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On some properties of the metric dimension

LADISLAV MIŠÍK JR., TIBOR ŽÁČIK

Abstract. In the paper two covering functions $N, M$ defined on a given compact metric space $K$ are studied; their binary logarithms are usually called $\varepsilon$-entropy and $\varepsilon$-capacity of this space, respectively. For a function $u$ with suitable properties a compact countable metric space, for which the function $u$ is the covering function, is constructed. By means of covering functions the both lower $\dim$ and upper $\overline{\dim}$ metric dimensions of $K$ are defined. It is shown that for a given compact metric space $K$ and every $\alpha \in [0, \overline{\dim} K]$ and $\beta \in [0, \dim K]$ there is a compact countable subspace $X$ of $K$ with the unique cluster point such that $\overline{\dim} X = \alpha$ and $\dim X \leq \beta$. Finally, it is shown that there exist compact spaces with arbitrary small $\overline{\dim}$ which are not isometrically embeddable into $\mathbb{R}^m$ for each $m \in \mathbb{N}$.

Keywords: Covering function, metric dimension, entropy dimension, limit capacity

Classification: Primary 54D20, 54F45; Secondary 51K99

Introduction.

There are two well-known numerical characteristics of the "massiveness" of metric spaces: topological dimension $\text{td}$, which is a natural number in any case, and Hausdorff dimension $\text{hd}$, which need not be an integer. In [PS] a new characteristic is defined, which is in [KT] called a lower metric dimension $\dim$. Hereby the upper metric dimension $\overline{\dim}$ was defined here. Both these dimensions are given by some integer-valued functions, the covering functions, defined for totally bounded subsets of a metric space. Binary logarithms of these functions are called an $\varepsilon$-entropy and an $\varepsilon$-capacity ([KT]) of the metric space, respectively. That is why the metric dimension ([CS], [H], [KT], [V]) is also called an entropy dimension ([B], [P], [Y]) or a limit-capacity ([M], [PT]). The basic relations of the different notions of the dimensions for a totally bounded metric space $K$, are given by the inequalities:

$$\text{td } K \leq \text{hd } K \leq \dim K \leq \overline{\dim} K,$$

(see e.g. [P], [V]).

The main difference between $\text{hd}$ and $\dim$ consists in the fact that $\text{hd } X = 0$ for a countable set $X$, while $\dim X$ can be positive. So $\dim$ and $\overline{\dim}$ can better control the partition of points of the metric space. On the other hand it is proved in [V] and [CS] that there exists a perfect subset $X \subset \mathbb{R}$ with prescribed Hausdorff and metric dimensions. In the general case a perfect subset $D$ of a complete metric space $A$ with $\text{hd } A = 0$ and $\dim D = \overline{\dim} A$ is constructed there. Note that in [CS] a slightly different notion of metric dimension is used. Some other different properties of $\text{hd}$ and $\dim$ can be found in [B] as well.

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The aim of this paper is to describe the behavior of the covering functions and some of the properties of $\dim$ and $\widetilde{\dim}$ for compact metric spaces. The main result of the first section is the fact that for every covering-like integer-valued function there is a compact, countable subspace of $l^\infty$ with the unique cluster point, covering function of which is the given function. Further, for compact subspaces of $\mathbb{R}^m$ upper bounds of the "jumps" in points of discontinuity of the covering function $N$ are shown. It is shown in the second section that every compact metric space contains a countable subset fulfilling some requirements given in advance. Then the consequence is the existence of a subset $X \subseteq K$ such that $\dim X = 0$ and $\widetilde{\dim} X = \dim K$. The last theorem of the paper says that in spite of finiteness of upper metric dimension of a metric space $K$ there need not exist an isometrical embedding of $K$ to any finite dimensional Euclidean space $\mathbb{R}^m$.

1. Covering functions $N$ and $M$.

Let $(K,d)$ be a nonempty compact metric space. For $p \in K$ and $r > 0$ denote by $B(p,r)$ an open ball centered in $p$ with radius $r$ and by $\overline{B(p,r)}$ its closure. Let $\mathbb{N}$ be the natural numbers and $\mathbb{R}$ the reals. Define the covering function $N(.,K) : \mathbb{R}^+ \to \mathbb{N}$, where $N(r,K)$ for every $r > 0$ denotes the least number of open balls with radius $r$ covering $K$.

The compactness of $K$ implies that $N(r,K)$ is finite for each $r > 0$, so the function $N(.,K)$ is well defined. In general this function is defined for totally bounded spaces. In this paper we shall need one more function $M(.,K) : \mathbb{R}^+ \to \mathbb{N}$:

For a set $F \subseteq K$ denote $\mu(F) = \inf \{d(x,y); x,y \in F, x \neq y\}$. We shall call a finite set $F$ $r$-discrete, for $r > 0$, if $\mu(F) > r$. Now the number $M(r,K)$ means the maximal cardinality of $r$-discrete subsets of $K$.

There are some similar functions defined for totally bounded metric spaces. Their properties and mutual relations can be found in [KT]. Note that in [KT] the functions $N$ and $M$ are defined dually in some sense: $N(r,K)$ is defined by means of closed sets of diameter $2r$ and to get $M(r,K)$ only finite sets $F$ with $\mu(F) > r$ are taken.

In the following Proposition 1 the basic properties of the functions $N$ and $M$ are summarized.

**Proposition 1.** Let $K$ be a compact metric space and $r > 0$. Then the following hold:

(i) Let $A \subseteq K$ be a compact set. Then $N(2r,A) \leq N(r,K)$, $M(r,A) \leq M(r,K)$.

(ii) Let $K = K_1 \cup K_2$, where $K_1, K_2$ are compact subsets of $K$. Then $N(r,K) \leq N(r,K_1) + N(r,K_2)$, $M(r,K) \leq M(r,K_1) + M(r,K_2)$.

(iii) If $K_1, K_2 \subseteq K$ are compact and $r \leq d(K_1,K_2)$, then $N(r,K_1 \cup K_2) = N(r,K_1) + N(r,K_2)$,

$M(r,K_1 \cup K_2) = M(r,K_1) + M(r,K_2)$.

(iv) If $F \subseteq K$ is an $r$-discrete set and $0 < p \leq r$ then $N(p,F) = M(p,F) = |F|$, where $|F|$ denotes the cardinality of $F$.

(v) $M(2r,K) \leq N(r,K) \leq M(r,K)$. 
PROOF: The proof is left to the reader.

Remarks. (a) Note that if \( A \subseteq K \) then \( M(r, A) \leq M(r, K) \) is valid although the inclusion \( A \subseteq K \) does not imply the inequality \( N(r, A) \leq N(r, K) \), as the following example shows: Let \( K = \{0,1,2\} \) and \( A = \{0,2\} \). Then \( B(1,2) \subseteq K \) and hence \( N(2, K) = 1 \), while \( N(2, A) = 2 \).

(b) One can show by induction that (ii) and (iii) holds for any finite number of subsets.

(c) With regard to the inequalities in (v),

\[
M(2r, K) \leq N(r, K) \leq M(r, K),
\]

we also call \( M \) the covering function.

The basic behavior of the functions \( N(., K) \) and \( M(., K) \) for a fixed compact \( K \) is given in the following:

**Theorem 1.** The function \( N(., K) : \mathbb{R}^+ \rightarrow N \) is piecewise constant, continuous on the left, nonincreasing and the set of all points of discontinuity of \( N \) can be arranged into a decreasing sequence \( \{r_n\} \), such that \( N(r, K) = 1 \) for \( r > r_1 \). \( K \) is infinite iff \( \{r_n\} \) is infinite, and then \( \lim_{n \to \infty} r_n = 0, \lim_{n \to \infty} N(r_n, K) = \infty \). The same is valid for the function \( M \).

**Proof:** Let \( \{B(y_i, r)\}_{i=1}^{N(r, K)} \) be a covering of \( K \). Then for \( s > r \) \( \{B(y_i, s)\}_{i=1}^{N(r, K)} \) is the covering of \( K \), too. Therefore the function \( N \) is nonincreasing and since the values of \( N \) are integers, \( N \) is piecewise constant. Denote \( Y = \{y_1, \ldots, y_{N(r, K)}\} \) and define a function \( f : K \rightarrow \mathbb{R}_0^+ \) by \( f(x) = d(x, Y) \). The function \( f \) is continuous and attains on \( K \) the maximum \( r^* < r \). Then for an arbitrary \( \rho \in (r^*, r) \) one has \( N(\rho, K) = N(r, K) \) and hence \( N \) is continuous on the left.

Let \( X = \{x_1, \ldots, x_{M(r, K)}\} \) be an \( r \)-discrete set of \( K \) and \( s < r \). Then \( X \) is also \( s \)-discrete set and therefore \( M(s, K) \geq M(r, K) \). So \( M \) is nonincreasing and piecewise constant. Let \( \{s_k\}_{k=1}^{\infty} \) be an increasing sequence of real numbers tending to \( r > 0 \) such that \( M(s_i, K) = M(s_j, K), i \neq j \), and denote by \( X_k \) the corresponding \( s_k \)-discrete set of \( K \). As \( 2^K \) is the compact metric space in Hausdorff metric \( h \), we can choose a convergent subsequence \( \{X_{k_i}\} \) in this metric tending to some set \( X_0 \). Then \( X_0 \) is a finite set with \(|X_0| = |X_{k_i}|, i \geq 1, \) and \( \mu(X_0) \geq r \). So the function \( M \) is continuous on the left.

The last statement of the theorem is obvious.

From the foregoing statement it follows that for a given compact \( K \) the functions \( N, M \) are uniquely determined by their points of discontinuity and by values in these points. The question is, if the opposite assertion is valid, i.e. if for every function \( u : \mathbb{R}^+ \rightarrow N \) with the properties from Theorem 1 we can find a compact metric space \( K \) such that \( u(r) = N(r, K) \) or \( u(r) = M(r, K) \). The following lemma shows the existence of such a space.

**Lemma 1.** Let \( \{r_n\}_{n=1}^{p} \) and \( \{k_n\}_{n=1}^{p} \), for \( p \in \mathbb{N} \) or \( p = \infty \), be two sequences of real numbers such that
\textbf{Theorem 2.} Let $u$ be the function from Lemma 1. Then there exists a countable, compact subspace $L_u$ of the space $l^\infty$, with unique cluster point $\hat{\theta}$, such that $u(r) = N(r, L_u) = M(r, K_u)$.

\textbf{Proof:} Recall that $l^\infty$ is the space of all bounded sequences of real numbers with the supremum metric $\rho$. Denote by $\varepsilon_j$ a sequence from $l^\infty$ having 1 on $j$-th place and 0 on $i$-th place for $i \neq j$; $\hat{\theta} = (0, 0, \ldots)$ is the zero element in $l^\infty$. Taking $K_u$ constructed in Lemma 1 it is sufficient to find an isometry $g : K_u \to l^\infty$. Define $g$ in the following way:

\begin{align*}
\begin{array}{ll}
g(x_0) = \hat{\theta}, \\
g(x_n^i) = r_n \cdot \varepsilon_{k_n-1+i}.
\end{array}
\end{align*}

It is well known that every metric space can be isometrically embedded into some Banach space. In our case there is one Banach space into which any space $K$ from Lemma 1 can be embedded.
We have
\[ \rho(g(x_0), g(x^i_n)) = \rho(\Theta, r_n \cdot \varepsilon_{k_n-1+i}) = r_n = d(x_0, x^i_n) \]
and
\[ \rho(g(x^i_n), g(x^j_m)) = \rho(r_n \cdot \varepsilon_{k_n-1+i}, r_m \cdot \varepsilon_{k_m-1+j}) = \max\{r_n, r_m\} = r_{\min\{n,m\}} = d(x^i_n, x^j_m). \]
It follows that \( g \) is an isometry and \( N(r, L_u) = N(r, g(K_u)) = N(r, K_u) = u(r) \), where \( L_u = g(K_u) \).

The compact subsets of \( \mathbb{R}^m \) play an important role in mathematics. That is why it would be useful to have some criterions for a given compact metric space to be isometrically embeddable into \( \mathbb{R}^m \) for some \( m \geq 1 \). In Theorem 3 we give only a necessary condition for it.

We shall consider the space \( \mathbb{R}^m \) with an arbitrary metric derived from some norm on \( \mathbb{R}^m \). This gives for any two such metrics \( d_1, d_2 \) the existence of constants \( m, M \) such that
\[ m \cdot d_1(x, y) \leq d_2(x, y) \leq M \cdot d_1(x, y) \]
for all \( x, y \in \mathbb{R}^m \). Moreover, each such metric \( d \) is invariant with respect to translation and
\[ d(\alpha x, \alpha y) = \alpha \cdot d(x, y) \]
for any \( \alpha \in \mathbb{R}^+ \) and \( x, y \in \mathbb{R}^m \). This implies, for an affine mapping \( f : \mathbb{R}^m \to \mathbb{R}^m \) given by
\[ f(x) = \alpha x + b, \ \alpha \in \mathbb{R}, \ b \in \mathbb{R}^m, \]
the equality
\[ (2) \]
\[ N(r, A) = N(\alpha r, f(A)), \]
whenever \( A \) is a compact subset of \( \mathbb{R}^m \) and \( r > 0 \).

**Proposition 2.** Let \( d_1, d_2 \) be metrics on \( \mathbb{R}^m \), let \( N_1, N_2 \) be the corresponding covering functions, let \( A \subset \mathbb{R}^m \) be a compact subset. Then

(i) there exists a constant \( k \in \mathbb{N} \) such that \( N_2(r, A) \leq k \cdot N_1(r, A), r > 0 \).

(ii) for an arbitrary \( c \in \mathbb{R}^+ \) there exists a constant \( l_c \in \mathbb{N} \) such that for \( r > 0 \)
\[ N_1(r, A) \leq l_c \cdot N_1(cr, A). \]

**Proof:** (i) Fix \( r > 0 \) and for \( x \in \mathbb{R}^m \) and \( i = 1, 2 \) put \( B_i(x, r) = \{ z \in \mathbb{R}^m ; d_i(x, z) < r \} \), call it a \( d_i \)-ball. Let \( \{ B_i(y_j, r) \}_{j=1}^{N_1(r, A)} \) be a covering of \( A \). Since \( d_1 \) and \( d_2 \) are topologically equivalent, \( \overline{B_1(x, \varepsilon)} \) is compact in \( d_2 \) for every \( x \in \mathbb{R}^m \) and \( \varepsilon > 0 \). Denote \( k = N_2(\frac{1}{2}, \overline{B_1(0, 1)}) \). This implies by (2) that \( \overline{B_1(y_j, r)} \), \( 1 \leq j \leq m \), can be covered by \( k \) \( d \)-balls with radius \( \frac{r}{2} \) and therefore \( \overline{B_1(y_j, r)} \cap A \) can be by Proposition 1 (i) surely covered by at most \( k \) \( d_2 \)-balls with radius \( r \). Then we can cover the whole set \( A \) by \( k \cdot N_1(r, A) \cdot d_2 \)-balls with radius \( r \) and hence \( N_2(r, A) \leq k \cdot N_1(r, A) \).

(ii) Put \( d_2 = c \cdot d_1 \) and apply (i).

**Lemma 2.** Let \( K \) be a compact metric space and let \( r' < r'' \) be two consecutive points of the discontinuity of the function \( N(., K) \). Denote \( q = N(r'', K) \). Then there exist points \( y_1, \ldots, y_q \) in \( K \) such that
\[ \bigcup_{i=1}^{q} B(y_i, r') \subset K \subset \bigcup_{i=1}^{q} \overline{B(y_i, r')}. \]
PROOF: Let \( \{\rho_k\} \) be a sequence of points from the interval \( (r', r'') \) tending to \( r' \). For every \( \rho_k \) there exists \( \{x^k_1, \ldots, x^k_q\} \subseteq K \) such that \( K \subseteq \bigcup_{i=1}^q B(x^k_i, r) \) for \( r \geq \rho_k \). Since \( K^q \) with the supremum metric is a compact metric space, we can choose a convergent subsequence of \( \{(x^k_1, \ldots, x^k_q)\}_{k=1}^\infty \) of points of \( K \); we can assume that the original sequence is convergent. Let \( \lim_{k \to \infty} (x^k_1, \ldots, x^k_q) = (y_1, \ldots, y_q) \) and \( r > r' \).

Since \( x^k_j \to y_j, \quad j = 1, \ldots, q \), and \( \rho_k \to r' \), there exist \( k_j \in N, \quad j = 1, \ldots, q \), such that \( B(x^k_j, \rho_k) \subset B(y_j, r) \) for \( k \geq k_j \). Then for \( k \geq \max\{k_1, \ldots, k_q\} \) we have \( K \subset \bigcup_{i=1}^q B(x^k_i, \rho_k) \subset \bigcup_{i=1}^q B(y_i, r) \). Therefore \( K \subset \bigcup_{i=1}^q B(y_i, r) \).

The following theorem shows that the jumps of the function \( N(., A) \) cannot be arbitrary when the set \( A \) is a compact subset of \( \mathbb{R}^m \), \( m \geq 1 \).

**Theorem 3.** For every \( m \in \mathbb{N} \) there exists \( k_m \in \mathbb{N} \) such that for every \( r > 0 \) and an arbitrary compact set \( A \subset \mathbb{R}^m \) the following inequality holds:

\[
N(r, A) \leq k_m \cdot \lim_{s \to r^+} N(s, A).
\]

**PROOF:** If \( r \) is not a point of discontinuity of \( N(., A) \) then (3) holds for \( k_m = 1 \). If \( r \) is a point of discontinuity, the Lemma 2 implies the existence of points \( y_1, \ldots, y_q \), where \( q = \lim_{s \to r^+} N(s, A) \), with \( A \subset \bigcup_{i=1}^q \overline{B(y_i, r)} \). Proposition 1 (i), (ii) and (2) then imply:

\[
N(r, A) \leq N\left(\frac{r}{2}, \bigcup_{i=1}^q \overline{B(y_i, r)}\right) \leq \sum_{i=1}^q N\left(\frac{r}{2}, \overline{B(y_i, r)}\right) = q \cdot N\left(\frac{1}{2}, \overline{B(0, 1)}\right).
\]

Denote \( k = N\left(\frac{1}{2}, \overline{B(0, 1)}\right) \). For such \( k \) we obtain (3).

2. **Metric dimensions \( \dim \) and \( \overline{\dim} \).**

There are more equivalent definitions of the lower ([B], [H], [V], [P]) and upper ([BT], [H], [V], [Y]) metric dimension. We will use the following definition. Let \( K \) be a compact metric space. Then we put

\[
\dim K = \liminf_{r \to 0^+} \frac{\log N(r, K)}{-\log r},
\]

\[
\overline{\dim} K = \limsup_{r \to 0^+} \frac{\log N(r, K)}{-\log r}.
\]

Taking into account the inequalities (1) one can replace the function \( N \) by the function \( M \) which can be sometimes useful.

**Proposition 3.** Let \( (K, d) \) be a compact metric space and let \( X \subset K \) be its compact subset. Then

(i) \( \dim X \leq \dim K \),

(ii) \( \overline{\dim} X \leq \overline{\dim} K \).
PROOF: We have, by Proposition 1 (i),
\[
\dim X = \liminf_{r \to 0^+} \frac{\log N(r, X)}{-\log r} \leq \liminf_{r \to 0^+} \frac{\log N(r, K)}{-\log r} = \dim K,
\]
which proves (i). (ii) can be proved in the same way. ■

**Proposition 4.** Let \( \mathcal{K} = \bigcup_{i=1}^{n} K_i \), where \( K_i \) are compact subsets of the metric space \( \mathcal{K} \). Then \( \overline{\dim} K = \max_{1 \leq i \leq n} \{ \dim K_i \} \).

**Proof:**
\[
\overline{\dim} K = \limsup_{r \to 0^+} \frac{\log N(r, K)}{-\log r} \leq \limsup_{r \to 0^+} \frac{\log(n \cdot \max_{1 \leq i \leq n} \{ N(r, K_i) \})}{-\log r} = \limsup_{r \to 0^+} \left( \frac{\log n}{-\log r} + \frac{\max_{1 \leq i \leq n} \{ \log N(r, K_i) \}}{-\log r} \right) = \max_{1 \leq i \leq n} \left\{ \limsup_{r \to 0^+} \frac{\log N(r, K_i)}{-\log r} \right\} = \max_{1 \leq i \leq n} \{ \dim K_i \}.
\]

**Remark.** A similar result is not valid for the lower metric dimension, for the counterexample see e.g. \([B]\).

**Corollary 1.** In each compact metric space \( (\mathcal{K}, d) \) there exists a point \( x_0 \) with the property \( \overline{\dim} K = \overline{\dim} (\mathcal{K}, x_0) \overset{\text{def}}{=} \inf \{ \overline{\dim} B(x_0, r) ; r > 0 \} \).

**Proof:** Put \( B_0 = \mathcal{K} \) and suppose that for each \( i = 1, 2, \ldots, n \) we have a closed ball \( B_i \) such that the radius of \( B_i \) is less than or equal to \( 2^{-i} \), \( B_i \subset B_{i-1} \) and \( \overline{\dim} B_i = \dim K \). Let us have a finite covering of \( B_n \) with balls of radius less than or equal to \( 2^{-(n+1)} \). Applying Proposition 4 we can choose a ball \( B_{n+1} \). Take \( x_0 \) to be the unique point in the intersection \( \bigcap_{n=1}^{\infty} B_n \).

By the Lemma 1 we can prescribe the function \( u(r) \) arbitrarily in spirit of Theorem 1, and we are able to find a compact metric space \( K \) with \( N(r, K) = M(r, K) = u(r) \). Now we may ask: What happens if we seek such a space only as a subspace of a given compact metric space? Although we have seen in Proposition 1 (i), Proposition 2 (ii) and Theorem 3 that we are not so free in prescribing the function \( N(r, K) \) or \( M(r, K) \) in this case; the relative great freedom will be stated in Lemma 3 and Theorem 4 where metric dimensions are concerned.

**Lemma 3.** Let \( (\mathcal{K}, d) \) be an infinite compact metric space. Let \( \{ \alpha_n \}_{n=1}^{\infty}, \{ \beta_n \}_{n=1}^{\infty} \) be two sequences from the interval \([0, \overline{\dim} \mathcal{K}]\) and let \( \{ \delta_n \}_{n=1}^{\infty}, \{ \alpha_n \}_{n=1}^{\infty} \) and \( \{ b_n \}_{n=1}^{\infty} \) be sequences of positive real numbers such that \( \delta_n \to 0, \alpha_n \to 0 \) and for each \( n \) the inequality \( b_{n+1} < \alpha_n < b_n \) holds. Then there exists a compact subspace \( X \subset \mathcal{K} \) with the unique cluster point and a decreasing sequence \( \{ \varepsilon_n \}_{n=1}^{\infty} \) of real numbers converging to 0 such that the following hold:
(i) \( \forall n \in \mathbb{N} \quad M(\varepsilon_n, X) \in \left((1/\varepsilon_n)^{\alpha_n - \delta_n}, (1/\varepsilon_n)^{\alpha_n + \delta_n}\right) \),

(ii) \( \forall \alpha > \limsup_{n \to \infty} \alpha_n \exists r_0 > 0; \forall r < r_0 \quad M(r, X) < (1/r)^\alpha \),

(iii) there exists a sequence \( \{k_n\}_{n=1}^\infty \) of natural numbers such that \( \forall n \in \mathbb{N} \) and \( \forall r \in (a_{k_n}, b_{k_n}) \) we have \( M(r, X) < (1/r)^{\beta_n} \).

**Proof:** Let \( x_0 \) be such a point that \( \dim K = \dim (K, x_0) \). For \( r \in \mathbb{R}^+ \) denote \( S_n(r) = \left((1/r)^{\alpha_n - \delta_n}, (1/r)^{\alpha_n + \delta_n}\right) \cap \mathbb{N} \). Note that for fixed \( n \) the cardinality of \( S_n(r) \) grows to infinity as \( r \) tends to 0. We shall construct consecutively by induction positive real numbers \( p_n, e_n, a_k \), a finite set \( K_n \subseteq K \) and a natural number \( k_n \) in such way that the following properties will be true for each \( n \in \mathbb{N} \):

(a) \( a_{k_n} < b_{k_n} < \varepsilon_n / 2 < 2p_n < \min \{a_{k_n-1}, d(x_0, x_n-1)\} \),

(b) \( |S_n(2\rho_n)| \geq 3 \),

(c) \( \max S_n(2\rho_n) > |X_{n-1}| + 1 \)

(d) \( \varepsilon_n = \max \{r \in (0, 2\rho_n]; M(r, \overline{B(x_0, \rho_n)}) + |X_{n-1}| \geq (1/r)^{\alpha_n - \delta_n} + 1\} \),

(e) \( X_n \) is an \( \varepsilon_n \)-discrete set,

(f) \( |X_n| \in S_n(\varepsilon_n) \) and \( |X_n| + 1 \in S_n(\varepsilon_n) \),

(g) \( |X_n| + 1 < (1/b_{k_n})^{\beta_n} \).

Put \( X_0 = 0 \) and \( k_0 = 1 \). Suppose that \( n \in \mathbb{N} \) and \( \rho_i, \varepsilon_i, X_i, k_i \) are constructed for all \( i \in \mathbb{N}, i < n \). Choose \( \rho_n \) fulfilling (b), (c) and, if \( n > 1 \), the last inequality in (a). Put \( B_n = \overline{B(x_0, \rho_n)} \). For each \( \eta < \dim B_n \) and \( d \in \mathbb{R} \) there are arbitrarily small \( r > 0 \) such that \( M(r, B_n) > (1/r)^n + d \). Now the special form of the function \( M \) (see Theorem 1) implies that the set in (d) has the greatest element. Denote this maximum by \( e_n \). Note that if \( 0 < p < q \) are real numbers then \( |S_n(p)| \geq |S_n(q)| - 1 \) and \( \max S_n(p) \geq \max S_n(q) \). Using this, (b) and (c) imply the existence of the minimal nonnegative integer \( q_n \) with

\[
\left(\frac{1}{\varepsilon_n}\right)^{\alpha_n - \delta_n} < q_n + |X_{n-1}| < \left(\frac{1}{\varepsilon_n}\right)^{\alpha_n + \delta_n} - 1.
\]

Now (d) implies \( q_n < M(\varepsilon_n, B_n) \) and therefore there exists a finite \( \varepsilon_n \)-discrete set \( F_n \subseteq B_n \) which does not contain \( x_0 \) with \( |F_n| = q_n \). The last inequality in (a) implies that \( F_n \) and \( X_{n-1} \) are disjoint. Put \( X_n = X_{n-1} \cup F_n \) and we can see that (e) and (f) are fulfilled. Finally choose \( k_n \) fulfilling the conditions (a) and (g). Put \( X = \bigcup_{n=1}^\infty X_n \cup \{x_0\} \). Using Proposition 1 we are going to prove (i)–(iii).

First note that for each \( n \in \mathbb{N} \):

\[
M(r, X) = M(r, X_n \cup X \setminus X_n) \leq M(r, X_n) + M(r, X \setminus X_n) \leq M(r, X_n) + M(r, B_{n+1})
\]

and for \( r \leq \varepsilon_n \) we have \( M(r, X_n) = |X_n| \), for \( r > 2\rho_{n+1} \) \( M(r, B_{n+1}) = 1 \).

(i) By the choice of \( X_n \) and by (f):

\[
\left(\frac{1}{\varepsilon_n}\right)^{\alpha_n - \delta_n} < |X_n| \leq M(\varepsilon_n, X_n) \leq M(\varepsilon_n, X) \leq |X_n| + 1 < \left(\frac{1}{\varepsilon_n}\right)^{\alpha_n + \delta_n},
\]
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and so \( M(\varepsilon_n, X) \in \left( (1/\varepsilon_n)^{\alpha_n-\delta_n}, (1/\varepsilon_n)^{\alpha_n+\delta_n} \right) \).

(ii) Let \( \alpha > \limsup \alpha_n \). Choose \( n_0 \) such that \( \alpha > \alpha_n + \delta \) for each \( n > n_0 \) and put \( r_0 = \varepsilon_{n_0} \). Now let \( r < r_0 \). Then there is an \( n > n_0 \) such that \( r \in (\varepsilon_{n+1}, \varepsilon_n] \). If \( r > 2\rho_{n+1} \) then using (f)

\[
M(r, X) \leq |X_n| + 1 < \left( \frac{1}{\varepsilon_n} \right)^{\alpha_n+\delta_n} \leq \left( \frac{1}{r} \right)^{\alpha_n+\delta_n} < \left( \frac{1}{r} \right)^{\alpha}.
\]

On the other hand if \( r \leq 2\rho_{n+1} \) then using (b) and (d)

\[
M(r, X) \leq M(r, B_{n+1}) + |X_n| < \left( \frac{1}{r} \right)^{\alpha_n+1-\delta_{n+1}} + 1 < \left( \frac{1}{r} \right)^{\alpha_n+1+\delta_{n+1}} < \left( \frac{1}{r} \right)^{\alpha}.
\]

(iii) Let \( r \in [a_k, b_k] \). Then using (a) and (g)

\[
M(r, X) \leq |X_n| + M(r, B_{n+1}) = |X_n| + 1 < \left( \frac{1}{b_{k_n}} \right)^{\beta_n}.
\]

\[\text{Theorem 4. Let } K \text{ be an infinite compact metric space, } \alpha \in [0, \dim K], \beta \in [0, \dim K], \text{ and } \alpha \geq \beta. \text{ There exists a countable, compact set } X \subset K \text{ with the unique cluster point such that } \dim X = \alpha \text{ and } \dim X \leq \beta. \]

\[\text{PROOF : Put } \alpha_n = \alpha, \beta_n = \beta, \delta_n = \frac{1}{n} \text{ for each } n \in \mathbb{N} \text{ and } a_n, b_n \text{ arbitrary with } a_n \to 0, b_{n+1} < a_n < b_n, \text{ and use Lemma 3. The conditions (i) and (ii) imply } \dim X = \alpha \text{ while (iii) implies } \dim X \leq \beta. \text{ This completes the proof.} \]

The following corollary says that each infinite compact metric space with positive upper metric dimension contains a simple infinite compact subspace which is "pathological" in the sense of distinction between upper and lower metric dimension.

\[\text{Corollary 2. Each infinite compact metric space } K \text{ contains a countable compact subspace } X \text{ with unique cluster point for which } \dim X = 0 \text{ and } \dim X = \dim K. \]

While the upper metric dimension for a countably compact subspace with the unique cluster point in Theorem 4 is prescribed exactly, for the lower metric dimension we have only the upper estimate. The following example shows that this restriction is substantial.

\[\text{Example. There is a compact } K \subset \mathbb{R} \text{ with } \dim K = 1 \text{ such that } \dim X = 0 \text{ for each compact subset } X \subset K \text{ with unique cluster point.} \]

\[\text{PROOF : According to } [H], \text{ there are subsets } F_1 \subset [0,1] \text{ and } F_2 \subset [2,3] \text{ with } \dim F_1 = \dim F_2 = 0 \text{ and } \dim (F_1 \cup F_2) = 1. \text{ Then put } K = F_1 \cup F_2. \]

One further pathological property of the metric dimension compared to the topological dimension is given in the following theorem.
Theorem 5. For each $0 < c < \infty$ there exists a compact metric space $K_c$ such that $\dim K_c = c$ and $K_c$ is not isometrically embeddable into $\mathbb{R}^m$ for any $m \in \mathbb{N}$.

PROOF: Let $\{k_m\}_{m=1}^{\infty}$ be a sequence of constants from Theorem 3. Let $\{c_m\}_{m=1}^{\infty}$ be a decreasing sequence of real numbers with $\lim_{m \to \infty} c_m = c$. Construct the sequences $\{l_m\}_{m=1}^{\infty}$ and $\{r_m\}_{m=1}^{\infty}$:

$$l_m = (k_1 + 1) \cdot (k_2 + 1) \cdot \ldots \cdot (k_m + 1)$$
$$r_m = l_m^{1/c_m}.$$

Defining the function

$$u(r) = \begin{cases} 1, & \text{for } r > r_1, \\ l_m, & \text{for } r_{m+1} < r \leq r_m, \end{cases}$$

this fulfills the conditions of Lemma 1 and so there exists a compact metric space $K_c$ (even countable with unique cluster point) such that $N(r, K_c) = u(r)$. Now $\dim K_c = c$, since $u(r) \leq (1/r)^{c_m}$ for $r \leq r_m \land u(r_m) = (1/r_m)^{c_m} > (1/r_m)^{c}$. Moreover,

$$N(r_1, K_c) = k_1 + 1 > k_1 \cdot \lim_{s \to r_1^+} N(s, K_c),$$

and for $m > 1$

$$N(r_m, K_c) = (k_m + 1) \cdot N(r_{m-1}, K_c) > k_m \cdot \lim_{s \to r_m^+} N(s, K_c),$$

and so by (3) $K_c$ cannot have an isometrical image in $\mathbb{R}^m$, $m \geq 1$.

REFERENCES


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