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Abstract. First, we prove in ZFC that no (von Neumann) regular ring derived from a group ring is an Ext-ring. Then we prove that the assertion "every regular Ext-ring is completely reducible" is consistent with ZFC.

Keywords: Regular ring, group ring, Whitehead property, Ext-ring

Classification: 16A30, 16A62

In [**5**], a project for studying associative rings by means of extension properties of modules was developed. The basic notion there was that of the Whitehead property of modules. A module N is said to have the Whitehead property provided either N is injective or, for each module M, $\mathrm{Ext}(M, N) = 0$ implies M is projective. In [**5**], rings such that each (left) module has the Whitehead property were called (left) Ext-rings. Their classification is a natural starting point of the above mentioned project.

The basic fact about Ext-rings is that they fall into two classes: the artinian, and the (von Neumann) regular ones (see [**5**, Theorem 3.4]). On the one hand, the theory of artinian Ext-rings is fairly developed: there is a complete description of the nonsingular and of the "small" singular ones (see [**5**, Theorem 8.1 and Theorems 4.1–4.6], respectively). On the other hand, examples of non-completely reducible regular Ext-rings are missing. In fact, [**5**, Theorem 6.3] shows that there are no examples of cardinality less that $2^{\aleph_0}$.

In the first section, we prove that the most promising candidates – derived from appropriate group rings by adding necessary nets of idempotents – actually fail to serve as the right examples. Roughly speaking, the main obstacle here is that the left "fundamental" ideal is non-trivial and it does not "support" the canonical simple module.

In the second section, we prove that the search for the examples of cardinality $\geq 2^{\aleph_0}$ in ZFC is in vain. Namely, the assertion "every regular left or right Ext-ring is completely reducible" is consistent with ZFC. The proof of this fact is based on a uniformization principle (Lemma 2.2) due to S. Shelah, which was kindly communicated to the author by prof. A. Mekler.

For the basic notation and facts used in the sequel, the reader is referred to [**1**] and [**2**].

1. Simple von Neumann regular rings derived from group rings.

Definition 1.1. Let K be a field, J a right linear space of dimension $2^{\aleph_0}$ over K and $Q = \mathrm{End}_K(J)$. Let $G = (G, \odot)$ be a (non-commutative) group such that the following conditions are satisfied: $\mathrm{card}(G) = 2^{\aleph_0}$; there exists a sequence of finite (non-commutative) groups $(G_n \mid n < \aleph_0)$ such that G is a dense subgroup of the group $\prod_{n<\aleph_0} G_n$ endowed with the product topology induced by the discrete topologies on G_n, $n < \aleph_0$. For each $n < \aleph_0$, put $p_n = \mathrm{card}(G_n)$ (> 1). Clearly, we can w.l.o.g. assume that the support of the group G_n is $p_n(= \{0, \dots, p_{n-1}\})$ and 0 is the null element of the additive group $G_n = (p_n, \odot_n)$.

For $n < \aleph_0$ denote by π_n the projection of G onto p_n. Let $\{b_h \mid h \in G\}$ be a basis of the right K-module J. For $g \in G$ define $a_g \in Q$ by $a_g b_h = b_{g \odot h}\ \forall h \in G$. Put $P_0 = \{0\}$ and, for each $0 < n < \aleph_0, P_n = \{(x_0, \dots, x_{n-1}) \mid x_i < p_i\ \forall i < n\}$.

Put $e_0 = 1 \in Q$ and, for $0 < n < \aleph_0$ and $x = (x_0, \dots, x_{n-1}) \in P_n$, define $e_x \in Q$ by $e_x b_h = b_h$ provided $h \in G$ and $h\pi_i = x_i$ for all $i < n$, and by $e_x b_h = 0$ otherwise. For $n < \aleph_0$ let $E_n = \{e_x \mid x \in P_n\}$.

Denote by R the subring of Q generated by $\{a_g \mid g \in G\} \cup \bigcup_{n<\aleph_0} E_n \cup K$. Clearly,

$$R = \{q \in Q \mid \exists n < \aleph_0 \forall x, y \in P_n \exists g_{yx} \in G \exists k_{yx} \in K : q = \sum_{x,y \in E_n} k_{yx} e_y a_{g_{yx}} e_x\}.$$

In the following lemma, we list the elementary facts about R we shall need in the sequel. Further details and properties can be found e.g. in [**4**, § 3].

Lemma 1.2. (i) *The ring R is simple non-completely reducible and directly finite, J is a simple faithful module, $K = \mathrm{End}(J)$ and $\dim(J) = 2^{\aleph_0}$.*

(ii) *Let $n < \aleph_0$, $g \in G$ and $e \in E_n$. Then $a_g e a_{(-g)} \in E_n$ and the mapping $\psi : E_n \to E_n$ defined by $e\psi = a_g e a_{(-g)}$ is bijective.*

(iii) *Let $0 < n < \aleph_0$, $g \in G$ and $e, f \in E_n$. Then $e = e_x$ and $f = e_y$ for some $x = (x_0, \dots, x_{n-1}) \in P_n$ and $y = (y_0, \dots, y_{n-1}) \in P_n$. Moreover, $f a_g e \neq 0$ iff $g\pi_i = y_i \odot_i (-x_i)$ for all $i < n$. If $f a_g e \neq 0$, then $f a_g = f a_g e = a_g e$.*

PROOF : Easy. ∎

Lemma 1.3. *The ring R is isomorphic to its opposite ring R^{op}.*

PROOF : Define $\xi : R \to R$ by $(\sum_{x,y \in P_n} k_{yx} e_y a_{g_{yx}} e_x)\xi = \sum_{x,y \in P_n} k_{yx} e_x a_{(-g_{yx})} e_y$. Then $\xi^2 = \xi$ and ξ is easily seen to be a ring anti-automorphism of R. ∎

Lemma 1.4. *Let H be any periodic subgroup of G and $I_H = \sum_{h \in H} R(1 - a_h)$. Then $I_H \neq R$. If H is infinite, then* $\mathrm{Hom}(R/I_H, J) = 0$.

PROOF : Assume there are $m < \aleph_0$ and $0 \neq h_i \in H, i < m$, such that $\sum_{i<m} R(1 - a_{h_i}) = R$. For each $i < m$, denote by ϱ_i the order of h_i. Take $n > 1$ such that for all $i, j < m$ there is some $k < n$ with $h_i\pi_k \neq h_j\pi_k$, and $q_n = \prod_{j<n} p_j \geq \varrho_i$ for all $i < m$. By 1.2 (ii), we can renumber the elements of E_n so that $E_n = \{f_i \mid i < q_n\}$ and $f_1 a_{h_0} = a_{h_0} f_0, \dots, f_{\varrho_0 - 1} a_{h_0} = a_{h_0} f_{\varrho_0 - 2}, f_0 a_{h_0} = a_{h_0} f_{\varrho_0 - 1}$. Then $f_1(1 - a_{h_0}) = -a_{h_0} f_0 (1 - a_{h_0}) \cdots - a_{(\varrho_0 - 1)h_0} f_2 (1 - a_{h_0})$. Moreover, since for each $i < m$ there is some $0 \neq x_i < q_n$ such that $a_{h_i} f_0 = f_{x_i} a_{h_i}$, we see that for each $i < m$, $f_{x_i}(1 - a_{h_i}) \in \sum_{x_i \neq k < q_n} R f_k (1 - a_{h_i})$. In particular, there are $s_i \in R, i < m$, such that $\sum_{i<m} s_i (\sum_{x_i \neq k < q_n} f_k (1 - a_{h_i})) = 1$, whence $f_0(\sum_{i<m} s_i(\sum_{x_i \neq k < q_n} f_k(1-$

$a_{h_i})f_0)) = f_0$ and for each $0 \neq j < q_n, f_0(\sum_{i<m} s_i(\sum_{x_i \neq k < q_n} f_k(1 - a_{h_i})f_j)) = 0$. By 1.2 (ii), we get $\sum_{i<m} f_0 s_i f_0$ and $\sum_{i<m} \sum_{x_i \neq j < q_n} f_0 s_i f_j(1-a_{h_i}) - \sum_{i<m} f_0 s_i f_0 = 0$. Hence $f_0 = \sum_{i<m} \sum_{x_i \neq j < q_n} f_0 s_i f_j(1 - a_{h_i})$. Since $s_i \in R$ for all $i < m$, there exist $p < \aleph_0$ and $e \in E_p$ such that $n \leq p, ef_0 = f_0 e = e$, and for each $i < m$ there are $k_i \in K$ and $g_i \in G$ such that $es_i = ek_i a_{g_i}$. Thus $e = \sum_{i<m} \sum_{x_i \neq j < q_n} k_i e a_{g_i} f_j(1 - a_{h_i})$. By 1.2 (iii), we infer that $e = \sum_{i<m} k_i e a_{g_i}(1 - a_{h_i})$. By 1.1, this implies that the identity $\sum_{i<m} k_i a_{g_i}(1 - a_{h_i}) = 1$ holds true in the group ring KG. But the left-hand side of the identity consists of an element of the fundamental ideal of KG, a contradiction.

If H is infinite, then $\{j \in J \mid (1 - a_h)j = 0 \, \forall h \in H\} = 0$ whence $\mathrm{Hom}(R/I_H, J) = 0$ ∎

Theorem 1.5. *The ring R is neither a left nor a right* Ext*-ring.*

PROOF : Assume R is a left Ext-ring. By 1.2 (i), R is simple and non-completely reducible whence, by [**5**, Theorem 3.4], R is regular.

First we prove that G is a periodic group. Assume g is a torsion-free element of G. Then $0 = \mathrm{Ker}(1 - a_g) \subsetneq J$. Take $r \in R$ such that $(1 - a_g)r(1 - a_g) = 1 - a_g$. Then $r(1 - a_g) = 1$ and $(1 - a_g)r = 1$, since R is directly finite. On the other hand, since g is torsion-free, we have $\mathrm{Im}(1 - a_g) \cap \{b_h \mid h \in G\} = 0$, a contradiction.

By 1.4, there is a maximal left ideal L containing I_G. Then $M = R/L$ is a non-projective simple module. Let N be a non-injective module. Denote by $I(N)$ the injective hull of N. Since the sequence

$$\mathrm{Hom}(M, I(N)/N) \to \mathrm{Ext}(M, N) \to \mathrm{Ext}(M, I(N)) = 0$$

is exact and $\mathrm{Ext}(M, N) \neq 0$, we conclude that the module $I(N)/N$ has a socle sequence with homogenous factors isomorphic to direct powers of M. Hence, all non-projective simple modules are isomorphic to M. In particular, $M \simeq J$, in contradiction with 1.4.

Thus R is not a left Ext-ring and 1.3 completes the proof. ∎

2. Consistency of the complete reducibility for von Neumann regular Ext-rings.

Definition 2.1. Let κ be an infinite cardinal such that $\mathrm{cf}(\kappa) = \aleph_0$. Let E be a stationary subset of κ^+ such that $E \subseteq \{\alpha < \kappa^+ \mid \mathrm{cf}(\alpha) = \aleph_0\}$. A sequence $(n_\nu \mid \nu \in E)$ is said to be a ladder system provided for each $\nu \in E \, (n_\nu(i) \mid i < \aleph_0)$ is a strictly increasing sequence of ordinals less than ν such that $\sup_{i<\aleph_0} n_\nu(i) = \nu$.

Lemma 2.2. *The following assertion is consistent with ZFC:*
UP*: "For each cardinal κ with $\mathrm{cf}(\kappa) = \aleph_0$ there are a stationary subset E of κ^+ satisfying $E \subseteq \{\alpha < \kappa^+ \mid \mathrm{cf}(\alpha) = \aleph_0\}$ and a ladder system $(n_\nu \mid \nu \in E)$ such that for every cardinal $\lambda < \kappa$ and every sequence $(h_\nu \mid \nu \in E)$ of mappings from $\aleph_0$ to λ there is a mapping $f : \kappa^+ \to \lambda$ satisfying $\forall \nu \in E \, \exists j < \aleph_0 \, \forall i > j : f(n_\nu(i)) = h_\nu(i)$."*

PROOF : By [**3**, § 2]. ∎

Definition 2.3. Let R be a regular left hereditary ring such that R is not completely reducible. Let $(e_i \mid i < \aleph_0)$ be a set of orthogonal idempotents of R. Let κ and E be as in 2.1. Let $(n_\nu \mid \nu \in E)$ be a ladder system. For each $\alpha < \kappa^+$ denote by π_α the α-th canonical projection of the module $R^{(\kappa^+)}$ onto R. Let $1_\alpha \in R^{(\kappa^+)}$ be such that $1_\alpha \pi_\alpha = 1$ and $1_\alpha \pi_\beta = 0$ for all $\beta \neq \alpha \in \kappa^+$. Define a module $\underline{M} = R^{(\kappa^+)}/G$, where $\underline{G} = \sum_{\nu,i} Rg_{\nu i}$, and for all $\nu \in E$ and all $i < \aleph_0 : g_{\nu i}\pi_{n_\nu(i)} = e_i$, $g_{\nu i}\pi_\nu = -e_i$ and $g_{\nu i}\pi_\alpha = 0$ otherwise.

Lemma 2.4. *Assume UP. Let R be a regular left hereditary ring such that R is not completely reducible. Let N be a non-injective module. Then N does not have the Whitehead property.*

PROOF : Let $(e_i \mid i < \aleph_0)$ be a set of orthogonal idempotents of the ring R. Put $\lambda = \mathrm{card}(N)$. Take $\kappa > \lambda$ such that $\kappa \geq \mathrm{card}(R)$ and $\mathrm{cf}(\kappa) = \aleph_0$. By UP, there are a stationary subset E of κ^+ such that $E \subseteq \{\alpha < \kappa^+ \mid \mathrm{cf}(\alpha) = \aleph_0\}$ and a ladder system $(n_\nu \mid \nu \in E)$ such that for every sequence $(h_\nu \mid \nu \in E)$ of mappings from $\aleph_0$ to λ there is a mapping $f : \kappa^+ \to \lambda$ satisfying $\forall \nu \in E \, \exists j_\nu < \aleph_0 \, \forall i > j_\nu : f(n_\nu(i)) = h_\nu(i)$. Let M and G be the modules corresponding to $R, (e_i \mid i < \aleph_0), \kappa, E$ and $(n_\nu \mid \nu \in E)$ by 2.3.

First, we prove that M is not projective. Put $M_0 = 0$ and for each $0 < \alpha < \kappa^+$ define $M_\alpha = \sum_{\beta<\alpha} R(1_\beta + G)$. Then $(M_\alpha \mid \alpha < \kappa^+)$ is a κ^+-filtration of M. Assume M is projective. Since R is regular there are a set of non-zero idempotents $\{f_\alpha \mid \alpha < \kappa^+\} \subseteq R$ and an isomorphism $\varphi : M \to \oplus \sum_{\alpha<\kappa^+} Rf_\alpha$. Put $N_0 = 0$ and, for each $0 < \alpha < \kappa^+$, $N_\alpha = \oplus \sum_{\beta<\alpha} Rf_\beta$. Then $(N_\alpha \mid \alpha < \kappa^+)$ is a κ^+-filtration of $\varphi(M)$. Since the set $C = \{\alpha < \kappa^+ \mid \mathrm{Im}(\varphi \restriction M_\alpha) = N_\alpha\}$ is closed and cofinal in κ^+, there exists some $\nu \in E \cap C$. In particular, $R/\oplus \sum_{i<\aleph_0} Re_i \simeq \varphi(M_{\nu+1})/\varphi(M_\nu) \subseteq \oplus \sum_{\alpha \geq \nu} Rf_\alpha$, whence the module $M_{\nu+1}/M_\nu$ is projective, a contradiction.

We shall prove that $\mathrm{Ext}(M, N) = 0$. Let $\varphi : \mathrm{Hom}(R^{(\kappa^+)}, N) \to \mathrm{Hom}(G, N)$ be given by $\varphi h = h \restriction G$ for all $h \in \mathrm{Hom}(R^{(\kappa^+)}, N)$. Since there is an exact sequence

$$\mathrm{Hom}(R^{(\kappa^+)}, N) \xrightarrow{\varphi} \mathrm{Hom}(G, N) \to \mathrm{Ext}(M, N) \to \mathrm{Ext}(R^{(\kappa^+)}, N) = 0,$$

it suffices to prove that φ maps onto. Let $x \in \mathrm{Hom}(G, N)$. Clearly, there is a bijection $b : N \to \lambda$. For all $\nu \in E$ and $i < \aleph_0$, define $h_\nu(i) = b(g_{\nu i}x) \in b(e_i N)$. For all $\nu \in E$ and $i \leq j_\nu$, put $\delta_{\nu i} = e_i b^{-1} f(n_\nu(i))$ provided there are $\mu \in E$ and $k < \aleph_0$ such that $j_\mu < k$ and $n_\nu(i) = n_\mu(k)$, and $\delta_{\nu i} = 0$ otherwise. Define a mapping $y \in \mathrm{Hom}(R^{(\kappa^+)}, N)$ by

$1_\alpha y = b^{-1} f(\alpha)$ provided there are $\nu \in E$ and $i < \aleph_0$ such that $j_\nu < i$ and $\alpha = n_\nu(i)$,
$1_\alpha y = \sum_{i \leq j_\alpha} (\delta_{\alpha i} - g_{\alpha i} x)$ provided $\alpha \in E$,
$1_\alpha y = 0$ otherwise.

Then $g_{\nu i}\, y = e_i\,(1_{n_\nu(i)}\, y - 1_\nu\, y) = g_{\nu i}\, x$ for all $\nu \in E$ and $i < \aleph_0$. Hence $\mathrm{Ext}(M, N) = 0$, and N does not have the Whitehead property. ■

Theorem 2.5. *The assertion "every regular left or right* Ext*-ring is completely reducible" is consistent with ZFC.*

PROOF : Assume UP. Let R be a regular non-completely reducible left Ext-ring. By [**5**, Theorem 6.2], R is left hereditary. By 2.4, no non-injective module has the Whitehead property, a contradiction. Dually, we prove that every regular right Ext-ring is completely reducible. Now, the result follows from 2.2. ■

Theorem 2.6. *The equivalence of the following three assertions is consistent with ZFC:*

(i) *R is a left non-singular left* Ext*-ring,*
(ii) *R is a right non-singular right* Ext*-ring,*
(iii) *either $R = S$ or $R = T$ or $R = S \boxplus T$, where S is a completely reducible ring and there is a skew field K such that T is Morita equivalent to the upper triangular matrix ring of degree 2 over K.*

PROOF : By 2.5 and [**5**, Theorems 3.4 and 8.1]. ■

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