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# A note on the Ramsey-type theorem of Erdös

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*Abstract.* If $\mathcal{F}$ is a normal filter over a cardinal $\kappa$ and $f : [\kappa]^2 \to 2$ is a colouring, then there is a set $A \subseteq \kappa$ that is either infinite and homogeneous in 0 or of positive $\mathcal{F}$-measure (= meets every $F \in \mathcal{F}$) and homogeneous in 1, respectively. If $\mathcal{F}$ is a filter of club sets over an ordinal of uncountable cofinality, the same holds. There are $\kappa$-complete filters not having this property.

*Keywords:* Normal filter, stationary set, partition relation

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Throughout this note, $\kappa$ and $\delta$ stand for infinite cardinal or ordinal, respectively, and $\omega$ denotes the first infinite cardinal. For a set $A$, we let $[A]^2 = \{\{x,y\} : x,y \in A, x \neq y\}$. If $f : [A]^2 \to \{0,1\}$ is a mapping, a set $B \subseteq A$ is called homogeneous in 0 (in 1) for $f$ if $f(\{x,y\}) = 0 (= 1)$ for each $\{x,y\} \in [B]^2$, respectively. $|A|$ denotes the cardinality of $A$ and $2 = \{0,1\}$.

For a filter $\mathcal{F}$ over $\delta$, $\mathcal{F}^* = \{\delta - F : F \in \mathcal{F}\}$ is the dual ideal to $\mathcal{F}$ and $\mathcal{F}^+ = \{A \subseteq \delta : A \notin \mathcal{F}^*\}$.

We deal with certain generalization of the Ramsey theorem. This famous theorem asserts that if $f : [\omega]^2 \to 2$ is a mapping such that each set $A \subseteq \omega$ homogeneous in 0 for $f$ is finite, then there is an infinite set $B \subseteq \omega$ homogeneous in 1 for $f$. Erdös, and Dushnik and Miller [**1**] generalized this, showing that if $f : [\kappa]^2 \to 2$ is as above, then there is a set $B \subseteq \kappa$ homogeneous in 1 such that $|B| = \kappa$. Rowbottom (see Kanamori and Magidor [**2**]) showed that if $\kappa$ admits a normal ultrafilter $\mathcal{U}$, then a very strong partition relation holds which implies that if $f : [\kappa]^2 \to 2$ is again as above, then there is $A \in \mathcal{U}$ homogeneous in 1 for $f$.

**1. Definition.** Let $\delta$ be an ordinal, $A \subseteq \delta$ and $\mathcal{F}$ a filter over $\delta$. We write

$$A \to (\omega, \mathcal{F}^+)^2$$

to abbreviate the formula:
"For each mapping $f : [A]^2 \to 2$ there is a set $B \subseteq A$ such that either $B$ is infinite and homogeneous in 0 for $f$ or else $B \in \mathcal{F}^+$ and $B$ is homogeneous in 1 for $f$."

All the mentioned assertions are of the type $\kappa \to (\omega, \mathcal{F}^+)^2$; the relevant filters are $\{A \subseteq \omega : |\omega - A| < \omega\}, \{A \subseteq \kappa : |\kappa - A| < \kappa\}$ and $\mathcal{U}$, respectively. First note that not every filter $\mathcal{F}$ over $\delta$ satisfies $\delta \to (\omega, \mathcal{F}^+)^2$.

**2. Fact.** Let $\kappa$ be an infinite cardinal. Then there is a cf$(\kappa)$-complete filter $\mathcal{F}$ over $\kappa$ such that $\kappa \not\to (\omega, \mathcal{F}^+)^2$.

Proof : Without loss of generality assume that $\kappa$ regular. Provide $\kappa \times \kappa$ by the product order and let $f\{x,y\} = 1$ for $x, y \in \kappa \times \kappa$, if $x < y$ or $y < x$ and $f\{x,y\} = 0$

otherwise. Since each decreasing sequence of ordinals is finite, each set homogeneous in 0 for $f$ is finite. It is routine to show that if $A \subseteq \kappa \times \kappa$ is homogeneous in 1 for $f$, then either $A \subseteq \kappa \times \alpha \cup \alpha \times \kappa$ for some $\alpha < \kappa$ or $|(\kappa \times \alpha \cup \alpha \times \kappa) \cap A| < \kappa$ for each $\alpha < \kappa$. Consequently, if we let $\mathcal{I}$ be the family of sets of the form $A \cup B$ where $A \subseteq \kappa \times \alpha \cup \alpha \times \kappa$ for some $\alpha < \kappa$ and $|(\kappa \times \alpha \cup \alpha \times \kappa) \cap B| < \kappa$ for each $\alpha < \kappa$, then each set homogeneous in 1 for $f$ is a member of $\mathcal{I}$. One can easily verify that $\mathcal{I}$ is a $\kappa$-complete ideal over $\kappa \times \kappa$ and that $\kappa \times \kappa \notin \mathcal{I}$. Hence $\mathcal{F} = \{\kappa \times \kappa - A : A \in \mathcal{I}\}$ is the required filter and $f$ destroys $\kappa \to (\omega, \mathcal{F}^+)^2$. ∎

The purpose of this note is to show that if $\mathcal{F}$ is a normal filter over a cardinal $\kappa$, then $\kappa \to (\omega, \mathcal{F}^+)^2$. Recall that $\mathcal{F}$ is called normal if $\{A \subseteq \kappa : |\kappa - A| < \kappa\} \subseteq \mathcal{F}$ and $\mathcal{F}$ is closed under diagonal intersections, i.e. $\Delta_{\alpha<\kappa} A_\alpha = \{\beta < \kappa : (\forall \alpha < \beta)(\beta \in A_\alpha)\} \in \mathcal{F}$ whenever $A_\alpha \in \mathcal{F}$ for each $\alpha < \kappa$.

**3. Theorem.** *Let $\kappa$ be an infinite cardinal, $\mathcal{F}$ a normal filter over $\kappa$ and $A \in \mathcal{F}^+$. Then $A \to (\omega, \mathcal{F}^+)^2$.*

PROOF : Let $f : [A]^2 \to 2$. For $x \in A$ put $C_0(x) = \{y \in A : f\{x, y\} = 0\}$ and $C_1(x) = \kappa - C_0(x)$. Consider the following condition.

(∗) For each $B \subseteq A$, if $B \in \mathcal{F}^+$, then $B \cap C_0(x) \in \mathcal{F}^+$ for some $x \in B$.

If (∗) is valid, put $A_0 = A$ and for each $n \in \omega$, find $x_n \in A_n$ with $A_n \cap C_0(x_n) \in \mathcal{F}^+$ and let $A_{n+1} = A_n \cap C_0(x_n)$. (∗) ensures this is possible for each $n \in \omega$. Let $B = \{x_n : n \in \omega\}$. Since $x_n \notin A_{n+1}, B$ is infinite. On the other hand, $x_{n+1} \in A_n \subseteq C_0(x_0) \cap \cdots \cap C_0(x_n)$, i.e. $f\{x_{n+1}, x_i\} = 0$ for each $n \in \omega$ and $i \leq n$. Hence $B$ is homogeneous in 0.

If (∗) fails, there is $B \subseteq A, B \in \mathcal{F}^+$ such that $B \cap C_0(x) \in \mathcal{F}^*$ for each $x \in B$. For $\alpha < \kappa$, let $A_\alpha = (\kappa - B) \cup C_1(\min(B - \alpha))$. Then $A_\alpha \in \mathcal{F}$, for $\kappa - A_\alpha = B \cap C_0(\min(B - \alpha))$ and $\min(B - \alpha) \in B$. Since $\mathcal{F}$ is normal, $\Delta_{\alpha<\kappa} A_\alpha \in \mathcal{F}$, and therefore $D = B \cap \Delta_{\alpha<\kappa} A_\alpha \in \mathcal{F}^+$. We show that $D$ is homogeneous in 1. Let $\alpha, \beta \in D$ and $\alpha < \beta$. Then, by the definition of $\Delta$, $\beta \in C_1(\min(B - \alpha)) = C_1(\alpha)$, as required. ∎

If $\mathcal{F}$ and $\mathcal{G}$ are two filters over $\kappa$ and $\mathcal{F} \subseteq \mathcal{G}$, then obviously $\mathcal{G}^+ \subseteq \mathcal{F}^+$. Hence:

**4. Corollary.** *Let $\kappa$ be an infinite cardinal and $\mathcal{F}$ a filter over $\kappa$ which is extendable to a normal filter. Then*

$$\kappa \to (\omega, \mathcal{F}^+)^2.$$

Maybe it is relevant to remark that the filter $\mathcal{F}$ from Fact 2 is not $\kappa^+$-saturated (for there is an almost disjoint family of cardinality $\geq \kappa^+$) and that this lack could be essential: It is known (see Kanamori and Magidor [**2**]) that a $\kappa^+$-saturated $\kappa$-complete filter $\mathcal{F}$ over $\kappa$ is "almost normal" in that there is an incompressible function $f \in^\kappa \kappa$ such that $\{A \subseteq \kappa : f^{-1}(A) \in \mathcal{F}\}$ is normal. So that it remains open, whether the $\kappa$-completeness and $\kappa^+$-saturatedness of $\mathcal{F}$ ensure $\kappa \to (\omega, \mathcal{F}^+)^2$.

We conclude this note with an application of Theorem 2 to stationary sets, which is similar to the theorem of Erdös, Dushnik and Miller, and in fact strengthens it for the case of $\kappa$ regular.

Recall that $F \subseteq \delta$ is called c.u.b. if $F$ is cofinal with $\delta$ and closed in the order topology. If the cofinality of $\delta$ is uncountable, then c.u.b. sets generate the filter which is usually denoted by $\mathrm{Cub}(\delta)$. If $\kappa$ is regular and uncountable, then $\mathrm{Cub}(\kappa)$ is a normal filter, see e.g. Kunen [**3**, II. 6. 14.]. The sets in $\mathrm{Cub}(\delta)^+$ are called stationary sets.

**5. Corollary.** *Let $\delta$ be an ordinal of uncountable cofinality and $A \subseteq \delta$ a stationary set. Then $A \to (\omega, \mathrm{Cub}(\delta)^+)^2$.*

PROOF : Let $\kappa$ be the cofinality of $\delta$. Then there is a cofinal set $C \subseteq \delta$ of order type $\kappa$. Let $t : \kappa \to C$ be the order isomorphism. For $\alpha < \kappa$ limit, put $g(\alpha) = \sup\{t(\beta) : b < \alpha\}$ and, for $\alpha < \kappa$ isolated, put $g(\alpha) = t(\alpha)$. One can easily compute that the map $g : \kappa \to \delta$ is increasing (and, in particular, one-to-one) and $g(\alpha) = \sup\{g(\beta) : \beta < \alpha\}$ for each $\alpha < \kappa$. Also $\sup g = \delta$. This shows that $g$ transfers $\mathrm{Cub}(\kappa)$ to $\mathrm{Cub}(\delta)$ and hence stationary sets to stationary sets.

For $f : [\delta]^2 \to 2$, we define $f^* : [\kappa]^2 \to 2$ by $f^*\{x, y\} = f\{gx, gy\}$. Let $A \subseteq \delta$ be stationary in $\delta$. Then $g^{-1}A = \{\alpha < \kappa : g\alpha \in A\}$ is stationary in $\kappa$ and according to Theorem 3 either (a) there is infinite $B \subseteq g^{-1}A$ homogeneous in 1 for $f$, or (b) there is stationary (in $\kappa$) $D \subseteq g^{-1}A$ homogeneous in 1 for $f$. In both (a) and (b), $g[B]\,(g[D])$ is homogeneous in 0 (in 1) for $f$, respectively. If (a) occurs, $g[B]$ is infinite, for $g$ is one-to-one. If (b) occurs, then the above mentioned property of $g$ ensures that $g[D]$ is stationary in $\delta$. ∎

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