Jiří Jelínek

Distinguishing example for the Tillmann product of distributions

Commentationes Mathematicae Universitatis Carolinae, Vol. 31 (1990), No. 4, 693--700

Persistent URL: http://dml.cz/dmlcz/106904

Comment.Math.Univ.Carolinae 31,4 (1990)693–700

Distinguishing example for the Tillmann product of distributions

JIŘÍ JELÍNEK

Abstract. The Tillmann product of Schwartz distributions is more general than the Kamiński product.

Keywords: Distribution, Tillmann product, Kamiński product

Classification: 46F05

Several ways to define the product of distributions by mollification have been investigated. Let S, T, W be distributions on $\mathbf{R}^n$. We say that $W = S \cdot T$ iff

$$(1) \qquad \langle W, \omega\rangle = \lim_{\varepsilon\downarrow 0}\langle (S * \varphi_\varepsilon)(T * \psi_\varepsilon), \omega\rangle$$

for all $\omega \in \mathcal{D}(\mathbf{R}^n)$ and for all nets $\{\varphi_\varepsilon\}_{\varepsilon>0}$, $\{\psi_\varepsilon\}_{\varepsilon>0}$ which vary in certain classes of nets of smooth functions and converge to the Dirac measure. Recently this definition with the condition

$$(2) \qquad \varphi_\varepsilon = \psi_\varepsilon$$

has become important because of its relations to the Colombeau algebras. With this conditions we call this product to be **Kamiński product** iff

$$(3) \qquad \varphi_\varepsilon(x) = \frac{1}{\varepsilon^n}\varphi(\frac{x}{\varepsilon}),$$

$\varphi \in \mathcal{D}$, $\int \varphi = 1$. Wawak [**2**] has proved that equivalently one can subject the nets to the conditions (2),

$$(4) \qquad \int \varphi_\varepsilon = 1,$$

$$(5) \qquad \operatorname{supp}\varphi_\varepsilon \to \{0\} \text{ as } \varepsilon \downarrow 0$$

and

$$(6) \qquad \int |x|^\alpha |(\frac{\partial}{\partial x})^\alpha \phi_\varepsilon(x)|\, dx \le M_\alpha$$

for all $\alpha \in \mathbf{N}_0^n$, $\varepsilon > 0$. Equivalently, the relations (1), (2) can be replaced by (see [**2**])

$$(7) \qquad \langle W, \omega\rangle = \lim_{\varepsilon\downarrow 0}\langle \frac{1}{2}[(S * \varphi_\varepsilon)T + S(T * \varphi_\varepsilon)], \omega\rangle .$$

The Kamiński definition is also equivalent to the Colombeau one [4] : W is associated with the product $S \odot T$ in the Colombeau algebra.

Further, in dimension $n = 1$, one can define the **Tillmann product** by (1) with (2) and (3) valid for one mollifier φ only,

$$\varphi(x) = \varrho(x) := \frac{1}{\pi(x^2+1)}.$$

As ϱ has not a compact support, the convolution in (1) is not defined for arbitrary $R, S \in \mathcal{D}'$. M. Oberguggenberger has shown that in the case $R, S \in \mathcal{D}'_{L^1}$, the existence of the Kamiński product implies its equality with the Tillmann product.

The aim of this paper is to give an example of distributions $S, T \in \mathcal{D}'(\mathbf{R}^1)$ $(S = \delta)$ having the Tillmann product but not the Kamiński product. So the Tillmann product is strictly more general. It gives the answer to one of the questions asked by M. Oberguggenberger during the International Conference on Generalized Functions... Dubrovnik 1987. I have presented this example on the International Conference on Generalized Functions... Oberwolfach 1989 and I thank M. Oberguggenberger and R. Wawak for having carefully read the example and proposed some simplifications of the calculation.

Definition 1. Put $T = \sum_{n=1}^{\infty} T_n \in \mathcal{D}'(\mathbf{R})$, where

$$T_n(x) = e^{\frac{\pi}{6}n}\left(\pi e^{-8^n}\delta(|x| - e^{-8^n}) - \frac{\sin(n(\ln|x| + 8^n))}{\ln|x| + 8^n}\right). \tag{8}$$

Here $\delta(|x| - a)$ for $a > 0$ denote $\delta(x-a) + \delta(x+a)$. Choose a function $\eta \in \mathcal{D}$, $\eta = 1$ on $[-1,1]$, $\eta = 0$ outside $[-2,2]$, $\eta \geq 0$ and denote

$$\varrho(x) = \frac{1}{\pi(x^2+1)}, \quad \sigma(x) = \frac{x^2}{\pi(x^6+1)}.$$

We have $\int \varrho = \int \sigma = 1$.

2. In the sequel φ_ε is meant always by (3) $(n = 1)$.

Note. We will prove that the Kamiński product $\delta \cdot T$ does not exist but the Tillmann product $\delta \cdot T = 0$. The last assertion means that

$$\lim_{\varepsilon \downarrow 0} \varrho_\varepsilon(T \ast \varrho_\varepsilon) = 0 \tag{9}$$

in $\mathcal{D}'$. We can replace the distribution T by the distribution with compact support ηT without changing the assertion above. Indeed, as

$$\left|\frac{\sin(n(\ln|x| + 8^n))}{\ln|x| + 8^n}\right| \leq \frac{1}{8^n} \quad \text{for } |x| > 1, \tag{10}$$

T is a bounded function outside $[-1,1]$.

Proposition 3. *T is a distribution of order 0 (and a bounded function outside $[-1,1]$).*

PROOF : Let us write

$$T_n = T_n\chi_{\{x;x>0\}} + T_n\chi_{\{x;x<0\}} =: T_n^+ + T_n^-.$$

For $\varphi \in \mathcal{D}([-A,A])$ we have

$$(11) \qquad \langle T_n^+, \varphi\rangle = e^{\frac{\pi}{6}n}\left(\pi e^{-8^n}\varphi(e^{-8^n}) - \int_0^\infty \frac{\sin(n(\ln x + 8^n))}{\ln x + 8^n}\cdot\varphi(x)\,dx\right)$$

and it is sufficient to estimate the integral.

$$\int\Big|\frac{\sin(n(\ln x + 8^n))}{\ln x + 8^n}\varphi(x)\Big|\,dx = \int_0^{e^{-4^n}} + \int_{e^{-4^n}}^A \le$$
$$\le e^{-4^n} n\cdot\max|\varphi(x)| + A\frac{1}{-4^n+8^n}\cdot\max|\varphi(x)|,$$

which gives the result. ■

Proposition 4. $\lim_{\varepsilon\downarrow 0}\langle T, \varrho_\varepsilon\rangle = 0$.

PROOF : for T^+. We have

$$(12) \qquad \varrho_\varepsilon(x) = \frac{1}{\pi x}\cdot\frac{1}{\frac{\varepsilon}{x}+\frac{x}{\varepsilon}}.$$

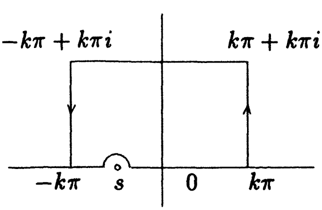
From (11) we obtain after the substitution $x = e^s\varepsilon$

$$(13) \quad \langle T_n^+, \varrho_\varepsilon\rangle =$$
$$= e^{\frac{\pi}{6}n}\left(\frac{1}{\frac{e^{-8^n}}{\varepsilon}+\frac{\varepsilon}{e^{-8^n}}} - \int_{-\infty}^{\infty}\frac{\sin(n(s+\ln\varepsilon+8^n))}{s+\ln\varepsilon+8^n}\cdot\frac{1}{\pi(e^s+e^{-s})}\,ds\right).$$

Writing

$$(14) \qquad \sin n(s+\ln\varepsilon+8^n) = \operatorname{Im} e^{\mathrm{i}n(s+\ln\varepsilon+8^n)}$$

we calculate the integral by the residue theorem integrating over the rectangle with vertexes $\pm k\pi$, $\pm k\pi + k\pi i$ $1(k\to\infty)$ but avoiding the simple pole at $s = -\ln\varepsilon - 8^n$.



The others simple poles are at $i(\frac{\pi}{2}+j\pi)$ $(j \in \mathbf{Z})$, where the function $\frac{1}{\pi(e^s+e^{-s})}$ has residues equal to $\frac{(-1)^j}{2\pi i}$. We obtain

$$\text{v.p.} \int_{-\infty}^{\infty} \frac{e^{in(s+\ln\varepsilon+8^n)}}{s+\ln\varepsilon+8^n} \cdot \frac{1}{\pi(e^s+e^{-s})} =$$

$$= \frac{\pi i}{\pi(e^{-\ln\varepsilon-8^n}+e^{\ln\varepsilon+8^n})} + \sum_{j=0}^{\infty} 2\pi i \cdot \frac{(-1)^j}{2\pi i} \cdot \frac{e^{in(i\frac{\pi}{2}+ij\pi+\ln\varepsilon+8^n)}}{i\frac{\pi}{2}+ij\pi+\ln\varepsilon+8^n},$$

which implies due to (13) and (14)

$$|\langle T_n^+, \varrho_\varepsilon\rangle| \le e^{\frac{\pi}{6}n} \cdot \sum_{j=0}^{\infty} \frac{e^{-n\frac{\pi}{2}} \cdot e^{-nj\pi}}{|i\frac{\pi}{2}+\ln\varepsilon+8^n|} \le$$

$$\le \frac{e^{-n(\frac{\pi}{2}-\frac{\pi}{6})}}{|i\frac{\pi}{2}+\ln\varepsilon+8^n|} \cdot \frac{1}{1-e^{-\pi}}.$$

and similarly for T_n^-.

Hence,

$$\lim_{\varepsilon\downarrow 0} |\langle T^+, \varrho_\varepsilon\rangle| \le 2 \lim_{\varepsilon\downarrow 0} \sum_{n=1}^{\infty} \frac{e^{-n(\frac{\pi}{2}-\frac{\pi}{6})}}{|i\frac{\pi}{2}+\ln\varepsilon+8^n|} \cdot \frac{1}{1-e^{-\pi}} =$$

$$= 2\sum_{n=1}^{\infty} \lim_{\varepsilon\downarrow 0} \frac{e^{-n(\frac{\pi}{2}-\frac{\pi}{6})}}{|i\frac{\pi}{2}+\ln\varepsilon+8^n|} \cdot \frac{1}{1-e^{-\pi}}$$

by the Lebesgue majorating theorem: the serie is majorated by the summable serie

$$\sum_{n=1}^{\infty} \frac{e^{-n(\frac{\pi}{2}-\frac{\pi}{6})}}{\frac{\pi}{2}} \cdot \frac{1}{1-e^{-\pi}}.$$

■

Remark 5. $\varrho_\varepsilon * \varrho_a = \varrho_{\varepsilon+a}$ (for ε, $a>0$).

Lemma. $\lim_{\varepsilon\downarrow 0}\langle (T * \varrho_\varepsilon)\varrho_\varepsilon, \varrho\rangle = 0$.

PROOF : This limit is equal to

$$\lim_{\varepsilon\downarrow 0}\langle T(x), (\frac{1}{\pi(x^2+1)} \cdot \frac{\varepsilon}{\pi(x^2+\varepsilon^2)}) * \frac{\varepsilon}{\pi(x^2+\varepsilon^2)}\rangle =$$

$$\lim_{\varepsilon\downarrow 0}\langle T(x), \frac{\varepsilon}{\pi^2(1-\varepsilon^2)}(\frac{1}{x^2+\varepsilon^2} - \frac{1}{x^2+1}) * \frac{\varepsilon}{\pi(x^2+\varepsilon^2)}\rangle.$$

Now the lemma follows from the remark and from the proposition above and from the fact that the functions $\varrho_{1+\varepsilon}$ form a bounded set in $\mathcal{E}$ and in L_1 (see Proposition 3). ■

Theorem 6. *The Tillmann product $\delta \cdot T = 0$.*

PROOF : By (9), we have to prove $\lim_{\varepsilon\downarrow 0}\langle \varrho_\varepsilon(T \ast \varrho_\varepsilon), \varphi\rangle = 0 \quad (\varphi \in \mathcal{D})$. As T is an even distribution, we can replace $\varphi(x)$ by $\frac{1}{2}(\varphi(x) + \varphi(-x))$, so we suppose without loss of generality that φ is even. Let us write $\varphi = \varphi(0) + \psi$ for to have $\psi(0) = \psi'(0) = 0$. Hence

$$\psi(x) \le A\frac{x^2}{1+x^2} \tag{15}$$

for some $A > 0$. Due to Proposition 4 and Remark 5 we have

$$\langle \delta \cdot T, 1\rangle = \lim \langle \varrho_\varepsilon(T \ast \varrho_\varepsilon), 1\rangle = 0,$$

so

$$\langle \delta \cdot T, \varphi\rangle = \langle \delta \cdot T, \psi\rangle = \lim_{\varepsilon\downarrow 0}\langle T, (\psi\varrho_\varepsilon) \ast \varrho_\varepsilon\rangle \,. \tag{16}$$

We are going to show that the last testing functions go to zero uniformly and in L_1, which will give the result due to Proposition 3. By (15)

$$|\psi(x)\varrho_\varepsilon(x)| \le A\frac{x^2}{1+x^2}\cdot\frac{\varepsilon}{\pi(x^2+\varepsilon^2)} = A\frac{1}{1+x^2}\cdot\frac{\varepsilon}{\pi}\cdot\frac{(\frac{x}{\varepsilon})^2}{(\frac{x}{\varepsilon})^2+1} \le \frac{A\varepsilon}{\pi},$$

so $|\psi\varrho_\varepsilon \ast \varrho_\varepsilon| \le \frac{A\varepsilon}{\pi}$, too. Further, due to Remark 5 and (15), $|\psi\varrho_\varepsilon \ast \varrho_\varepsilon| \le A\varrho_{2\varepsilon}$. ■

Proposition 7. *For the function $\sigma(x) = \frac{x^2}{\pi(x^6+1)}$ the assertion $\lim_{\varepsilon\downarrow 0}\langle T, \sigma_\varepsilon\rangle = 0$ is not true.*

PROOF : We have

$$\sigma_\varepsilon(x) = \frac{1}{\pi x}\cdot\frac{1}{(\frac{x}{\varepsilon})^3 + (\frac{\varepsilon}{x})^3}\,. \tag{17}$$

From (11) we obtain after the substitution $x = \varepsilon e^s$

$$\langle T_n^+, \sigma_\varepsilon\rangle = \\ = e^{\frac{\pi n}{6}}\left(\frac{1}{(\frac{e^{-8^n}}{\varepsilon})^3 + (\frac{\varepsilon}{e^{-8^n}})^3} - \int_{-\infty}^{\infty}\frac{\sin n(s+\ln\varepsilon+8^n)}{s+\ln\varepsilon+8^n}\cdot\frac{ds}{\pi(e^{3s}+e^{-3s})}\right).$$

The function $\frac{1}{\pi(e^{3s}+e^{-3s})}$ has simple poles at

$$s = i(\frac{\pi}{6} + j\frac{\pi}{3}) \quad (j \in \mathbb{Z})$$

with residues equal to $\frac{(-1)^j}{6\pi i}$. Using the same method as in the proof of Proposition 4, we obtain

$$\langle T_n^+, \sigma_\varepsilon \rangle = -e^{\frac{\pi n}{6}} \operatorname{Im} \sum_{j=0}^{\infty} 2\pi i \frac{e^{in(i\frac{\pi}{6}+ij\frac{\pi}{3}+\ln\varepsilon+8^n)}}{i\frac{\pi}{6}+ij\frac{\pi}{3}+\ln\varepsilon+8^n} \cdot \frac{(-1)^j}{6\pi i}\,.$$

Choose $\varepsilon_m = e^{-8^m} (m = 1, 2, \dots)$ and denote

$$A_{nj} = \frac{(-1)^j}{3} \cdot \frac{e^{-nj\frac{\pi}{3}} \cdot e^{in(8^n-8^m)}}{i\frac{\pi}{6}+ij\frac{\pi}{3}+8^n-8^m}\,.$$

So we have

$$\langle T^+, \sigma_\varepsilon \rangle = -\sum_{n=1}^{\infty}\sum_{j=0}^{\infty} \operatorname{Im} A_{nj} \tag{18}$$

with following estimations:

$$\sum_{n=1}^{m-1}\sum_{j=0}^{\infty} |A_{nj}| \le \sum_{n=1}^{m-1}\sum_{j=0}^{\infty} \frac{1}{3}\cdot\frac{e^{-j\frac{\pi}{3}}}{\frac{1}{2}8^m} \le \frac{2}{3}\cdot\frac{m}{8^m}\cdot\frac{1}{1-e^{-\frac{\pi}{3}}}\,,$$

$$A_{m,0} = \frac{1}{3}\frac{1}{i\frac{\pi}{6}}\,,$$

$$\sum_{j=1}^{\infty} |A_{mj}| \le \sum_{j=1}^{\infty} \frac{1}{3}\cdot\frac{e^{-mj\frac{\pi}{3}}}{\frac{\pi}{6}} \le \frac{2}{\pi}\cdot\frac{e^{-m\frac{\pi}{3}}}{1-e^{-\frac{\pi}{3}}}\,,$$

$$\sum_{n=m+1}^{\infty}\sum_{j=0}^{\infty} |A_{nj}| \le \sum_{n=m+1}^{\infty}\sum_{j=0}^{\infty} \frac{1}{3}\cdot\frac{e^{-j\frac{\pi}{3}}}{\frac{1}{2}8^n} = \frac{2}{3}\cdot\frac{1}{1-e^{-\frac{\pi}{3}}}\cdot\frac{8^{-m-1}}{1-\frac{1}{8}}\,.$$

From (18) and from the above estimations one sees that

$$\lim_{m\to\infty} \langle T^+, \sigma_{\varepsilon_m} \rangle = \lim(-\operatorname{Im} A_{m,0}) = \frac{2}{\pi}\,.$$

∎

Remark 8. The sets of functions

$$\left\{x \mapsto \frac{1}{\varepsilon^2}\varrho(\frac{x}{\varepsilon});\quad 0<\varepsilon\le 1\right\}$$

$$\left\{x \mapsto \frac{1}{\varepsilon^4}\sigma(\frac{x}{\varepsilon});\quad 0<\varepsilon\le 1\right\}$$

restricted on the interval (2, 8), are bounded in $\mathcal{B}((2,8))$. It means

$$\left|\left(\frac{\partial}{\partial x}\right)^\alpha \left(\frac{1}{\varepsilon^2}\varrho(\frac{x}{\varepsilon})\right)\right| \le M_\alpha \quad (x\in(2,8),\ \alpha\in\mathbf{N}_0,\ 0<\varepsilon\le 1)$$

and similarly for σ.

Proposition. *If a distribution $F \in \mathcal{E}'(\mathbf{R})$ has a value equal to ν at* 0, *then*

$$\lim_{\varepsilon\downarrow 0}\langle F, \varrho_\varepsilon\rangle = \lim_{\varepsilon\downarrow 0}\langle F, \sigma_\varepsilon\rangle = \nu.$$

PROOF : Without loss of generality we can suppose that

$$\operatorname{supp} F \subset \{x; |x| \le 1\}. \tag{19}$$

By the assumption, we have

$$\lim_{\varepsilon\downarrow 0}\langle F, \varphi_\varepsilon\rangle = \nu \cdot \int \varphi \quad \text{for } \varphi \in \mathcal{D}. \tag{20}$$

Let us decompose (as for η see Definition 1)

$$\varrho(x) = \varphi(k_0, x) + \sum_{k=k_0}^{\infty} \varrho(k, x), \tag{21}$$

where

$$\varphi(k_0, x) = \varrho(x)\cdot \eta(2^{-k_0}x), \tag{22}$$

$$\varrho(k, x) = \varrho(x)\left(\eta(2^{-k-1}x) - \eta(2^{-k}x)\right). \tag{23}$$

So

$$\langle F, \varrho_\varepsilon\rangle = \langle F, \varphi_\varepsilon(k_0, \bullet)] + \sum_{k=k_0}^{\infty}\langle F(x), \frac{1}{\varepsilon}\varrho(k, \frac{x}{\varepsilon})\rangle. \tag{24}$$

By the above remark, the set of functions

$$\{x \mapsto (2^{k-1})^2\varrho(k, 2^{k-1}x);\quad k = 1, 2, \dots\}$$

is bounded in $\mathcal{D}([-8,-2]\cup[2,8])$. The set of distributions

$$\{\mathcal{D}([-8,-2]\cup[2,8]) \ni \varphi \mapsto \langle F, \varphi_\varepsilon\rangle;\quad \varepsilon > 0\}$$

is bounded in $\mathcal{D}'([-8,-2]\cup[2,8])$, because $\varepsilon \mapsto \langle F, \varphi_\varepsilon\rangle$ is a continuous function which goes to 0 as $\varepsilon \downarrow 0$ and which is equal to zero for $\varepsilon > \frac{1}{2}$ (due to (19)). This implies that for some $c > 0$ (independent on ε, k)

$$\left|\langle F(x), \frac{1}{\varepsilon}(2^{k-1})^2\varrho(k, 2^{k-1}\frac{x}{\varepsilon})\rangle\right| \le c,$$

or, putting ε instead of $2^{-k+1}\varepsilon$,

$$\left|\langle F(x), \frac{1}{\varepsilon}\varrho(k, \frac{x}{\varepsilon})\rangle\right| \le 2^{-k+1}c.$$

From this, by (24) we obtain

$$\left|\langle F, \varrho_\varepsilon\rangle - \langle F, \varphi_\varepsilon(k_0, \bullet)\rangle\right| \le \frac{2^{-k_0+1}c}{1-\frac{1}{2}}.$$

Passing to the limit as $\varepsilon_0 \downarrow 0$ and $k_0 \to \infty$, we obtain from (20) and (22) the result for the function ϱ. As for σ, the proof is similar. ∎

Theorem 9. *If $T = \check{T} \in \mathcal{E}'$ is an even distribution for which the Kamiński product with δ exists, then T has at 0 a value ν (and it's known that $\delta \cdot T = \nu\delta$).*

PROOF : Using (7) for the definition of the Kamiński product for $\omega = 1$ in a neighbourhood of zero, we obtain the existence of the limit (and its independence on $\varphi \in \mathcal{D}$, $\int \varphi = 1$)

$$\lim_{\varepsilon \downarrow 0} \frac{1}{2}[\langle T, \varphi_\varepsilon \rangle + T \ast \varphi_\varepsilon(0)] = \lim_{\varepsilon \downarrow 0} \frac{1}{2}[\langle T, \varphi_\varepsilon \rangle + \langle T, \check{\varphi}_\varepsilon \rangle] = \lim_{\varepsilon \downarrow 0} \langle T, \varphi_\varepsilon \rangle .$$

■

Consequences 10. From Propositions 4, 7, 8 we obtain that T has not a value at 0. By Theorem 9, the Kamiński product $\delta \cdot T$ does not exist, while the Tillmann product $\delta \cdot T = 0$ by Theorem 6.

REFERENCES

[1] Colombeau J.F., *Elementary Introduction to New Generalized Functions*, North Holland, Amsterdam, New York, Oxford, 1985.

[2] Wawak R., *On the Colombeau Product of Distributions*, will be published in the proceedings of the conference held in Katowice in June 88.

[3] Oberguggenberger M., *Product of distributions: Nonstandard methods*, Zeitschr. Anal. Anw. **7**, No. 4 (1988), 347–365.

[4] Jelínek J., *Characterization of the Colombeau product of distributions*, Comment. Math. Univ. Carolinae **27** (1986), 377–394.

Department of Mathematics, Charles University, Sokolovská 83, 186 00 Praha 8, Czechoslovakia

(*Received* April 17, 1990)