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On the Hammerstein integral equations in Banach spaces

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Abstract. This paper contains some existence theorems for L^p_r -solutions of nonlinear integral equations in Banach spaces. In our assumptions and proofs we employ measures of noncompactness.

Keywords: Integral equations, measures of noncompactness

Classification: 45N05

1. Introduction.

Assume that E, F are Banach spaces and D is a compact domain in the Euclidean space R^{ν} . Denote by $L^p_r = L^p_r(D, E)$ ($p > 1$) the space of all strongly measurable functions $u : D \rightarrow E$ with $\int_D r(t)\|u(t)\|^p dt < \infty$, provided with the norm $\|u\|_{p,r} = (\int_D r(t)\|u(t)\|^p dt)^{1/p}$, where $r : D \rightarrow R$ is a nonnegative, bounded and integrable function such that $\text{mes}\{t \in D : r(t) = 0\} = 0$.

In this paper we give sufficient conditions for the existence of a solution $x \in L^p_r$ of the integral equation

$$(1) \quad x(t) = g(t) + \lambda \int_D r(s)K(t, s)f(s, x(s)) ds$$

or

$$(2) \quad x(t) = g(t) + \int_0^t r(s)K(t, s)f(s, x(s)) ds.$$

Our results extend some theorems from the papers [8], [9], concerning L^p -solutions. Throughout this paper we shall assume that:

- 1° p, q are real numbers such that $p, q > 1$ and $p \geq \min(q, 2)$; let $l = \frac{q}{q-1}, m = \max(p, l)$ and let k be a number such that $1 < k \leq \infty$ and $\frac{1}{k} + \frac{1}{m} + \frac{1}{p} = 1$;
- 2° $g \in L^p_r$;
- 3° $(s, x) \rightarrow f(s, x)$ is a function from $D \times E$ into F such that
 - (i) f is strongly measurable in s and continuous in x ;
 - (ii) $\|f(s, x)\| \leq a(s) + b\|x\|^{p/q}$ for $s \in D$ and $x \in E$, where $a \in L^q_r(D, R)$ and $b \geq 0$;
- 4° K is a strongly measurable function from $D \times D$ into the space of continuous linear mappings $F \rightarrow E$ and

$$\int \int_{D \times D} r(s)r(t)\|K(t, s)\|^m ds dt < \infty.$$

Denote by α and α_p the Kuratowski measures of noncompactness in E and L^p , respectively. For any set V of functions from D into E denote by v the function defined by $v(t) = \alpha(V(t))$ for $t \in D$ (under the convention that $\alpha(A) = \infty$ if A is unbounded), where $V(t) = \{x(t) : x \in V\}$.

Without loss of generality we shall always assume that all functions from $L^1(D, E)$ or $L^1_r(D, E)$ are extended to R^v by putting $u(t) = 0$ outside D .

Before passing to further considerations we shall quote two lemmas.

Lemma 1 (Heinz [5]). *Let V be a countable set of strongly measurable functions $D \rightarrow E$ such that there exists $M \in L^1(D, R)$ such that $\|x(t)\| \leq M(t)$ for all $x \in V$ and $t \in D$. Then the corresponding function v is integrable and $\alpha(\{\int_D x(t) dt : x \in V\}) \leq 2 \int_D v(t) dt$.*

Lemma 2 (Szuffla [8]). *Let V be a countable set of strongly measurable functions $D \rightarrow E$ such that*

- (i) *there exists $M \in L^p(D, R)$ such that $\|x(t)\| \leq M(t)$ for all $x \in V$ and $t \in D$;*
- (ii) *$\lim_{h \rightarrow 0} \sup_{x \in V} \int_D \|x(t+h) - x(t)\|^p dt = 0$.*

Then

$$\alpha_p(v) \leq 2 \left(\int_D v^p(t) dt \right)^{1/p}$$

Let us recall that in the last twenty years the measure of noncompactness has been employed for differential and integral equations by many authors (see [1], [3], [4], [6], [8], [9]).

2. The existence of L^p_r -solutions.

Theorem 1. *Let h be a nonnegative function belonging to $L^k_r(D, R)$. If*

$$(3) \quad \alpha(f(t, X)) \leq h(t)\alpha(X) \quad \text{for } t \in D$$

and for each bounded subset X of E , then there exists a positive number ϱ such that for any $\lambda \in R$ with $|\lambda| < \varrho$, the equation (1) has at least one solution $x \in L^p_r$.

PROOF : For simplicity put

$$Q = \left(\int_D r(t) \left(\int_D r(s) \|K(t, s)\|^l ds \right)^{p/l} dt \right)^{1/p},$$

$$S = \left(\int_D r(t) \left(\int_D r(s) \|K(t, s)\|^m ds \right)^{p/m} dt \right)^{1/p}$$

and $k(t) = \|K(t, \cdot)\|_{l,r}$. It follows from 1° and 4° that $Q, S < \infty$ and $k \in L^p_r$.

Choose a positive number ϱ such that

$$\varrho \leq \min \left(\sup_{\eta > 0} \frac{\eta - \|g\|_{p,r}}{Q(\|a\|_{q,r} + b\eta^{p/q})}, \frac{1}{2S\|h\|_{k,r}} \right).$$

Fix $\lambda \in \mathbb{R}$ with $|\lambda| < \rho$ and choose $c > 0$ satisfying the inequality $\|g\|_{p,r} + |\lambda|Q(\|a\|_{q,r} + bc^{p/q}) \leq c$. Let $B = \{x \in L_r^p : \|x\|_{p,r} \leq c\}$. We define a mapping F by

$$(4) \quad F(x)(t) = g(t) + \lambda \int_D r(s) K(t, s) f(s, x(s)) ds$$

for $x \in B, t \in D$.

By the Fubini theorem for vector functions, the Hölder inequality for L_r^p spaces and $1^\circ - 4^\circ$, for any $x \in B$ the function $F(x)$ is strongly measurable on D and

$$(5) \quad \|F(x)(t)\| \leq M(t) \quad \text{for } t \in D,$$

where $M(t) = \|g(t)\| + |\lambda|k(t)(\|a\|_{q,r} + bc^{p/q})$. It is clear that F is a mapping $B \rightarrow B$. Using the standard argument it can be shown that $1^\circ - 4^\circ$ imply the continuity of F .

Furthermore, from the conditions $1^\circ - 4^\circ$ and the Hölder inequality it follows that

$$\|r(t+h)F(x)(t+h) - r(t)F(x)(t)\| \leq d(t, h) \quad \text{for all } x \in B,$$

where

$$d(t, h) = \begin{cases} r(t)M(t) & \text{if } t \in D \text{ and } t+h \notin D \\ \|r(t+h)g(t+h) - r(t)g(t)\| + \\ + |\lambda|(\|a\|_{q,r} + bc^{p/q}) \left(\int_D r(s) \|r(t+h)K(t+h, s) - \right. \\ \left. - r(t)K(t, s)\|^l ds \right)^{1/l} & \text{if } t, t+h \in D. \end{cases}$$

Let $r(t) \leq R$ for $t \in D$. Since

$$\begin{aligned} & \lim_{h \rightarrow 0} \int_D \|r(t+h)K(t+h, s) - r(t)K(t, s)\|^m dt = 0 \quad \text{for a.e. } s \in D, \\ & \lim_{h \rightarrow 0} \int_D \left[\int_D r(s) \|r(t+h)K(t+h, s) - r(t)K(t, s)\|^l ds \right]^{m/l} dt \leq \\ & \leq \lim_{h \rightarrow 0} \int_D [(\text{mes } D)^{\dagger - \frac{1}{m}} \int_D r(s) \|r(t+h)K(t+h, s) - \\ & - r(t)K(t, s)\|^m ds] dt = \\ & = (\text{mes } D)^{\dagger - \frac{1}{m}} \lim_{h \rightarrow 0} \int_D r(s) \left[\int_D \|r(t+h)K(t+h, s) - \right. \\ & \left. - r(t)K(t, s)\|^m dt \right] ds = 0 \end{aligned}$$

and $p \leq m$, we see that

$$(6) \quad \lim_{h \rightarrow 0} \int_D d^p(t, h) dt = 0.$$

Moreover

$$(7) \quad \int \int_{D \times \Omega_\eta} d^p(t, s) dt ds$$

where Ω_η means the closed ball in R^p with center 0 and radius η . This implies that

$$(8) \quad \limsup_{h \rightarrow 0} \int_D \|r(t+h)F(x)(t+h) - r(t)F(x)(t)\|^p dt = 0.$$

Let V be a countable subset of B such that

$$(9) \quad V \subset \overline{\text{conv}(F(V) \cup \{0\})}.$$

Then $rV \subset L^p$ and by (5)

$$(10) \quad \|x(t)\| \leq M(t) \quad \text{for } x \in V$$

and for a.e. $t \in D$. Hence $\|f(s, x(s))\| \leq \eta(s) = a(s) + bM^{p/q}(s)$ for $x \in V, s \in D$ and $\eta \in L^q_r(D, R)$. By the Hölder inequality, from this we deduce that for fixed $t \in D$ the function $s \rightarrow \|K(t, s)\| \eta(s)$ belongs to $L^1_r(D, E)$. Moreover, from (10) and Lemma 1 it follows that the function $t \rightarrow \alpha(rV(t))$ is measurable on D and $\alpha(V(t)) \leq 2M(t)$ for a.e. $t \in D$.

Therefore, by Lemma 1 and (3), we obtain

$$\begin{aligned} v(t) &\leq \alpha(F(V)(t)) \leq \alpha(\{\lambda \int_D r(s)K(t, s)f(s, x(s)) ds : x \in V\}) \leq \\ &\leq 2|\lambda| \int_D \alpha(\{r(s)K(t, s)f(s, x(s)) : x \in V\}) ds \leq \\ &\leq 2|\lambda| \int_D r(s)\|K(t, s)\| h(s)v(s) ds \quad \text{for a.e. } t \in D. \end{aligned}$$

Consequently, by the Hölder inequality, we have

$$v(t) \leq 2|\lambda| \|h\|_{k,r} \|v\|_{p,r} \left(\int_D r(s)\|K(t, s)\|^m ds \right)^{1/m}$$

As the above inequality holds for a.e. $t \in D$, we get

$$\|v\|_{p,r} \leq 2|\lambda| S \|h\|_{k,r} \|v\|_{p,r}.$$

Since $|\lambda|2S\|h\|_{k,r} < 1$, from this we infer that $\|v\|_{p,r} = 0$. It follows from (8), (9) and Lemma 2 that $\alpha_p(rV) \leq 2\|rv\|_p \leq 2R^{1-\frac{1}{p}}\|v\|_{p,r} = 0$, i.e. rV is relatively compact in L^p .

We shall show that V is relatively compact in L^p_r . Let $u_n \in V$ for $n = 1, 2, \dots$. Then there exists a subsequence (u_{n_k}) of (u_n) such that $\lim_{k \rightarrow \infty} \|ru_{n_k} - u_0\|_p = 0$

for some $u_0 \in L^p$. Put $u(t) = u_0(t)/r(t)$ for $t \in D$ such that $r(t) \neq 0$. Without loss of generality (by passing to a subsequence if necessary) we may assume that $\lim_{k \rightarrow \infty} (r(t)u_{n_k}(t) - r(t)u(t)) = 0$ for a.e. $t \in D$. From (10) it is clear that $\|u_{n_k}(t) - u(t)\| \leq 2M(t)$ for a.e. $t \in D$. Thus, by Lebesgue theorem,

$$\begin{aligned} \lim_{k \rightarrow \infty} \|u_{n_k} - u\|_{p,r}^p &= \lim_{k \rightarrow \infty} \int_D r(t) \|u_{n_k}(t) - u(t)\|^p dt = \\ &= \int_D r(t) \lim_{k \rightarrow \infty} \|u_{n_k}(t) - u(t)\|^p dt = 0. \end{aligned}$$

Hence, V is relatively compact in L_r^p . Applying now the Mönch fixed point theorem [7, Theorem 2.2] we conclude that there exists $x \in B$ such that $x = F(x)$. Obviously x is a solution of (1). ■

3. The Volterra–Hammerstein integral equation.

Consider now the equation (2), with $D = [0, d]$. Choose $\eta \in (0, \frac{1}{2})$ and an interval $J = [0, a] \subset D$ in such a way that for each ε , $0 \leq \varepsilon \leq \eta$, the maximal continuous solution z_ε of the integral equation

$$z(t) = \varepsilon + 2^{p-1} \int_0^t r(s) (\|g(s)\| + k(s) \|a\|_{q,r} + bk(s) z^{1/q}(s))^p ds$$

is defined on J and $z_\varepsilon(t) \leq z_0(t) + 1$ for $t \in J$. Let $c = \max_{t \in J} (z_0(t) + 1)^{1/p}$, $L_r^p = L_r^p(J, E)$, $B = \{x \in L_r^p : \|x\|_{p,r} \leq c\}$ and $U = \{x \in L_r^p : \|x\|_{p,r} \leq \eta\}$. Put $F(x)(t) = g(t) + \int_0^t r(s) K(t, s) f(s, x(s)) ds$ for $x \in L_r^p$, $t \in J$. Then $\|F(x)(t)\| \leq M(t)$ and $\|r(t+h)F(x)(t+h) - r(t)F(x)(t)\| \leq d(t, h)$ for $x \in B$, $t \in J$, where d satisfies (6) and (7). Moreover F is a continuous mapping $L_r^p \rightarrow L_r^p$.

Theorem 2. *Let h be a nonnegative function belonging to $L_r^k(D, R)$. If 1°–4° hold and $\alpha(f(t, X)) \leq h(t)\alpha(X)$ for $t \in D$ and for each bounded subset X of E , then the equation (2) has at least one solution $x \in L_r^p$.*

PROOF : For any positive integer n we define a function $u_n : J \rightarrow E$ by

$$u_n(t) = \begin{cases} g(t) & \text{for } 0 \leq t \leq a_n \\ g(t) + \int_0^{t-a_n} r(s) K(t, s) f(s, u_n(s)) ds & \text{for } a_n < t \leq a, \end{cases}$$

where $a_n = a/n$. From the Hölder inequality and 3° it follows that

$$\|u_n(t)\| \leq \|g(t)\| + k(t) \|a\|_{q,r} + bk(t) \left(\int_0^t r(s) \|u_n(s)\|^p ds \right)^{1/q}$$

and

$$\begin{aligned} (11) \quad \|u_n(t) - g(t) - \int_0^t r(s) K(t, s) f(s, u_n(s)) ds\| &\leq \\ &\leq k_n(t) (\|a\|_{q,r} + b \left(\int_0^t r(s) \|u_n(s)\|^p ds \right)^{1/q}) \quad \text{for } t \in J, \end{aligned}$$

where

$$k_n(t) = \begin{cases} k(t) & \text{for } 0 \leq t \leq a_n \\ \|K(t, \cdot)\chi_{[t-a_n, t]}\|_{t,r} & \text{for } a_n < t \leq a. \end{cases}$$

Putting $w_n(t) = \int_0^t r(s) \|u_n(s)\|^p ds$ we see that

$$w_n(t) \leq \int_0^t r(s) (\|g(s)\| + k(s)\|a\|_{q,r} + bk(s)w_n^{1/q}(s))^p ds.$$

By the theorem on integral inequalities this implies $w_n(t) \leq z_0(t) + 1 \leq c^p$ for $t \in J$. Hence $u_n \in B$ and $\|u_n(t)\| \leq \|g(t)\| + k(t)(\|a\|_{q,r} + bc^{p/q}) = M(t)$. Moreover $\lim_{n \rightarrow \infty} k_n(t) = 0$ and $k_n(t) \leq k(t)$ for a.e. $t \in J$. By (11), $\lim_{n \rightarrow \infty} (u_n(t) - F(u_n)(t)) = 0$ for a.e. $t \in J$ and $\lim_{n \rightarrow \infty} \|u_n - F(u_n)\|_{p,r} = 0$. Arguing similarly as in the proof of Theorem 1, we can show that the set $\{u_n : n = 1, 2, \dots\}$ is relatively compact in L_r^p . Thus we can find a subsequence (u_{n_k}) of (u_n) which converges in L_r^p to a limit u . Consequently

$$\|u - F(u)\|_{p,r} = \lim_{k \rightarrow \infty} \|u_{n_k} - F(u_{n_k})\|_{p,r} = 0,$$

which proves that u is a solution of (2). ■

Combining the proofs of Theorem 2 and Theorem from [9], we can prove the following Aronszajn-type

Theorem 3. *Under the assumptions of Theorem 2, the set S of all solutions $x \in L_r^p$ of (2) is a compact R_δ , i.e. S is homeomorphic to the intersection of a decreasing sequence of compact absolute retracts.*

REFERENCES

- [1] Ambrosetti A., *Un teorema di esistenza per le equazioni differenziali negli spazi di Banach*, Rend. Sem. Mat. Univ. Padova **39** (1967), 349-369.
- [2] Azbrieliev N.V., Caljuk Z.B., *Ob integralnykh nieravienstvach*, Matem. Sbornik **56** (1962), 325-342.
- [3] Daneš J., *On Densifying and Related Mappings and Their Application in Nonlinear Functional Analysis*, Theory of Nonlinear Operators, Akademie-Verlag, Berlin 1974, 15-56.
- [4] Deimling K., *Ordinary Differential Equations in Banach Spaces*, Lecture Notes in Math. 596, Berlin, Heidelberg, New York, 1977.
- [5] Heinz H.P., *On the behaviour of measures of noncompactness with respect to differentiation and integration of vector-valued functions*, Nonlinear Analysis **7** (1983), 1351-1371.
- [6] Krasnoselskii M.A., Zabreiko P.P., Pustyl'nik E.I., Sobolevskii P.E., *Integral Operators in Spaces of Integrable Functions*, Moskva 1966.
- [7] Mönch H., *Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces*, J. Nonlinear Analysis **4** (1980), 985-999.

- [8] Szufia S., *Existence theorems for L^p -solutions of integral equations in Banach spaces*, Publ. Inst. Math. **40**, 54 (1986), 99-105.
- [9] Szufia S., Appendix to the paper *Existence theorems for L^p -solutions of integral equations in Banach spaces*, Publ. Inst. Math. **43**, 57 (1988), 113-116.

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