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ON A CLASS OF VARIATIONAL PROBLEMS DEFINED BY POLYNOMIAL LAGRANGIANS

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1. The purpose of this short remark is to study a class of first order variational problems arising in a natural way from differential forms on the total spaces of fibred manifolds. We introduce these problems similarly to the classical papers by Lepage [6] and modern approach of Hermann [1], [2], Śniatycki [8], Trautman [9], and the author [4], [5]. Our results can be briefly paraphrazed as follows. If ϱ is a differential form from a considered class (the *Lagrangian*), then a variational description is given of those critical sections γ of the variational problem defined by ϱ, on which the exterior derivative $d\varrho$ vanishes, $d\varrho \circ \gamma = 0$. It is shown, in particular, that the equation $d\varrho \circ \gamma = 0$ for γ can be understood as a consequence of certain symmetry requirements on the critical sections, in the sense of a definition by Trautman [10].

In Sections 2 and 3 we have collected some necessary information on the variational problems in fibred manifolds. Sections 4–6 are devoted to the definition and main properties of the class of variational problems we are busy with, and we summarize the results in Section 7.

2. Let $\pi : Y \to X$ be a finite dimensional fibred manifold with oriented base space X, $\dim X = n$, $\dim Y = n + m$. Put $\mathscr{J}^0 Y = Y$ and denote by $\mathscr{J}^r Y$ the r-jet prolongation of π, i.e., the manifold of all r-jets of local sections of π together with the natural projection $\pi_r : \mathscr{J}^r Y \to X$, and by $\pi_{rs} : \mathscr{J}^r Y \to \mathscr{J}^s Y$, $s \leqq r$, the natural projection of jets. Write j^r for the r-jet extension map. If W is a subset of X we denote by $\Gamma_W(\pi)$ the set of all local sections of π defined on a neighbourhood of W (not necessarily the same for all sections).

We shall work with the following definition.

Definition 1. We say that there is given an *rth-order variational problem* $(\pi, \varrho, \mathscr{V})$, if we have the following objects:

1. A fibred manifold $\pi : Y \to X$ with oriented base space X, $\dim X = n$, $\dim Y = n + m$.
2. A differential n-form ϱ on $\mathscr{J}^r Y$.
3. A vector space $\mathscr{V}$ of π-vertical vector fields on Y.

The n-form ϱ is called the *Lagrangian* for π, and the space $\mathscr{V}$ is said to define *admissible variations* for the rth-order variational problem $(\pi, \varrho, \mathscr{V})$.

Let us comment the definition. If $\Omega \subset X$ is a compact submanifold with boundary, of the same dimension as X, we can consider the function

$$\text{(1)} \qquad \Gamma_\Omega(\pi) \ni \gamma \to \int_\Omega j^r\gamma^*\varrho \in R$$

($j^r\gamma^*\varrho$ being the pull-back of ϱ), the *action* of the Lagrangian ϱ, mapping sections of π into the field R of real numbers. Each $\xi \in \mathscr{V}$ generates, in the well-known sense, a one-parameter group α_t of transformations of the manifold Y, and at the same time assignes to each section γ_0 of π a one-parameter family of sections $\gamma_t = \alpha_t \circ \gamma_0$. The families $t \to \gamma_t$ (labelled by ξ) may be regarded from the variational point of view as "*slight deformations*" of γ_0. The study of the behaviour of the action (1) under such "slight one-parameter deformations" represents the main problem of the calculus of variations in fibred manifolds.

Let Ω be an n-dimensional compact submanifold of X with boundary, oriented by the induced orientation, let $\gamma \in \Gamma_\Omega(\pi)$. Let $\xi \in \mathscr{V}$ and denote by α_t its one-parameter group. The vector field ξ gives rise to a function

$$(-\varepsilon, \varepsilon) \ni t \to \int_\Omega j^r\gamma_t^*\varrho \in R$$

defined for some $\varepsilon > 0$ and called the *variation of the action* (1) (induced by the vector field ξ). Let $j^r\xi$ denote the r-jet prolongation of ξ (see, e.g., [3]) defined by

$$j^r\xi(j^r_x\gamma) = \left\{\frac{\mathrm{d}}{\mathrm{d}t}\, j^r_x\alpha_t\gamma\right\}_0$$

(the derivative with respect to t is taken at the point $t = 0$). If we denote by $\vartheta(j^r\xi)\,\varrho$ the Lie derivative of the Lagrangian ϱ with respect to $j^r\xi$, then obviously

$$\left\{\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (j^r\alpha_t\gamma)^*\varrho\right\}_0 = \int_\Omega j^r\gamma^*\vartheta(j^r\xi)\,\varrho,$$

and it is natural to define:

Definition 2. Let γ be a section of π defined on an open subset U of X. We call γ a *critical section*, or an *extremal*, of the variational problem $(\pi, \varrho, \mathscr{V})$, if the condition

$$\int_\Omega j^r\gamma^*\vartheta(j^r\xi)\,\varrho = 0$$

holds for each n-dimensional compact submanifold Ω of X with boundary (provided with the induced orientation), and for all $\xi \in \mathscr{V}$.

3. Let λ be a π_1-horizontal n-form on $\mathscr{J}^1 Y$, (x_i, y_σ) some fibre coordinates on Y, $(x_i, y_\sigma, z_{i\sigma}, z_{ij\sigma})$ the corresponding fibre coordinates on $\mathscr{J}^2 Y$. If λ has an expression

$$\lambda = \mathscr{L}\, \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x$$

then the *Euler form* associated to λ, $\mathscr{E}(\lambda)$, is defined by

$$\mathscr{E}(\lambda) = \mathscr{E}_\sigma(\lambda) \,.\, \omega_\sigma \wedge \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}y_n.$$

where

$$\mathscr{E}_\sigma(\lambda) = \frac{\partial \mathscr{L}}{\partial y_\sigma} - \frac{\partial^2 \mathscr{L}}{\partial x_k \, \partial z_{k\sigma}} - \frac{\partial^2 \mathscr{L}}{\partial y_\lambda \, \partial z_{k\sigma}} \,.\, z_{k\lambda} - \frac{\partial^2 \mathscr{L}}{\partial z_{l\lambda} \, \partial z_{k\sigma}} \,.\, z_{kl\lambda}$$

$$\omega_\sigma = \mathrm{d}y_\sigma - z_{i\sigma}\, \mathrm{d}x_i.$$

In these formulas (as well as throughout this paper) the standard summation convention is understood unless otherwise explicitly designated.

Consider a variational problem $(\pi, \lambda, \mathscr{V})$, where $\mathscr{V}$ is the set of all π-vertical vector fields of compact support. It is known that a section γ of π is a critical section of $(\pi, \lambda, \mathscr{V})$ if and only if it satisfies the *Euler–Lagrange equation*

$$\mathscr{E}(\lambda) \circ j^2\gamma = 0$$

equivalent with the system $\mathscr{E}_\sigma(\lambda) \circ j^2\gamma = 0$, $1 \leqq \sigma \leqq m$, of second-order partial differential equations.

4. Let now ϱ be an n-form on Y. There exists one and only one Lagrangian for π, defined on $\mathscr{J}^1 Y$ and π_1-horizontal, $\boldsymbol{h}(\varrho)$, such that

$$j^1\gamma^* \boldsymbol{h}(\varrho) = \gamma^* \varrho$$

for all sections γ of π (see, e.g., [5]). In this paper we wish to give a description of the variational problems defined by forms of the type $\boldsymbol{h}(\varrho)$, and show that this class of variational problems admits a simple characterization of certain critical sections in terms of the exterior derivative of the initial differential form ϱ.

Suppose that in some fibre coordinates (x_i, y_σ), $1 \leqq i \leqq n$, $1 \leqq \sigma \leqq m$, on Y the n-form ϱ has an expression

$$\varrho = f_0 \, \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n + \\ + \sum_{r=1}^{n} \sum_{s_1 < \ldots < s_r} \sum_{\sigma_1, \ldots, \sigma_r} \frac{1}{r!} f_{\sigma_1}^{s_1} \cdots {}_{\sigma_r}^{s_r} \, \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}y_{\sigma_1} \wedge \ldots \wedge \mathrm{d}y_{\sigma_r} \wedge \ldots \wedge \mathrm{d}x_n,$$

where $\mathrm{d}y_{\sigma_j}$ stands on s_j-th place and the functions $f_{\sigma_1}^{s_1} \cdots {}_{\sigma_r}^{s_r}$ are antisymmetric in the subscripts. Then if γ is a section of π we have

$$\gamma^* \varrho = \left(f_0 + \sum_{r=1}^{n} \sum_{s_1 < \ldots < s_r} \sum_{\sigma_1, \ldots, \sigma_r} f_{\sigma_1}^{s_1} \cdots {}_{\sigma_r}^{s_r} \frac{\partial (y_{\sigma_1} \circ \gamma)}{\partial x_{s_1}} \cdots \frac{\partial (y_{\sigma_r} \circ \gamma)}{\partial x_{s_r}} \right) \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n$$

which shows that in the corresponding fibre coordinates $(x_i, y_\sigma, z_{i\sigma})$ on $\mathscr{J}^1 Y$ the n-form $\boldsymbol{h}(\varrho)$ has the expression

$$\boldsymbol{h}(\varrho) = \mathscr{L} \, \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n,$$

where

$$\mathscr{L} = f_0 + \sum_{r=1}^{n} \sum_{s_1 < \ldots < s_r} \sum_{\sigma_1, \ldots, \sigma_r} f_{\sigma_1}^{s_1} \cdots {}_{\sigma_r}^{s_r} z_{i_1 \sigma_1} \cdots z_{i_r \sigma_r}.$$

Notice that the functions $f_0, f^{s_1}_{\sigma_1} \cdots {}^{s_r}_{\sigma_r}$, are independent of $z_{i\sigma}$; this property is obviously invariant under changes of fibre coordinates on Y and the corresponding changes of the fibre coordinates on $\mathscr{J}^1 Y$. This shows that the variational problems we have introduced belong to the class of the so called polynomial variational problems in fibred manifolds studied by Palais [7].

5. Consider the n-form $\boldsymbol{h}(\varrho)$ expressed as in Section 4. Then we have the following:

Proposition 1. *Let γ be a section of π such that $\mathrm{d}\varrho$ vanishes on γ, i.e., $\mathrm{d}\varrho \circ \gamma = 0$. Then γ is a critical section of the variational problem $(\pi, \boldsymbol{h}(\varrho), \mathscr{V})$, where $\mathscr{V}$ is the set of all π-vertical vector fields of compact support.*

Proof. After some calculation we can obtain the following coordinate expression for the Euler form:

$$\mathscr{E}_\sigma \boldsymbol{h}(\varrho) = \frac{\partial f_0}{\partial y_\sigma} - \frac{\partial f^k_\sigma}{\partial x_k} + \sum_{r=1}^{n-1} \sum_{s_1 < \ldots < s_r} \sum_{\sigma_1, \ldots, \sigma_r} \left(\frac{\partial f^{s_1}_{\sigma_1} \cdots {}^{s_r}_{\sigma_r}}{\partial y_\sigma} - \right.$$

$$- \sum_{s<s_1} \frac{\partial f^{s\, s_1}_{\sigma\sigma_1} \cdots {}^{s_r}_{\sigma_r}}{\partial x_s} - \sum_{s_1<s<s_2} \frac{\partial f^{s_1\, s s_2}_{\sigma_1 \sigma\sigma_2} \cdots {}^{s_r}_{\sigma_r}}{\partial x_s} - \ldots - \sum_{s>s_r} \frac{\partial f^{s_1}_{\sigma_1} \cdots {}^{s_r\, s}_{\sigma_r \sigma}}{\partial x_s} -$$

$$\left. - \frac{\partial f^{s_1 s_2}_{\sigma\, \sigma_2} \cdots {}^{s_r}_{\sigma_r}}{\partial y_{\sigma_1}} - \frac{\partial f^{s_1 s_2\, s_3}_{\sigma_1 \sigma\, \sigma_3} \cdots {}^{s_r}_{\sigma_r}}{\partial y_{\sigma_2}} - \ldots - \frac{\partial f^{s_1}_{\sigma_1} \cdots {}^{s_{r-1} s_r}_{\sigma_{r-1} \sigma_r}}{\partial y_{\sigma_r}} \right) z_{s_1\sigma_1} \cdots z_{s_r \sigma_r} +$$

$$+ \left(\frac{\partial f^1_{\sigma_1} \cdots {}^n_{\sigma_n}}{\partial y_\sigma} - \frac{\partial f^{12}_{\sigma\sigma_2} \cdots {}^n_{\sigma_n}}{\partial y_{\sigma_1}} - \frac{\partial f^{1\,23}_{\sigma_1 \sigma\sigma_3} \cdots {}^n_{\sigma_n}}{\partial y_{\sigma_2}} - \ldots - \frac{\partial f^1_{\sigma_1} \cdots {}^{n-1\, n}_{\sigma_{n-1} \sigma}}{\partial y_{\sigma_n}} \right) z_{1\sigma_1} \cdots z_{n\sigma_n}.$$

Similarly

$$\mathrm{d}\varrho = \left(\frac{\partial f_0}{\partial y_\sigma} - \frac{\partial f^k_\sigma}{\partial x_k} \right) \mathrm{d}y_\sigma \wedge \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n + \sum_{r=1}^{n-1} \sum_{s_1 < \ldots < s_r} \sum_{\sigma_1, \ldots, \sigma_r} \frac{1}{r!} \left(\frac{\partial f^{s_1}_{\sigma_1} \cdots {}^{s_r}_{\sigma_r}}{\partial y_\sigma} - \right.$$

$$\left. - \frac{1}{r+1} \left(\sum_{s<s_1} \frac{\partial f^{s\, s_1}_{\sigma\sigma_1} \cdots {}^{s_r}_{\sigma_r}}{\partial x_s} + \sum_{s_1<s<s_2} \frac{\partial f^{s_1\, s s_2}_{\sigma_1 \sigma\sigma_2} \cdots {}^{s_r}_{\sigma_r}}{\partial x_s} + \ldots + \sum_{s>s_r} \frac{\partial f^{s_1}_{\sigma_1} \cdots {}^{s_r\, s}_{\sigma_r \sigma}}{\partial x_s} \right) \right) \times$$

$$\times \mathrm{d}y_\sigma \wedge \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}y_{\sigma_1} \wedge \ldots \wedge \mathrm{d}y_{\sigma_r} \wedge \ldots \wedge \mathrm{d}x_n + \frac{1}{n!} \frac{\partial f^1_{\sigma_1} \cdots {}^n_{\sigma_n}}{\partial y_\sigma} \mathrm{d}y_\sigma \wedge \mathrm{d}y_{\sigma_1} \wedge \ldots \wedge \mathrm{d}y_{\sigma_n}.$$

Performing necessary antisymmetrization and comparing the two expressions we obtain our assertion.

Note that for the class of Lagrangians we consider, the Euler form can be regarded as defined on $\mathscr{J}^1 Y$.

It is clear that if we want the condition $\mathrm{d}\varrho \circ \gamma = 0$ to follow from the system $\mathscr{E}_\sigma(\lambda) \circ j^1\gamma = 0$, $1 \leqq \sigma \leqq m$, of the Euler–Lagrange equations for γ, then we must regard this system as a system of partial differential equations with respect to the variables x_i and y_σ, and of algebraic nature in the variables $z_{i\sigma}$. For this sake we define:

Definition 3. A section δ of π_1 is said to be a *prolongation* of a section γ of π, if $\pi_{10} \circ \delta = \gamma$.

The following is an immediate consequence of this definition and the formulas from the proof of Proposition 1:

Proposition 2. *If all prolongations δ of a section γ of π satisfy the condition $\mathscr{E}(\lambda) \circ \delta =$* $= 0$, then $\mathrm{d}\varrho \circ \gamma = 0$.

6. In the sequel we shall be busy with a variational interpretation of Proposition 2. It suggests that we should consider for this an appropriate variational problem for sections of the 1-jet prolongation π_1 of the fibred manifold $\pi : Y \to X$. Accordingly, we shall examine the variational problem $(\pi_1, \boldsymbol{h}(\varrho), \mathscr{V}_1)$ of order 0 defined by the following objects:

1. The fibred manifold $\pi_1 : \mathscr{J}^1 Y \to X$.
2. The differential n-form $\boldsymbol{h}(\varrho)$, where ϱ is an n-form on Y.
3. The set $\mathscr{V}_1$ of all 1-jet prolongations of π-vertical vector fields of compact support.

The following is a direct consequence of the fact that "admissible variations" of the problem $(\pi_1, \boldsymbol{h}(\varrho), \mathscr{V}_1)$ are essentially the same as "admissible variations" of the initial problem $(\pi, \varrho, \mathscr{V})$.

Proposition 3. *A section δ of π_1 is a critical section of the variational problem $(\pi_1, \boldsymbol{h}(\varrho), \mathscr{V}_1)$ if and only if $\mathscr{E}(\boldsymbol{h}(\varrho)) \circ \delta = 0$.*

Proof. If a π-vertical vector field ξ is expressed as

$$\xi = \xi_\sigma \frac{\partial}{\partial y_\sigma},$$

then the Lie derivative $\vartheta(j^1\xi)\, \lambda$ of a π_1-horizontal n-form λ is expressed as

$$\vartheta(j^1\xi)\, \lambda = \left(\mathscr{E}_\sigma(\lambda) \,.\, \xi_\sigma + \frac{\partial}{\partial x_k}\left(\frac{\partial \mathscr{L}}{\partial z_{k\sigma}}\, \xi_\sigma\right) + \frac{\partial}{\partial y_\lambda}\left(\frac{\partial \mathscr{L}}{\partial z_{k\sigma}}\, \xi_\sigma\right) . \, z_{k\lambda} + \right.$$
$$\left. + \frac{\partial}{\partial z_{i\lambda}}\left(\frac{\partial \mathscr{L}}{\partial z_{k\sigma}}\, \xi_\sigma\right) . \, z_{ki\lambda}\right) \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n$$

(see, e.g., [3]). We follow here our notation of Section 3. Condition $\mathscr{E}(\boldsymbol{h}(\varrho)) \circ \delta = 0$ now follows from the Stokes' formula for integration of differential forms and from Definition 2.

The desired variational interpretation of sections γ of π such that $\mathrm{d}\varrho \circ \gamma = 0$ can now be obtained by means of certain symmetry requirements on sections of the variational problem $(\pi_1, \boldsymbol{h}(\varrho), \mathscr{V}_1)$.

Definition 4. Let $(\pi, \varrho, \mathscr{V})$ be an rth-order variational problem, γ a critical section of the problem. An automorphism α of Y satisfying $\pi \circ \alpha = \pi$ is called a *symmetry transformation* of γ, if $\alpha \circ \gamma$ is again a critical section of $(\pi, \varrho, \mathscr{V})$.

We apply this definition to automorphisms of $\mathscr{J}^1 Y$ (over X), permuting the set of prolongations of sections of π (in the sense of Definition 3). Let δ be a section of π_1 and $\mathscr{A}_\delta$ denote the set of all automorphisms of $\mathscr{J}^1 Y$ such that

$$\pi_{10} \circ \alpha \circ \delta = \pi_{10} \circ \delta.$$

This means that $\mathscr{A}_\delta$ contains just those automorphisms of $\mathscr{J}^1 Y$ that leave the section $\pi_{10} \circ \delta = \gamma$ of π unchanged but deform the section δ (over $\pi_{10} \circ \delta$). The following is a direct consequence:

Proposition 4. *Let δ be a critical section of $(\pi, \boldsymbol{h}(\varrho), \mathscr{V})$ such that each $\alpha \in \mathscr{A}_\delta$ is a symmetry transformation of δ. Then* $\mathrm{d}\varrho \circ \pi_{10} \circ \delta = 0$.

Proof. For δ satisfying assumptions of Proposition 4 the relation $\mathscr{E}(\boldsymbol{h}(\varrho)) \circ \alpha \circ \delta = 0$ must hold for all $\alpha \in \mathscr{A}_\delta$ (Proposition 3). Comparing with the formulas of Section 5 for $\mathscr{E}(\boldsymbol{h}(\varrho))$ and $\mathrm{d}\varrho$ and using the condition $\pi_{10} \circ \alpha \circ \delta = \pi_{10} \circ \delta$ we obtain, since the functions f_0 and $f^{s_1}_{\sigma_1} \dots {}^{s_r}_{\sigma_r}$. remain unchanged by α, $\mathrm{d}\varrho \circ \pi_{10} \circ \delta = 0$.

7. We are now in a position to summarize our results.

Theorem. *Let $\pi : Y \to X$ be a fibred manifold with oriented base space X,* $\dim X = n$, *$\pi_1 : \mathscr{J}^1 Y \to X$ its δ-jet prolongation. Suppose that we have an n-form ϱ on Y, and denote by $\mathscr{V}$ the space of all π-vertical vector fields of compact support, and by $\mathscr{V}_1$ the space of δ-jet prolongations of all vector fields from $\mathscr{V}$. Then the following three conditions are equivalent:*

1. *For the section γ of π the condition* $\mathrm{d}\varrho \circ \gamma = 0$ *holds.*
2. *The section γ of π is a critical section of the variational problem $(\pi, \boldsymbol{h}(\varrho), \mathscr{V})$ such that each its prolongation δ is a critical section of the variational problem $(\pi_1, \boldsymbol{h}(\varrho), \mathscr{V}_1)$.*
3. *The 1-jet prolongation $j^1\gamma$ of the section γ of π is a critical section of the variational problem $(\pi_1, \boldsymbol{h}(\varrho), \mathscr{V}_1)$ such that each automorphism α of π_1 satisfying the condition $\pi_{10} \circ \alpha j^1\gamma = \gamma$ is a symmetry transformation of $j^1\gamma$.*

REFERENCES

[1] R. Hermann: *Differential geometry and the calculus of variations,* New York (1968).
[2] R. Hermann: *Geometry, physics and systems,* New York (1973).
[3] D. Krupka: *Lagrange theory in fibred manifolds,* Rep. Math. Phys. 2 (1971), 121—133.
[4] D. Krupka: *On generalized invariant transformations,* Rep. Math. Phys. 5 (1974), 355—360.
[5] D. Krupka: *A geometric theory of ordinary first order variational problems in fibered manifold.* I. Critical sections, J. Math. Anal. Appl. 49 (1975), 180—206.
[6] Th. H. J. Lepage: *Sur les champs géodésiques du Calcul des Variations,* Bull. Acad. Roy. Belg., Cl. Sci. V, Sér. 22 (1936), 716, 1036.

[7] R. S. Palais: *Manifolds of sections of fiber bundles and the calculus of variations,* in ''Nonlinear Functional Analysis,'' Proc. Sympos. Pure Math., Vol. XVIII, Part 1, Chicago, Ill., 1968, pp. 195—205, Providence RI, 1970.
[8] J. Śniatycki: *On the geometric structure of classical field theory in Lagrangian formulation,* Proc. Camb. Phil. Soc. (1970), 68, 475—484.
[9] A. Trautman: *Invariance of Lagrangian systems,* in "General Relativity", papers in honour of J. L. Synge, Oxford (1972).

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