $\dot{f}$ are dense in $\mathcal{P}$ below $p$.*

The statement of this lemma is clear by the $\sigma$-completeness of $\mathcal{P}$.

**Definition 2.5.** Given $p, q \in \mathcal{P}$ we write $p \propto q$ iff $T_p \cap T_q <_{\mathrm{On}} T_p \setminus T_q <_{\mathrm{On}} T_q \setminus T_p$, $\mathrm{type}_{<_{\mathrm{On}}} T_p = \mathrm{type}_{<_{\mathrm{On}}} T_q$ and denoting by $\pi$ the unique $<_{\mathrm{On}}$-preserving bijection between $T_p$ and $T_q$ we have

(a) $\alpha \prec_p \beta$ iff $\pi(\alpha) \prec_q \pi(\beta)$ for each $\alpha, \beta \in T_p$,

(b) $f^p_x(\alpha,\beta) = f^q_x(\pi(\alpha),\pi(\beta))$ for each $\alpha, \beta \in T_p$ and $x \in 2^*$,

that is, $\pi$ is an isomorphism between $p$ and $q$.

If $p \propto q$ and $\alpha \in T_p \cup T_q$ put

$$\tilde{\alpha} = \begin{cases} \pi^{-1}(\alpha) & \text{if } \alpha \in T_q, \\ \alpha & \text{otherwise.} \end{cases}$$

**Lemma 2.6.** *If $p \propto q$ then there is an $r \in \mathcal{P}$ such that $T_r = T_p \cup T_q$, $\prec_r = \prec_p \cup \prec_q$ and $r \leq p$, $r \leq q$.*

PROOF : Take $T = T_p \cup T_q$, $\prec = \prec_p \cup \prec_q$ and $f_x^- = f_x^p \cup f_x^q$ for $x \in 2^*$. Choose pairwise different natural numbers $n(x, \alpha, \delta, k) \in f^p_{1\frown x}(\tilde{\alpha}, \tilde{\delta})$ where $\langle x, \alpha, \delta, k\rangle$ ranges over $2^* \times T \times T \times \omega$. It can be easily done because the set $2^* \times T \times T \times \omega$ is countable.

A pair $\langle \alpha, \delta\rangle \in T \times T$ is called *old* iff it is an element of $\text{dom}(f_x^-)$, and *new* otherwise. Now for each $x \in 2^*$ define the function $f_x : T \times T \to [\omega]^\omega$ by setting

$$f_x(\alpha, \delta) = \begin{cases} f_x^-(\alpha, \delta) & \text{if } \langle \alpha, \delta\rangle \text{ is old,} \\ \{n(x, \alpha, \delta, k) : k \in \omega\} & \text{if } \langle \alpha, \delta\rangle \text{ is new.} \end{cases}$$

Taking $r = \langle T, \prec, \langle f_x : x \in 2^*\rangle\rangle$ it is sufficient to show that $r \in \mathcal{P}$. Obviously (A)–(C) hold for $r$. To check (D) fix $\langle \alpha, \beta, \gamma\rangle \in V(\langle T, \prec\rangle)$, $\alpha <_{\text{On}} \delta \in T$, $k, m, n \in \omega$. If not exactly one of the pairs $\langle \alpha, \delta\rangle$, $\langle \beta, \delta\rangle$ and $\langle \gamma, \delta\rangle$ is new then (D) holds by the construction of $r$. Since it is impossible that $\langle \alpha, \delta\rangle$ is new and both $\langle \beta, \delta\rangle$ and $\langle \gamma, \delta\rangle$ are old, we can assume, without loss of generality, that $\langle \gamma, \delta\rangle$ is the new pair. So

$$f_{0^k}(\alpha, \delta) \cap f_{0^m}(\beta, \delta) \cap f_{0^n}(\gamma, \delta) \subset f^p_{0^k}(\tilde{\alpha}, \tilde{\delta}) \cap F^p_{1\frown 0^n}(\tilde{\gamma}, \tilde{\delta}) = \emptyset,$$

because $p$ satisfies (E). So $r$ satisfies (D).

Finally we check (E). Fix $\alpha, \beta \in T$, $\alpha \prec \beta$, $\alpha <_{\text{On}} \delta \in T$, $n \in \omega$ and $x \in 2^* \setminus 1^*$. We can assume that exactly one of the pairs $\langle \alpha, \delta\rangle$ and $\langle \beta, \delta\rangle$ is new or (E) holds. Thus $\langle \beta, \delta\rangle$ must be the new pair. Then

$$f_{0^n}(\alpha, \delta) \cap f_x(\beta, \delta) \subset f^p_{0^n}(\tilde{\alpha}, \tilde{\delta}) \cap f^p_{1\frown x}(\tilde{\beta}, \tilde{\delta}) = \emptyset,$$

for $1\frown x \in 2^* \setminus 1^*$and $p$ satisfies (E). So $r \in P$ is proved. ∎

**Lemma 2.7.** *Assume that $p \propto q \propto r$ with $T_p \cap T_q = T_p \cap T_r = T_q \cap T_r$. Let $\pi$ and $\rho$ be the unique order preserving bijections between $T_p$ and $T_q$, and between $T_p$ and $T_r$, respectively. Assume that $\nu \in T_p \setminus T_q$ with $b_{T_p}(\nu) \cap (T_p \setminus T_q) = \emptyset$. Let $\mu = \pi(\nu)$ and $\theta = \rho(\nu)$. Then there is a condition $t \in \mathcal{P}$ such that $t \leq p, q, r$ and $\langle \nu, \mu, \theta\rangle \in \text{V}(\langle T_t, \prec_t\rangle)$.*

PROOF : Let $A = T_p \cap T_q$. Write $T = T_p \cup T_q \cup T_r$ and let $\prec$ be the partial ordering on $T$ generated by the set $\prec_p \cup \prec_q \cup \prec_r \cup \{\langle \nu, \mu\rangle, \langle \nu, \theta\rangle\}$. It is easy to see that $\mathcal{T} = \langle T, \prec_{\mathcal{T}}\rangle$ is a tree and $\langle \nu, \mu, \theta\rangle \in \text{V}(\langle T, \prec\rangle)$. Given $\alpha \in T$ take

$$\tilde{\alpha} = \begin{cases} \pi^{-1}(\alpha) & \text{if } \alpha \in T_q, \\ \rho^{-1}(\alpha) & \text{if } \alpha \in T_r, \\ \alpha & \text{otherwise.} \end{cases}$$

Pick distinct natural numbers $n(x,\alpha,\delta,k)$ where $\langle x,\alpha,\delta,k\rangle$ ranges over $2^*\times T\times T\times\omega$ in such a way that $n(x,\alpha,\delta,k)\in f^p_{0\frown x}(\tilde\alpha,\tilde\delta)$ provided $\alpha=\nu$ and $n(x,\alpha,\delta,k)\in f^p_{1\frown x}(\tilde\alpha,\tilde\delta)$ otherwise. It can be easily done because the set $2^*\times T\times T\times\omega$ is countable. Take now $f^-_x=f^p_x\cup f^q_x\cup f^r_x$ for $x\in 2^*$. A pair $\langle\alpha,\delta\rangle\in T\times T$ is said *old* iff it is an element of $\mathrm{dom}(f^-_\emptyset)$, and *new* otherwise. For each $x\in 2^*$ define the function $f_x:T\times T\to[\omega]^\omega$ by setting

$$f_x(\alpha,\delta)=\begin{cases} f^-_x(\alpha,\delta) & \text{if } \langle\alpha,\delta\rangle \text{ is old,}\\ \{n(x,\alpha,\nu,k):k\in\omega\} & \text{if } \langle\alpha,\delta\rangle \text{ is new.}\end{cases}$$

Taking $t=\langle T,\prec,\langle f_x:x\in 2^*\rangle\rangle$ it is enough to show that $t\in\mathcal{P}$. Obviously (A)–(C) hold for $t$. Next we check (D). Suppose that $\langle\alpha,\beta,\gamma\rangle\in V(\langle T,\prec\rangle)$, $\alpha<_{\mathrm{On}}\delta\in T$, $k,m,n\in\omega$. We can assume that exactly one of the pairs $\langle\alpha,\delta\rangle$, $\langle\beta,\delta\rangle$ and $\langle\gamma,\delta\rangle$ is new or (D) holds by the construction of $t$. We must distinguish two cases.

**Case 1.** $\langle\alpha,\delta\rangle$ is new .

Since $\langle\alpha,\delta\rangle$ is the only new pair and $\alpha\prec\beta$ it follows that $\alpha=\nu$ and either $\beta,\gamma,\delta\in T_q\setminus A$ or $\beta,\gamma,\delta\in T_r\setminus A$. So $\beta\not\parallel_T\gamma$ implies that $\tilde\beta\not\parallel_{T_p}\tilde\gamma$ and $\langle\nu,\tilde\beta,\tilde\gamma\rangle\in V(\langle T_p,\prec_p\rangle)$. Thus

$$f_{0^k}(\alpha,\delta)\cap f_{0^m}(\beta,\delta)\cap f_{0^n}(\gamma,\delta)\subset f^p_{0^{k+1}}(\nu,\tilde\delta)\cap f^p_{0^m}(\tilde\beta,\tilde\delta)\cap f^p_{0^n}(\tilde\gamma,\tilde\delta)=\emptyset.$$

**Case 2.** $\langle\alpha,\delta\rangle$ is old.

Without loss of generality we can assume that $\langle\beta,\delta\rangle$ is the new pair. If $\beta=\nu$ then $\delta\in(T_q\setminus A)\cup(T_r\setminus A)$, $\alpha\in A$ and $\tilde\gamma\not\parallel_{T_p}\nu$ for $\gamma\not\parallel_T\nu$. Thus $\langle\tilde\alpha,\tilde\beta,\tilde\gamma\rangle=\langle\alpha,\nu,\tilde\gamma\rangle\in V(T_p,\prec_p)$ and so

$$f_{0^k}(\alpha,\delta)\cap f_{0^m}(\beta,\delta)\cap f_{0^n}(\gamma,\delta)\subset f^p_{0^k}(\tilde\alpha,\tilde\delta)\cap f^p_{0^{m+1}}(\tilde\beta,\tilde\delta)\cap f^p_{0^n}(\tilde\gamma,\tilde\delta)=\emptyset.$$

If $\beta\neq\nu$ then

$$f_{0^k}(\alpha,\delta)\cap f_{0^m}(\beta,\delta)\cap f_{0^n}(\gamma,\delta)\subset f^p_{0^k}(\tilde\alpha,\tilde\delta)\cap f^p_{1\frown 0^m}(\tilde\beta,\tilde\delta)=\emptyset$$

because (E) holds for $p$. So $t$ satisfies (D).

Finally we check (E). Suppose that $\alpha,\beta\in T$, $\alpha\prec\beta$, $\alpha<_{\mathrm{On}}\delta\in T$, $n\in\omega$ and $x\in 2^*\setminus 1^*$. We can assume that exactly one of the pairs $\langle\alpha,\delta\rangle$ and $\langle\beta,\delta\rangle$ is new. If $\langle\alpha,\delta\rangle$ is the new pair then we have $\alpha=\nu$ because $\langle\beta,\delta\rangle$ is old and $\alpha\prec\beta$ . So

$$f_{0^n}(\alpha,\delta)\cap f_x(\beta,\delta)\subset f^p_{0^{n+1}}(\tilde\alpha,\tilde\delta)\cap f^p_x(\tilde\beta,\tilde\delta)=\emptyset.$$

If $\langle\alpha,\delta\rangle$ is old, then

$$f_{0^n}(\alpha,\delta)\cap f_x(\beta,\delta)\subset f^p_{0^n}(\tilde\alpha,\tilde\delta)\cap f^p_{1\frown x}(\tilde\beta,\tilde\delta)=\emptyset,$$

because (E) holds for $p$. Thus (E) is also satisfied by t. This shows that $t\in\mathcal{P}$, which completes the proof of the Lemma 2.7. ∎

**Proof of Theorem 2.1.** Since CH holds, every subset of $P$ with cardinality $\omega_2$ contains two elements, $p$ and $q$, with $p \propto q$. So, by Lemma 2.7, $p$ and $q$ are compatible, that is, $\mathcal{P}$ satisfies the $\omega_2$-chain-condition.

Let $\mathcal{G}$ be any $\mathcal{P}$-generic filter over $V$. Take $T^* = \bigcup\{T_p : p \in \mathcal{G}\}$ , $\prec^* = \bigcup\{\prec_p : p \in \mathcal{G}\}$, $\mathcal{T}^* = \langle T^*, \prec^* \rangle$ and $F^* = \bigcup\{f_\emptyset^p : p \in \mathcal{G}\}$. By Lemma 2.2 it follows that $T^* = \omega_2$. For each $\delta \in \omega_2$ choose a function $F_\delta : \delta \to \omega$ with $F_\delta(\alpha) \in F^*(\alpha, \delta)$. Now $F_\delta$ shows that the tree $\langle \delta, \prec^* \restriction \delta \rangle$ is special. Indeed, given $\alpha$, $\beta$, $\gamma \in \delta$ choose $p \in \mathcal{G}$ with $\alpha$, $\beta$, $\gamma \in T_p$ and apply (D) for $p$ taking $k = m = n = 0$.

Next we show that $\mathcal{T}^*$ is not special. Let us remark that this implies height$(\mathcal{T}^*) \geq \omega_2$. Since height$(\mathcal{T}^*) \leq \omega_2$ by $\prec^* \subset <_{\mathrm{On}}$, this proves height$(\mathcal{T}^*) = \omega_2$ as well.

Assume on the contrary that

$$p \Vdash \text{“}\dot{f} : \hat{\omega}_2 \to \hat{\omega}_1 \text{ specializes } \mathcal{T}^*\text{”}.$$

For each $\alpha < \omega_2$ choose a condition $p_\alpha \leq p$ which is strong for $\dot{f}$ with $\alpha \in T_{p_\alpha}$. Since CH holds, we can find a set $Y \in [\omega_2]^{\omega_2}$ such that (1) $\{T_{p_\xi} : \xi \in Y\}$ forms a $\Delta$-system with kernel $A$, and (2) $p_\xi \propto p_\eta$ whenever $\xi, \eta \in Y$ with $\xi <_{\mathrm{On}} \eta$. Since $\xi \in T_{p_\xi}$, it follows that $T_{p_\xi} \setminus A \neq \emptyset$ for each $\xi \in Y$.

Take $c_\xi = \min_{<_{\mathrm{On}}}(T_{p_\xi} \setminus A)$ for $\xi \in Y$. Choose $\zeta < \xi < \eta$ from $Y$ and $\sigma \in \omega_1$ such that $p_\theta \Vdash \text{“}\dot{f}(\hat{c}_\theta) = \hat{\sigma}\text{”}$ for each $\theta \in \{\zeta, \xi, \eta\}$. By Lemma 2.7 there is a $t \in \mathcal{P}$ with $t \leq p_\zeta, p_\xi, p_\eta$ and $\langle c_\zeta, c_\xi, c_\eta \rangle \in \mathrm{V}(\langle T_t, \prec_t \rangle)$. So

$$t \Vdash \text{“}\dot{f}(\hat{c}_\zeta) = \dot{f}(\hat{c}_\xi) = \dot{f}(\hat{c}_\eta) = \hat{\sigma},\ \langle \hat{c}_\zeta, \hat{c}_\xi, \hat{c}_\eta \rangle \in \mathrm{V}(\mathcal{T}^*) \text{ and } \dot{f} \text{ specializes } \mathcal{T}^*\text{”}.$$

Contradiction, $\mathcal{T}^*$ is not special.

To prove that $\mathcal{T}^*$ is Aronszajn assume on the contrary that $p \Vdash$ “*$\dot{b}$ is an $\omega_2$-branch in $\mathcal{T}^*$*”. Denote by $\mathcal{T}^*_\alpha$ the $\alpha^{\text{th}}$-level of $\mathcal{T}^*$. For each $\alpha < \omega_2$ choose a condition $p_\alpha \leq p$ and a $\gamma_\alpha \in \omega_2$ with $\gamma_\alpha \in T_{p_\alpha}$ and $p_\alpha \Vdash \text{“}\dot{b} \cap \mathcal{T}^*_\alpha = \{\hat{\gamma}_\alpha\}\text{”}$. By standard $\Delta$-system arguments we can find $\alpha <_{\mathrm{On}} \beta <_{\mathrm{On}} \omega_2$ such that $p_\alpha \propto p_\beta$ and $\pi(\gamma_\alpha) = \gamma_\beta$, where $\pi$ is the unique $<_{\mathrm{On}}$-preserving bijection between $T_{p_\alpha}$ and $T_{p_\beta}$. By Lemma 2.6 $p_\alpha$ and $p_\beta$ have a common extension $r$ in $\mathcal{P}$ with $\prec_r = \prec_{p_\alpha} \cup \prec_{p_\beta}$. Then $r \Vdash \text{“}\dot{b} \cap \mathcal{T}^*_\alpha = \hat{\gamma}_\alpha$ *and* $\dot{b} \cap \mathcal{T}^*_\beta = \hat{\gamma}_\beta\text{”}$, and so $\gamma_\alpha \neq \gamma_\beta$. Therefore $\gamma_\alpha \in T_{p_\alpha} \setminus T_{p_\beta}$ and $\gamma_\beta \in T_{p_\beta} \setminus T_{p_\alpha}$. Thus $r \Vdash \text{“}\hat{\gamma}_\alpha \not\parallel_{\mathcal{T}^*} \hat{\gamma}_\beta\text{”}$, which is a contradiction because the elements of any branch are pairwise comparable.

Finally we prove that the levels of $\mathcal{T}^*$ have cardinalities $\omega_1$. By way of contradiction assume that $\alpha < \omega_2$ and $p \Vdash \text{“}|\mathcal{T}^*_\alpha| = \omega_2\text{”}$. Fix a $\mathcal{P}$-name $\dot{h}$ such that $p \Vdash$ “$\dot{h}(\nu)$ *is the height of $\nu$ in $\mathcal{T}^*$ for $\nu \in \omega_2$*”. Choose a set $Y \in [\omega_2]^{\omega_2}$ and a condition $p_\xi \leq p$ for each $\xi \in Y$ such that $p_\xi \Vdash \text{“}\xi \in \mathcal{T}^*_\alpha\text{”}$ and $p_\xi$ is strong for $\dot{h}$. By standard arguments we can assume that the set $\{T_{p_\xi} : \xi \in Y\}$ forms a $\Delta$-system with kernel $A$ and that $p_\xi \propto p_\eta$ for each $\xi < \eta \in Y$. Take $c_\xi = \min_{\prec_{p_\xi}}((b_{T_{p_\xi}}(\xi) \cup \{\xi\}) \setminus A)$ for $\xi \in Y$ and define the function $g : Y \to \alpha$ by $p_\xi \Vdash \text{“}\dot{h}(\hat{c}_\xi) = \widehat{g(\xi)}\text{”}$. Pick $\zeta < \xi < \eta \in Y$ with $g(\zeta) = g(\xi) = g(\eta)$. By Lemma 2.7 we have a condition $t$ such that $t \leq p_\zeta, p_\xi, p_\eta$ and $c_\zeta \prec_t c_\xi$. But it means that $t \Vdash$ “height$(\hat{c}_\zeta) =$ height$(\hat{c}_\xi)$ *and* $\hat{c}_\zeta \prec^* \hat{c}_\xi$”, which is a contradiction. Therefore the levels of $\mathcal{T}^*$ have sizes $< \omega_2$, which completes the proof of Theorem 2.1. ∎

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