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The approximation of functions in the sense of Tchebychev. II

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This paper gives a certain generalization of the (classical) Haar condition and the corresponding theory of the approximation.

The detailed knowledge of all the theory, the notation and the terminology given in the paper [1] is necessary for understanding this paper.

1. THE HAAR DECOMPOSITION CONDITION

Assumption (for § 1.). Let $B$ be a set, $n \in N$, $S = R$ or $S = C$, let $\text{card } B \geq n$. Let $\mathcal{M}$ be a decomposition of the set $B$. Let $\omega \in \mathcal{M} \cup \{\emptyset\}$.

Definition 1. Let $V$ be an $n$-dimensional subspace of $S^B$. We shall say that $V$ satisfies the Haar decomposition condition (with respect to $B$, $\mathcal{M}$, $\omega$) iff every non-trivial polynomial $Q \in V$ has at most $n - 1$ zeros in distinct classes of $\mathcal{M} - \{\omega\}$.

Remark. If $\text{card } (\mathcal{M} - \{\omega\}) \leq n - 1$, then every $n$-dimensional subspace of $S^B$ satisfies the Haar decomposition condition.

Theorem 1. Let $\text{card } (\mathcal{M} - \{\omega\}) > n$. Let $V$ be a subspace of $S^B$ generated by functions $Q_1, \ldots, Q_n \in S^B$. Then the following assertions are equivalent:

(1) $Q_1, \ldots, Q_n$ form a basis of $V$ and $V$ satisfies the Haar decomposition condition.

(2) If $x_1, \ldots, x_n \in B - \omega$ are in distinct classes of $\mathcal{M}$, then $\det Q_k(x_j) \neq 0$.

Proof. The proof of the assertion is simple.

Theorem 2. Let $\text{card } (\mathcal{M} - \{\omega\}) \geq n$. Let $V$ be a subspace of $S^B$, $\dim V \leq n$. Then the following assertions are equivalent:

(1) $\dim V = n$ and $V$ satisfies the Haar decomposition condition.

(2) If $x_1, \ldots, x_n \in B - \omega$ are in distinct classes of $\mathcal{M}$ and if $y_1, \ldots, y_n \in S$ are arbitrary, then there exists exactly one $P \in V$ such that $P(x_j) = y_j$ for $j = 1, \ldots, n$. 

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(3) If \( 1 \leq m \leq n \) and if \( x_1, \ldots, x_m \in B - \omega \) are in distinct classes of \( M \), then \( \dim_{\{x_1, \ldots, x_m\}} V = m \).

**Proof.** We shall prove that (1) implies (3); the rest of the proof is simple. Let (1) hold, let \( x_1, \ldots, x_m \in B - \omega \) be in distinct classes. If \( m = n \), then the assertion \( \dim_{\{x_1, \ldots, x_n\}} V = n \) follows from Theorem 1(2) and from Theorem 23(2) of [1]. Let \( m < n \); we can add such points \( x_{m+1}, \ldots, x_n \in B - \omega \) that the points \( x_1, \ldots, x_n \) are in distinct classes and hence \( \dim_{\{x_1, \ldots, x_n\}} V = n \). By Theorems 23(4) and 23(1) of [1], we have
\[
\dim_{\{x_1, \ldots, x_n\}} V = \dim_{\{x_1, \ldots, x_m\}} V + (n - m) = n \quad \text{and} \quad \dim_{\{x_1, \ldots, x_m\}} V = m \quad \text{and (3) is valid.}
\]

**Remark.** If \( M = \{\{x\}/x \in B\} \) and \( \omega = 0 \), then the Haar decomposition condition is equivalent to the (classical) Haar condition (see [2], p. 25).

**Theorem 3.** Let \( D \subset B \). Let us denote \( N = \{x \cap D/x \in M\} - \{\emptyset\}, \ k = \omega \cap D \). Then \( N \) is a decomposition of \( D \) and \( x \in N \cup \{\emptyset\} \).

Let \( \text{card}(N - \{x\}) \geq n \). Let \( V \) be an \( n \)-dimensional subspace of \( S^B \) satisfying the Haar decomposition condition with respect to \( B, M, \omega \); let us denote \( W = \{Q_d/Q \in V\} \). Then \( W \) is an \( n \)-dimensional subspace of \( S^B \) satisfying the Haar decomposition condition with respect to \( D, N, x \).

**Proof.** The assertion is obvious.

### 2. THE QUOTIENT FUNCTION \( p(x, y) \)

**Assumption** (for § 2.). Let \( B \) be a set, \( n \in N, S = R \) or \( S = C \), let \( \text{card} B \geq n \). Let \( M \) be a decomposition of \( B \); let \( ~ \) denote the equivalence on \( B \) corresponding to \( M \). Let \( \omega \in M \cup \{\emptyset\} \).

Let us suppose that for each \( x, y \in B - \omega \) of the same class of \( M \) there is given a fixed non-zero number \( p(x, y) \in S \). If \( x, y, z \in B - \omega \) and \( x \sim y \) and \( y \sim z \), let the relation \( p(x, y) = p(x, z) = p(y, z) \) hold.

Let us denote \( Y = Y(B, M, \omega, p, S) = \{g \in S^B/g(x) = 0 \text{ for all } x \in \omega, g(x) = p(x, y) \cdot g(y) \text{ for } x, y \in B - \omega \text{ and } x \sim y\} \). (In what follows we shall deal only with the functions of \( Y \).)

**Theorem 4.** (1) We have \( p(x, x) = 1 \) for all \( x \in B - \omega \).

(2) If \( x, y \in B - \omega \) and \( x \sim y \), then \( p(x, y) \cdot p(y, x) = 1 \).

(3) \( Y \) is a subspace of \( S^B \).

(4) If \( M = \{\{x\}/x \in B\} \) and \( \omega = \emptyset \), then \( Y = S^B \).

(5) Let us choose for each class \( \alpha \in M - \{\omega\} \) a fixed point \( x_\alpha \in \alpha \) and a number \( c_\alpha \in S \). Then there exists one and only one \( g \in Y \) such that \( g(x_\alpha) = c_\alpha \) for all \( \alpha \in M - \{\omega\} \).

**Proof.** (1) \( p(x, x) = p(x, x) \cdot p(x, x) \) and \( p(x, x) \neq 0 \), hence \( p(x, x) = 1 \).
(2) We have \( p(x, y) \cdot p(y, x) = p(x, x) = 1 \).

(5) Let \( g \in Y \) be such that \( g(x_\alpha) = c_\alpha \) for all \( \alpha \in \mathcal{M} - \{\omega\} \).

Then \( g(x) = 0 \) for all \( x \in \omega \). If \( x \in B - \omega \), then there exists one and only one \( \alpha \in \mathcal{M} - \{\omega\} \) such that \( x \in \alpha \); we have \( g(x) = p(x, x_\alpha) \cdot g(x_\alpha) = p(x, x_\alpha) \cdot c_\alpha \). Hence there exists at most one \( g \in Y \) such that \( g(x_\alpha) = c_\alpha \) for all \( \alpha \in \mathcal{M} - \{\omega\} \).

On the other hand, let us define \( g \in S^B \) by the relations: \( g(x) = 0 \) for all \( x \in \omega \), \( g(x) = p(x, x_\alpha) \cdot c_\alpha \) for \( x \in \alpha \) where \( \alpha \in \mathcal{M} - \{\omega\} \). Then \( g \in Y \) and \( g(x_\alpha) = c_\alpha \) for all \( \alpha \in \mathcal{M} - \{\omega\} \).

**Definition 2.** A point \( x \in B \) will be called a significant point iff \( x \in B - \omega \) and \( |p(y, x)| \leq 1 \) for all \( y \in B - \omega \) such that \( y \sim x \).

**Theorem 5.** Let \( V \) be an \( n \)-dimensional subspace of \( Y, f \in Y \).

1. We have \( \text{card} (\mathcal{M} - \{\omega\}) \geq n \).
2. If \( x \in \omega \), then \( Q(x) - f(x) = 0 \) for all \( Q \in V \).
3. If \( x, y \in B - \omega \) and \( x \sim y \), then \( Q(y) - f(y) = p(y, x) \cdot [Q(x) - f(x)] \) for all \( Q \in V \).
4. Let \( x \) be a significant point. If \( y \sim x \), then \( |Q(y) - f(y)| \leq |Q(x) - f(x)| \) for all \( Q \in V \).
5. Let \( P \in V \) and \( 0 < \|P - f\| < + \infty \). Let \( x \in B \) be such a point that \( |P(x) - f(x)| = \|P - f\| \) (such a point is called an extreme point of \( B \)). Then \( x \) is a significant point.

**Proof.** (1) Let \( Q_1, \ldots, Q_n \) form a basis of \( V \). By Theorem 21 or [1], there exist points \( x_1, \ldots, x_n \in B \) such that \( \det Q_k(x_j) \neq 0 \). Evidently \( x_j \notin \omega \) for \( j = 1, \ldots, n \). Let us admit that \( x_i \sim x_j \) and \( i \neq j \). Then \( Q_k(x_i) = p(x_i, x_j) \cdot Q_k(x_j) \) for \( k = 1, \ldots, n \), hence \( \det Q_k(x_i) = 0 \), which is a contradiction. Therefore \( x_1, \ldots, x_n \) are in distinct classes of \( \mathcal{M} - \{\omega\} \), hence \( \text{card} (\mathcal{M} - \{\omega\}) \geq n \).

(5) Necessarily \( x \in B - \omega \). Let us admit that there exists \( y \in B \) such that \( y \sim x \) and \( |p(y, x)| > 1 \). Then (3), \( |P(y) - f(y)| = |p(y, x)\cdot|P(x) - f(x)| > |P(x) - f(x)| = \|P - f\| \), which is a contradiction.

**Theorem 6.** Let \( V \) be an \( n \)-dimensional subspace of \( Y \) satisfying the Haar decomposition condition. Let \( f \in Y \), let us denote \( \mu = \min_{Q \in V} \|Q - f\| \).

1. Let \( M \neq \emptyset \) be a minimal set (i.e. \( \mu > 0, f \notin V \)). Then:
   a) \( M \cap \omega = \emptyset \);
   b) the points of \( M \) are in distinct classes of \( \mathcal{M} - \{\omega\} \);
   c) if \( x \in M \), then \( x \) is a significant point;
   d) \( \text{card} M \geq n + 1 \) (and if \( S = R \), then \( \text{card} M = n + 1 \));
   e) \( \dim_{\mathcal{M}} V = n \).

2. Suppose that there exists a minimal set \( M \neq \emptyset \). If \( P \in V \) and \( \|P - f\| = \mu \), then \( P \) has at least \( n + 1 \) extreme points in distinct classes of \( \mathcal{M} - \{\omega\} \).
(3) Suppose that there exists a minimal set \( M \). Then there exists one and only one \( P \in V \) such that \( \| P - f \| = \mu \).

**Proof.** (1) a) Let us admit that \( x \in M \cap \omega \). We have \( \| Q - f \|_{M - \{x\}} = \| Q - f \|_M \) for all \( Q \in V \), hence \( \mu(M - \{x\}) = \mu(M) \), which is a contradiction. Hence \( M \cap \omega = \emptyset \).

b) Let us admit that \( x, y \in M \) and \( x \sim y \). Since \( p(x, y) \cdot p(y, x) = 1 \), we may assume \( |p(x, y)| \leq 1 \). By Theorem 5(3), \( |Q(x) - f(x)| \leq |Q(y) - f(y)| \) for all \( Q \in V \), which is in contradiction with Theorem 16(1) of [1].

c) Let \( x \in M \), let \( P \in V \) be such a polynomial that \( \| P - f \| = \mu \). By Theorems 9(4) and 17 of [1], we have \( \| P(x) - f(x) \| = \mu = \| P - f \| \). Since \( \mu > 0 \), \( x \) is a significant point by Theorem 5(5).

d) Let us admit that \( \text{card } M = m \leq n \). By a), b) and Theorem 2, we have \( \dim_M V = m \). By Theorem 24 of [1], we have \( \mu = \mu(M) = 0 \), which is a contradiction. Hence \( \text{card } M \geq n + 1 \).

e) By a), b), d) and by Theorem 2, we have \( \dim_B V = n \) even for each subset \( D \subseteq M \) with at least \( n \) points. Hence \( \dim_B V = n \), too.

(2) By Theorems 9(4) and 17 of [1], we have \( |P(x) - f(x)| = \mu \) for all \( x \in M \). The assertion follows now from (1a), (1b), (1d).

(3) If \( M = \emptyset \), then \( f \in V \) and the assertion is evident. If \( M \neq \emptyset \), then \( \dim_M V = n \) by (1e) and the assertion follows from Theorem 20(3) of [1].

**Remark.** Theorem 6(3) is a generalization of the classical Haar theorem, namely of the assertion of the sufficiency (see Theorem 19 of [2]). We can generalize also the assertion of the necessity (see Theorem 20 of [2]); we need, however, stronger assumptions. Theorem 7 is not used in the following theory.

**Theorem 7.** Suppose that there exists a number \( d > 0 \) such that for each \( \alpha \in \mathcal{M} - \{\omega\} \) there exists a point \( z_\alpha \in \alpha \) such that \( |p(x, z_\alpha)| \leq d \) for all \( x \in \alpha \).

Let \( D \) be such a subset of \( B \) that \( p(x, y) = 1 \) for \( x, y \in D - \omega \) and \( x \sim y \). Let \( \mathcal{T} \) be a topology on \( D \). Let us denote \( \mathcal{N} = \{ \alpha \cap D : \alpha \in \mathcal{M} \} - \{\emptyset\} \); then \( \mathcal{N} \) is a decomposition of \( D \). Let us denote \( \mathcal{F} = \{ \mathcal{A} \subseteq \mathcal{N} | \cup \mathcal{A} \in \mathcal{T} \} \); then \( \mathcal{F} \) is a topology on \( \mathcal{N} \). Suppose that \( (\mathcal{N}, \mathcal{F}) \) is a compact Hausdorff T-space.

Let \( V \) be an \( n \)-dimensional subspace of \( Y \) not satisfying the Haar decomposition condition (with respect to \( B, \mathcal{M}, \omega \)). Let \( P \) be a non-trivial polynomial of \( V \) having zeros \( x_1, \ldots, x_n \) in distinct classes \( \alpha_1, \ldots, \alpha_n \in \mathcal{M} - \{\omega\} \). Suppose that \( P \) is bounded in \( B \) and continuous in \( D \) with respect to the topology \( \mathcal{T} \).

Then there exists a function \( f \in Y \) continuous in \( D \) with respect to \( \mathcal{T} \) which has infinitely many polynomials of the best approximation in \( V \).

**Proof.** We give only the principle ideas:

1. We may assume \( \| P \| = \frac{1}{d} \), \( x_k = z_{\alpha_k} \) and \( x_k \in D \) for \( \alpha_k \cap D \neq \emptyset \).
2. There exist $b_1, \ldots, b_n \in S$ not all zero such that $\sum_{j=1}^n b_j Q(x_j) = 0$ for all $Q \in V$.

3. There exist a function $g \in S^D$ continuous in $D$ with respect to $\mathcal{F}$ with the following properties: $g(x) = 0$ for all $x \in D \cap \omega$; $g(x) = g(y)$ for $x, y \in D - \omega$ and $x \sim y$; $g(x_k) = \text{sign } b_k$ for $x_k \in D - \neq 0$; $|g(x)| \leq 1$ for all $x \in D$.

4. Let us define $f \in S^D$ in this way: $f(x) = 0$ for $x \in \omega$ and for $x \in x, x \cap D = \emptyset$, $x \notin \{x_1, \ldots, x_n\}$; $f(x) = p(x, z_k) \cdot g(z_k) \cdot (1 - |P(z_k)|)$ for $x \in x, x \cap D \neq \emptyset$; $f(x) = p(x, z_k) \cdot (\text{sign } b_k) \cdot (1 - |P(z_k)|)$ for $x \in x_k, x_k \cap D = \emptyset$. Then $\mu = \min \|Q - f\| = 1$ and $\|aP - f\| = 1$ for all $a \in S$ such that $|a| \leq 1$.

Remark. In Theorem 20 of [2] there are the following assumptions: $B$ is a compact Hausdorff T-space, $V$ is an $n$-dimensional subspace of $C(B)$ not satisfying the (classical) Haar condition. We take $M = \{x/ x \in B\}$, $\omega = \emptyset$, $D = B$. Then $\mathcal{N} = \mathcal{M}$ and $(\mathcal{N}, \mathcal{F})$ is a compact Hausdorff T-space. If $x \sim y$, then $x = y$ and $p(x, y) = 1$.

By Theorem 7, there exists $f \in C(B)$ having infinitely many polynomials of the best approximation in $V$.

3. THE APPROXIMATION

Assumption (for § 3.). Let $n \in \mathbb{N}, S = R$. Let $D$ be a set, $\mathcal{N}$ a decomposition of $D$ ($\sim$ the corresponding equivalence on $D), x \in \mathcal{N} \cup \{\emptyset\}$. Let us suppose that for each $x, y \in D - x$ of the same class of $\mathcal{N}$ there is given a fixed non-zero number $q(x, y) \in R$.

If $x, y, z \in D - x$ and $x \sim y$ and $y \sim z$, let the relation $q(x, z) = q(x, y) \cdot q(y, z)$ hold.

Let $B$ be a subset of $D$. Let us denote $\mathcal{M} = \{x \cap B/ x \in \mathcal{N}\} - \{\emptyset\}$, $\omega = x \cap B$. Let us suppose card $(\mathcal{M} - \{\omega\}) \geq n + 1$.

Let $W$ be an $n$-dimensional subspace of $Y(D, \mathcal{N}, x, q, R)$ satisfying the Haar decomposition condition with respect to $D, \mathcal{N}, x$. Let $Q_1, \ldots, Q_n$ form a basis of $W$.

Suppose that there are given an interval $I \subset R^*$, a set $I \subset D - x$ and a one-one mapping $\xi$ of $J$ onto $I$. Let every $Q \in W$ have the following property: if $Q[\xi(s)]$ is non-zero in a subinterval $(c, d) \subset J$, then $Q[\xi(c)] \cdot Q[\xi(d)] > 0$. (The same is true e.g. when $Q[\xi(s)]$ is continuous in $J$.)

Let $f \in R^B$ be such a function that $f(x) = 0$ for all $x \in \omega$ and $f(x) = q(x, y) \cdot f(y)$ for $x, y \in B - \omega, x \sim y$.

Remark. (1) $\mathcal{M}$ is a decomposition of $B$, $\omega \in \mathcal{M} \cup \{\emptyset\}$.

(2) If $x, y \in B$ and $x \sim y$, then we define $p(x, y) = q(x, y)$. The function $p$ satisfies the requirements of the Assumption for § 2 with respect to $B, \mathcal{M}, \omega$. We have $Y(B, \mathcal{M}, \omega, p, R) = \{g \in R^B/ g(x) = 0 \text{ for } x \in \omega, g(x) = q(x, y) \cdot g(y) \text{ for } x, y \in B - \omega \text{ and } x \sim y\}$, i.e. $f \in Y(B, \mathcal{M}, \omega, p, R)$.

(3) Let us denote $V = \{Q_B/ Q \in W\}$. We can easily prove (by Theorem 3 etc.) that $V$ is an $n$-dimensional subspace of $Y(B, \mathcal{M}, \omega, p, R)$ satisfying the Haar decomposition condition with respect to $B, \mathcal{M}, \omega$. 

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(4) Let us denote \( \mu = \min_{Q \in V} ||Q - f|| \). If \( Q \in W \), let us denote \( ||Q - f|| = \sup_{x \in B} |Q(x) - f(x)| = ||Q_B - f|| \). Then \( \mu = \min_{Q \in W} ||Q - f|| \), too.

(5) The restrictions of the functions \( Q_1, \ldots, Q_n \) to the set \( B \) form a basis of \( V \). When we apply the theorems of [1] and of § 1 and § 2, we must realize that under the basis of \( V \) these restrictions must be understood. However, in the theorems and formulae we shall speak only about the polynomials of \( W \).

(6) For \( x, y \in I \) let us denote: \( x < y \) iff \( \xi^{-1}(x) < \xi^{-1}(y) \), \( x \leq y \) iff \( x < y \) or \( x = y \).

(7) If \( B = D \), then \( \mathcal{M} = \mathcal{N} \), \( \omega = \pi \), \( p = q \), \( V = W \), too. If we consider such a case, we shall speak only about \( B, \mathcal{M}, \omega, p, V \).

(8) If \( I \subset (D - x) \cap \mathbb{R}^* \) is an interval and if each polynomial \( Q \in W \) is continuous in \( I \), we take mostly \( J = I \), \( \xi(x) \equiv x \). Then \( x < y \) iff \( x < y \).

(9) All these assumptions and constructions are necessary for the applications; see § 4.

**Theorem 8.** Let \( x_1 < \ldots < x_{n+1} \) be such points in \( I \) that \( x_1 \leq x \leq x_{n+1} \) and \( x \sim x_k \) implies \( x = x_k \) (for each \( x \in I \) and \( k = 1, \ldots, n+1 \)). For \( k = 1, \ldots, n+1 \) let us denote

\[
C_k = (-1)^{k-1} \begin{vmatrix} Q_1(x_1) & \cdots & Q_1(x_{k-1}) & Q_1(x_k) & Q_1(x_{k+1}) & \cdots & Q_1(x_{n+1}) \\ \vdots & & & & & & \vdots \\ Q_n(x_1) & \cdots & Q_n(x_{k-1}) & Q_n(x_k) & Q_n(x_{k+1}) & \cdots & Q_n(x_{n+1}) \end{vmatrix}.
\]

Then the numbers \( C_1, \ldots, C_{n+1} \) are non-zero and alternate their signs.

**Proof.** Let \( k \in \{1, \ldots, n\} \). For all \( x \in D \) let us put

\[
Q(x) = \begin{vmatrix} Q_1(x_1) & \cdots & Q_1(x_{k-1}) & Q_1(x_k) & Q_1(x_{k+2}) & \cdots & Q_1(x_{n+1}) \\ \vdots & & & & & & \vdots \\ Q_n(x_1) & \cdots & Q_n(x_{k-1}) & Q_n(x_k) & Q_n(x_{k+2}) & \cdots & Q_n(x_{n+1}) \end{vmatrix}.
\]

Then \( Q \in W \). If \( s \in \langle \xi^{-1}(x_k), \xi^{-1}(x_{k+1}) \rangle \), then the points \( x_1, \ldots, x_{k-1}, \xi(s), x_{k+2}, \ldots, x_{n+1} \) are in distinct classes of \( \mathcal{N} - \{\xi\} \), hence \( Q[\xi(s)] \neq 0 \) by Theorem 1. Hence (by the Assumption) \( Q(x_k) \cdot Q(x_{k+1}) > 0 \). We have \( C_k = (-1)^{k-1} Q(x_{k+1}) \)

\( C_{k+1} = (-1)^k Q(x_k) \), hence \( C_k \cdot C_{k+1} < 0 \).

**Remark.** If each class \( \alpha \in \mathcal{N} - \{\xi\} \) has at most one point in the set \( \{x \in I/x_1 \leq x \leq x_{n+1}\} \), then the assumption of Theorem 8 is fulfilled.

**Theorem 9.** Let \( P \in W \) have the following property: there exist points \( x_1 < \ldots < x_{n+1} \) in \( I \) such that \( x_1 \leq x \leq x_{n+1} \) and \( x \sim x_k \) implies \( x = x_k \) (for \( x \in I, k = 1, \ldots, n+1 \)), points \( t_1, \ldots, t_{n+1} \in B \) and a number \( h \in \{-1, +1\} \) such that for \( k = 1, \ldots, n+1 \) we have \( t_k \sim x_k \) and

\[
P(t_k) - f(t_k) = h \cdot \text{sign } q(t_k, x_k) \cdot (-1)^k \cdot d_k, \quad \text{where } d_k \geq 0.
\]
(1) For $k = 1, \ldots, n + 1$ let us denote
\[
D_k = (-1)^{k-1} \left| \begin{array}{ccc}
Q_1(t_1) & \cdots & Q_1(t_{k-1}) \\
\vdots & & \vdots \\
Q_n(t_1) & \cdots & Q_n(t_{k-1}) \\
\end{array} \right|
\]
Then $\mu \geq \mu(t_1, \ldots, t_{n+1}) = \frac{\sum |D_k| \cdot |P(t_k) - f(t_k)|}{\sum |D_k|} \geq \min_{k=1, \ldots, n+1} |P(t_k) - f(t_k)|$.

(2) Let us define the numbers $C_1, \ldots, C_n$ as in Theorem 8. Then $\mu(t_1, \ldots, t_{n+1}) = \frac{\sum |C_k| \cdot |q(x_k, t_k)| \cdot |P(t_k) - f(t_k)|}{\sum |C_k|}$. 

(3) If $|P(t_k) - f(t_k)| = ||P - f||$ for $k = 1, \ldots, n + 1$, then $||P - f|| = \mu$.

Proof. Let us denote $w = q(t_1, x_1) \cdots q(t_{n+1}, x_{n+1})$. Let $k \in \{1, \ldots, n + 1\}$. Then we have $Q_i(t_k) = q(t_k, x_k) \cdot Q_i(x_k)$ for $i = 1, \ldots, n$, hence $D_k = q(t_1, x_1) \cdots q(t_{n-1}, x_{n-1}) \cdot q(t_k, x_k) \cdot \cdots \cdot q(t_{n+1}, x_{n+1}) \cdot C_k = \frac{w}{q(t_k, x_k)} \cdot C_k = w \cdot q(x_k, t_k) \cdot C_k$. By Theorem 8, there exists $a \in \{-1, +1\}$ such that $\operatorname{sign} C_k = a \cdot (-1)^k$ for $k = 1, \ldots, n + 1$, hence $\operatorname{sign} D_k = w \cdot \operatorname{sign} q(x_k, t_k) \cdot a \cdot (-1)^k$. Let us denote $b = a \cdot h \cdot \operatorname{sign} w$. Then for $k = 1, \ldots, n + 1$ we have $b \cdot D_k = \operatorname{sign} q(x_k, t_k) \cdot a \cdot (-1)^k \cdot h \cdot \operatorname{sign} q(x_k, t_k) \cdot (-1)^k \cdot d_k = |D_k| \cdot d_k \geq 0$. Therefore (1) follows from Theorem 28(6) of [1] (we take $t_k, D_k$ instead of $x_k, C_k$).

(2) follows from (1), if we substitute $|D_k| = |w| \cdot |q(x_k, t_k)| \cdot |C_k|$, (3) follows from (1).

**Theorem 10.** Let $P \in W$ have the property: there exist points $x_1 < \ldots < x_{n+1}$ in $I \cap B$ such that $x_1 \leq x \leq x_{n+1}$ and $x \sim x_k$ implies $x = x_k (x \in I, k = 1, n + 1)$ and a number $h \in \{-1, +1\}$ such that for $k = 1, \ldots, n + 1$ we have
\[
P(x_k) - f(x_k) = h \cdot (-1)^k \cdot d_k,
\]
where $d_k \geq 0$.

(1) Let us define $C_1, \ldots, C_{n+1}$ as in Theorem 8. Then $\mu \geq \mu(t_1, \ldots, t_{n+1}) = \frac{\sum |C_k| \cdot |P(x_k) - f(x_k)|}{\sum |C_k|} \geq \min_{k=1, \ldots, n+1} |P(x_k) - f(x_k)|$.

(2) If $|P(x_k) - f(x_k)| = ||P - f||$ for $k = 1, \ldots, n + 1$, then $||P - f|| = \mu$.

Proof. Theorem 10 follows from Theorem 9. We take $t_k = x_k$, hence $q(t_k, x_k) = 1, C_k = D_k$.

**Theorem 11.** Let $M = \{t_1, \ldots, t_{n+1}\}$ be a minimal set (see Theorem 6(1)). Suppose that there exist such points $x_1 < \ldots < x_{n+1}$ in $I$ that $t_k \sim x_k$ for $k = 1, \ldots, n + 1$. Let $P \in W$ and $||P - f|| = \mu$.

(1) Let us define $C_1, \ldots, C_{n+1}$ as in Theorem 8. Then there exists $b \in \{-1, +1\}$ such that for $k = 1, \ldots, n + 1$
\[
P(t_k) - f(t_k) = b \cdot \operatorname{sign} q(t_k, x_k) \cdot \operatorname{sign} C_k \cdot ||P - f||.
\]

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(2) Let \( x_1 \leq x \leq x_{n+1} \) and \( x \sim x_k \) imply \( x = x_k(x \in I, k = 1, \ldots, n + 1) \).

a) Then there exists a number \( h \in \{-1, +1\} \) such that for \( k = 1, \ldots, n + 1 \) we have

\[
P(t_k) - f(t_k) = h \cdot \text{sign } q(t_k, x_k) \cdot (-1)^k \cdot \|P - f\|.
\]

b) Let \( u_1, \ldots, u_{n+1} \in B \) be such points that \( u_k \sim t_k \) for \( k = 1, \ldots, n + 1 \). Then for \( k = 1, \ldots, n + 1 \) we have

\[
P(u_k) - f(u_k) = h \cdot |q(u_k, t_k)| \cdot \text{sign } q(u_k, x_k) \cdot (-1)^k \cdot \|P - f\|.
\]

c) If \( x_1, \ldots, x_{n+1} \in B \), then for \( k = 1, \ldots, n + 1 \) we have

\[
P(x_k) - f(x_k) = h \cdot |q(x_k, t_k)| \cdot (-1)^k \cdot \|P - f\|.
\]

\[\text{Proof.}\] Let us define \( D_1, \ldots, D_{n+1} \) as in Theorem 9, let us denote \( w = q(t_1, x_1) \ldots q(t_{n+1}, x_{n+1}) \). Then \( D_k = w \cdot q(x_k, t_k) \cdot C_k \) for \( k = 1, \ldots, n + 1 \) (see proof of Theorem 9). By Theorem 31(2) of [1] (where we take \( t_k, D_k \) instead of \( x_k, C_k \)), there exists \( a \in \{-1, +1\} \) such that \( P(t_k) - f(t_k) = a \cdot \text{sign } D_k \cdot \|P - f\| = a \cdot \text{sign } w \cdot \text{sign } q(x_k, t_k) \cdot \text{sign } C_k \cdot \|P - f\| \) for \( k = 1, \ldots, n + 1 \). Let us put \( b = a \cdot \text{sign } w \); since \( \text{sign } q(x_k, t_k) = \text{sign } (q(t_k, x_k)) \), our assertion is valid.

(2a) By Theorem 8, there exists \( c \in \{-1, +1\} \) such that \( \text{sign } C_k = c \cdot (-1)^k \) for \( k = 1, \ldots, n + 1 \). Let us denote \( h = b \cdot c \); the assertion follows now from (1).

(2b) \( P(u_k) - f(u_k) = q(u_k, t_k) \cdot |P(t_k) - f(t_k)| = |q(u_k, t_k)| \cdot \text{sign } q(u_k, t_k) \cdot h \cdot \text{sign } q(t_k, x_k) \cdot (-1)^k \cdot \|P - f\| = h \cdot |q(u_k, t_k)| \cdot \text{sign } q(u_k, x_k) \cdot (-1)^k \cdot \|P - f\| \).

(2c) follows from (2b) for \( u_k = x_k \).

**Theorem 12.** (1) Suppose that \( \alpha \cap B \neq \emptyset \) implies \( \alpha \cap I \neq \emptyset \) for each \( \alpha \in \mathcal{N} - \{x\} \).

Let \( M \neq \emptyset \) be a minimal set. Then there exist (significant) points \( t_1, \ldots, t_{n+1} \in B \) (in distinct classes of \( \mathcal{N} - \{x\} \)) and points \( x_1 < \ldots < x_{n+1} \) in \( I \) such that \( M = \{t_1, \ldots, t_{n+1}\} \) and \( t_k \sim x_k \) for \( k = 1, \ldots, n + 1 \).

(2) Suppose that \( \alpha \cap B \neq \emptyset \) implies \( \text{card } (\alpha \cap I) \leq 1 \) for each \( \alpha \in \mathcal{N} - \{x\} \). If \( x_1 < \ldots < x_{n+1} \) are arbitrary points in \( I \) and if there exist points \( t_1, \ldots, t_{n+1} \in B \) such that \( t_k \sim x_k \) for \( k = 1, \ldots, n + 1 \), then \( x_1 \leq x \leq x_{n+1} \) and \( x \sim x_k \) implies \( x = x_k \).

\[\text{Proof.}\] (1) By Theorem 6(1), \( M \) has exactly \( n + 1 \) points which are significant and are in distinct classes of \( \mathcal{M} - \{\omega\} \); let us denote them by \( t_1, \ldots, t_{n+1} \). For \( k = 1, \ldots, n + 1 \) let \( x_k \in \mathcal{N} \) be the class containing \( t_k \). Then \( x_k \neq x \), \( x_k \cap B \neq \emptyset \), hence \( x_k \cap I \neq \emptyset \). Let us choose \( x_k \alpha \propto \cap I \) arbitrarily. The points \( x_1, \ldots, x_{n+1} \) are distinct; we may assume that the points \( t_1, \ldots, t_{n+1} \) are denoted so that \( x_1 < \ldots < x_{n+1} \).

(2) Let \( k \in \{1, \ldots, n + 1\} \). Let \( x_k \in \mathcal{N} \) be the class containing \( x_k \). Then \( x_k \neq x \) and \( x_k \cap B \neq \emptyset \), hence \( x_k \cap I = \{x_k\} \) and the validity of the assertion is proved.
Theorem 13. Suppose that $\alpha \cap B \neq \emptyset$ implies $\text{card}(\alpha \cap I) = 1$ for each $\alpha \in \mathcal{N} - \{x\}$. Suppose that there exists a minimal set, let $P \in W$.

Then $\| P - f \| = \mu$ iff there exist points $t_1, \ldots, t_{n+1} \in B$ (in distinct classes of $\mathcal{N} - \{x\}$), points $x_1 < \ldots < x_{n+1}$ in $I$ and a number $h \in \{-1, +1\}$ such that for $k = 1, \ldots, n + 1$ we have $t_k \sim x_k$ and

$$P(t_k) - f(t_k) = h \cdot \text{sign} q(t_k, x_k) \cdot (-1)^k \cdot \| P - f \|.$$ 

Proof. Let the latter condition be fulfilled. Then we have $\| P - f \| = \mu$ by Theorems 12(2) and 9(3).

Let $\| P - f \| = \mu = 0$. Since card $(\mathcal{M} - \{x\}) \geq n + 1$, there exist distinct classes $\alpha_1, \ldots, \alpha_{n+1} \in \mathcal{N} - \{x\}$ such that $\alpha_k \cap B \neq \emptyset$ for $k = 1, \ldots, n + 1$. Let $\{x_k\} = \alpha_k \cap I$, $t_k \in \alpha_k \cap B$. By a renumeration we can attain that $x_1 < \ldots < x_{n+1}$ and the assertion holds.

Let $\| P - f \| = \mu > 0$. Then the assertion follows from Theorems 12(1), 12(2) and 11(2a).

Theorem 14. Suppose that there exists a minimal set. Then there exists one and only one $P \in W$ such that $\| P - f \| = \mu$.

Proof. By Theorem 6(3) there exists exactly one $Q \in V$ such that $\| Q - f \| = \mu$. Since card $(\mathcal{M} - \{x\}) \geq n + 1$, two distinct polynomials of $W$ cannot coincide in $B$ (see Theorem 2). If $P \in W$ is the only polynomial for which $P_B = Q$, then $P$ is the only polynomial of $W$ such that $\| P - f \| = \mu$.

Theorem 15. Let a subset $A \subset B$ be compact with respect to some topology, let the function $| Q - f |$ be continuous in $A$ for any $Q \in W$. Suppose that if $\alpha \in \mathcal{N} - \{x\}$ and $\alpha \cap B \neq \emptyset$, then there exists a significant point $x \in \alpha \cap A$. Then $A$ is a representative subset (and there exists a minimal set).

Proof. Let $x \in B - \omega$, let $\alpha \in \mathcal{N}$ be the class containing $x$. Then $\alpha \neq x$, $\alpha \cap B \neq \emptyset$, hence there exists a significant point $y \in \alpha \cap B$. As $| q(x, y) | \leq 1$, we have $| Q(x) - f(x) | \leq | Q(y) - f(y) |$ for all $Q \in W$.

Let $x \in \omega$. As $A \neq \emptyset$, we can choose arbitrary $y \in A$ and then $| Q(x) - f(x) | = 0 \leq | Q(y) - f(y) |$ for all $Q \in W$.

Lemma. Let $x, y \in D$ be such points that $| Q(x) - f(x) | \leq | Q(y) - f(y) |$ for all $Q \in W$. Then there exists a number $d \in R$ such that $| d | \leq 1$, $f(x) = d \cdot f(y)$ and $Q(x) = d \cdot Q(y)$ for all $Q \in W$. (The proof is not difficult and we do not give it here.)

Theorem 16. Let $A \subset B$ be a representative subset.

(1) If $x \in B - \omega$, then there exists $y \in A$ such that $x \sim y$ and $| q(x, y) | \leq 1$.

(2) Let the class $\alpha \in \mathcal{N} - \{x\}$ contain at least one significant point (of course with respect to $p$). Then there is a significant point in $\alpha \cap A$, too.
Proof. (1) Let $x \in B - c^0$; let $y \in A$ be such a point that $|Q(x) - f(x)| \leq |Q(y) - f(y)|$ for all $Q \in W$. By Lemma, there exists $d \in R$ such that $|d| \leq 1$ and $Q(x) = d \cdot Q(y)$ for all $Q \in W$. Then $\dim_{x,y} W \leq 1$ and hence $x \sim y$ by Theorem 2. Then $q(x, y) = d$ and $|q(x, y)| \leq 1$.

(2) Let $x \in \alpha$ be a significant point, let $y$ be the point mentioned in (1). Then $|q(x, y)| = 1$. If $z \in \alpha \cap B$, then $|p(z, y)| = |p(z, x)| \cdot |p(x, y)| = |p(z, x)| \leq 1$, hence $y$ is a significant point, too.

4. APPLICATIONS

A. The (Classical) Haar Condition

Assumption. Let $S = R$, $n \in N$, $a, b \in R^*$, $a < b$. Let $W$ be a $n$-dimensional subspace of $C[a, b)$, let every non-trivial polynomial $Q \in W$ have at most $n - 1$ zeros in $<a, b>$ (the Haar condition). Let $B \subset <a, b>$ be compact, card $B \geq n + 1$, let $f \in C(B)$. Let us denote $\mu = \min_{Q \in W} |Q - f|$.

Remark. We take $D = <a, b>$, $\mathcal{N} = \{x | x \in <a, b>\}$, $\alpha = \emptyset$. We have, card $(\mathcal{M} - \{\omega\}) = \text{card } B \geq n + 1$. $W$ is a $n$-dimensional subspace of $Y(D, \mathcal{N}, \alpha, q, R) = R^{<a,b>}$ satisfying the Haar decomposition condition with respect to $D, \mathcal{N}, \alpha$. We take $I = J = <a, b>$; then card $(\alpha \cap I) = 1$ for all $\alpha \in \mathcal{N}$. Since $B$ is a representative subset, there exists a minimal set.

Remark. As $x \sim y$ implies $x = y$, it is not necessary to define $q$ explicitly; we always have $q(x, y) = 1$. The situation will be similar in the other applications; moreover, if $x \sim y$ and $x \neq y$, it is sufficient to define $q(x, y)$; we have $q(y, x) = \frac{1}{q(x, y)}$.

Theorem 17. (1) Let $P \in W$ have the property: there exist points $x_1 < ... < x_{n+1}$ in $B$ such that the numbers $P(x_k) - f(x_k)$ ($k = 1, ..., n + 1$) alternate their signs. Then $\mu \geq \mu(\{x_1, ..., x_{n+1}\}) \geq \min_{k=1, ..., n+1} |P(x_k) - f(x_k)|$.

(2) Let $P \in W$. Then $\|P - f\| = \mu$ iff there exist points $x_1 < ... < x_{n+1}$ in $B$ and $h \in \{-1, +1\}$ such that $P(x_k) - f(x_k) = h \cdot (-1)^k \cdot \|P - f\|$ for $k = 1, ..., n + 1$.

(3) There exists one and only one $P \in W$ such that $\|P - f\| = \mu$.

Proof. (1) follows from Theorems 12(2) and 10(1); (2) follows from Theorem 13 (where $t_k \sim x_k$ implies $t_k = x_k$); (3) follows from Theorem 14.

Remark. If we introduce a basis $Q_1, ..., Q_n$ of $W$, we can get a better estimation in (1) from Theorems 9 and 10. The same will be true of the other applications.
B. Functions with Zero Values at the End Points

Assumption. Let $S = \mathbb{R}$, $n \in \mathbb{N}$, $a, b \in \mathbb{R}^*$, $a < b$. Let $W$ be an $n$-dimensional subspace of $C\langle a, b \rangle$, let $Q(a) = 0$ for all $Q \in W$ and let every non-trivial polynomial $Q \in W$ have at most $n - 1$ zeros in $(a, b)$. Let $B \subset \langle a, b \rangle$ be compact, $\text{card } (B - \{a\}) \geq n + 1$. Let $f \in C(B)$ and $f(a) = 0$ in case $a \in B$. Let us denote $\mu = \min_{Q \in W} \| Q - f \|$. Let $\mathcal{N} = \{x \mid x \in \langle a, b \rangle\}$, $\alpha = \{a\}$; $g$ is defined implicitly. We have $\text{card } (\mathcal{N} - \{\alpha\}) \leq n + 1$. $W$ is an $n$-dimensional subspace of $Y(D, \mathcal{N}, \alpha, g, R) = \{g \in R^{\langle a, b \rangle} \mid g(a) = 0\}$ satisfying the Haar decomposition condition with respect to $D, \mathcal{N}, \alpha$. If $x \in \omega$, then $x = a$ and $x \in B$, hence $f(x) = 0$. We take $I = J = \langle a, b \rangle$, $\xi(s) \equiv s$. Then $\alpha \cap I = \emptyset$, card $(\alpha \cap I) = 1$ for all $\alpha \in \mathcal{N} - \{\alpha\}$. As $B$ is a representative subset, there exists a minimal set.

Theorem 18. (1) Let $P \in W$ have this property: there exist points $x_1 < \ldots < x_{n+1}$ in $B - \{a\}$ such that the numbers $P(x_k) - f(x_k)$ $(k = 1, \ldots, n + 1)$ alternate their signs. Then $\mu \geq \mu((x_1, \ldots, x_{n+1})) \geq \min_{k=1, \ldots, n+1} |P(x_k) - f(x_k)|$.

(2) Let $P \in W$. Then $\| P - f \| = \mu$ if there exist points $x_1 < \ldots < x_{n+1}$ in $B - \{a\}$ and $h \in \{-1, +1\}$ such that $P(x_k) - f(x_k) = h \cdot (-1)^k \cdot \| P - f \|$ for $k = 1, \ldots, n + 1$.

(3) There exists one and only one $P \in W$ such that $\| P - f \| = \mu$.

Proof. (1) follows from Theorems 12(2) and 10(1); (2) follows from Theorems 10(2) and 13 (we have $t_k = x_k$); (3) follows from Theorem 14.

Remark. (1) If we examine the functions being of zero value at $b$, we get similar results.

(2) We can also examine the functions having zero values at both $a$ and $b$. We assume that $Q(a) = Q(b) = 0$ for all $Q \in W$, every non-trivial polynomial $Q \in W$ has at most $n - 1$ zeros in $(a, b)$, $\text{card } (B - \{a, b\}) \geq n + 1$, $f(a) = 0$ in the case $a \in B$ and $f(b) = 0$ in the case $b \in B$. We take $\alpha = \{a, b\}$, $I = J = \langle a, b \rangle$ etc. Theorem 17 will hold also in this case, only the points $x_1 < \ldots < x_{n+1}$ will be in $B - \{a, b\} = B \cap \langle a, b \rangle$.

C. Functions with Proportional Values at the End Points

Assumption. Let $S = \mathbb{R}$, $n \in \mathbb{N}$, $a, b \in \mathbb{R}^*$, $a < b$, $d \in \mathbb{R}$, $d \neq 0$. Let $W$ be an $n$-dimensional subspace of $C\langle a, b \rangle$, let $Q(a) = d \cdot Q(b)$ for all $Q \in W$ and let each non-trivial polynomial $Q \in W$ have at most $n - 1$ zeros in $(a, b)$. Let $B \subset \langle a, b \rangle$ be compact, let $\text{card } B \geq n + 2$ in the case $a, b \in B$ and $\text{card } B \geq n + 1$ in the other cases. Let $f \in C(B)$ and $f(a) = d \cdot f(b)$ in the case $a, b \in B$. Let us denote $\mu = \min_{Q \in W} \| Q - f \|$.
Remark. We take \( D = \langle a, b \rangle, N = \{(x) / x \in (a, b) \} \cup \{a, b\}, x = \emptyset, q(a, b) = d \). We have \( \text{card} (M - \{\omega\}) \geq n + 1 \). \( W \) is an \( n \)-dimensional subspace of \( Y(D, N, x, q, R) = \{g \in R^{a, b} / g(a) = d \cdot g(b)\} \) satisfying the Haar decomposition condition with respect to \( D, N, x \) (as \( Q(b) = 0 \) iff \( Q(a) = 0 \)). The function \( f \) satisfies the requirements. Let us put \( I = J = \langle a, b \rangle, \xi(s) = s \). If \( x_1 < ... < x_{n+1} \) are such points in \( \langle a, b \rangle \) that \( x_1 > a \) or \( x_{n+1} < b \), then \( x_1 \leq x \leq x_{n+1} \) and \( x \sim x_k \) implies \( x = x_k \). If \( x \in N \), then \( \alpha \cap I \neq \emptyset \). Since \( B \) is a representative subset, there exists a minimal set.

**Theorem 19.** (1) Let \( P \in W \) have the following property: there exist points \( x_1 < ... < x_{n+1} \) in \( B \) such that either \( x_1 > a \) or \( x_{n+1} < b \) and the numbers \( P(x_k) - f(x_k) (k = 1, ..., n + 1) \) alternate their signs. Then \( \mu \geq \mu (\{x_1, ..., x_{n+1}\}) \geq \min \{ P(x_k) - f(x_k) \} \).

(2) Let \( P \in W \). Then \( \| P - f \| = \mu \) iff there exist points \( x_1 < ... < x_{n+1} \) in \( B \) and a number \( h \in \{-1, +1\} \) such that either \( x_1 > a \) or \( x_{n+1} < b \) and \( P(x_k) - f(x_k) = h \cdot (-1)^k \cdot \| P - f \| \) for \( k = 1, ..., n + 1 \).

(3) There exists one and only one \( P \in W \) such that \( \| P - f \| = \mu \).

**Proof.** (1) follows from Theorem 10(1); (3) follows from Theorem 14. As for (2): Let \( \| P - f \| = \mu > 0 \). Let \( x_1 < ... < x_{n+1} \) be the points in \( B \) which form a minimal set. Then either \( x_1 > a \) or \( x_{n+1} < b \) (else \( x_1 \sim x_{n+1} \)) and the assertion follows from Theorem 11(2c) (we take \( t_k = x_k \)).

**Remark.** Let \( a, b \in B \). Let \( P \in W \), \( \| P - f \| = \mu > 0 \); then the points \( x_1 < ... < x_{n+1} \) of Theorem 19(2) are significant by Theorem 5(5). Hence, if \( |d| < 1 \), then \( x_1 > a \); if \( |d| > 1 \), then \( x_{n+1} < b \).

**Theorem 20.** We have \( \text{sign} d = (-1)^{n-1} \).

**Proof (we give only the principle ideas).** Let \( Q_1, ..., Q_n \) form a basis of \( W \), let us choose points \( x_1, ..., x_{n-1} \) such that \( a_1 < x_1 < ... < x_{n-1} < b \). For all \( x \in \langle a, b \rangle \) let us put

\[
Q(x) = \begin{vmatrix}
Q_1(x) & Q_1(x_1) & \cdots & Q_1(x_{n-1}) \\
Q_n(x) & Q_n(x_1) & \cdots & Q_n(x_{n-1})
\end{vmatrix}.
\]

Then \( Q \in W \), \( Q(x) \neq \emptyset \) for \( x \in \langle a, b \rangle - \{x_1, ..., x_{n+1}\} \). We can prove that \( Q \) changes the sign at each point \( x_k \): Let e.g. \( Q(x) > 0 \) for \( 0 < |x - x_k| \leq u \). Let \( T \in W \) be such that \( T(x_k) = 1 \) and \( T(x_j) = 0 \) for \( j \neq k \). Then there exists \( c > 0 \) such that \( Q - cT \) has two zeros in \( (x_k - u, x_k) \cap (x_k, x_k + u) \): of course \( x_1, ..., x_{k-1}, x_{k+1}, ..., x_{n-1} \) are zeros of \( Q - cT \), too, which is a contradiction. Hence \( \text{sign} Q(b) = (-1)^{n-1} \cdot \text{sign} Q(a) = (-1)^{n-1} \cdot \text{sign} d \cdot \text{sign} Q(b) \), i.e. \( \text{sign} d = (-1)^{n-1} \).
D. Functions with Proportional values at m Points

Assumption. Let $S = R$, $n \in N$, $a, b \in R^*$, $a < b$, $m \in N$. Let $B \subset \langle a, b \rangle$ be compact, card $B \geq n + 1$. Let us consider distinct points $z_1, \ldots, z_m \in \langle a, b \rangle - B$ and non-zero numbers $d_1, \ldots, d_m \in R$. Let $W$ be an $n$-dimensional subspace of $C\langle a, b \rangle$, let $Q(z_k) = d_k$. $Q(z_1)$ for $k = 2, \ldots, m$ and for all $Q \in W$, let each non-trivial polynomial $Q \in W$ have at most $n - 1$ zeros in $\langle a, b \rangle - \{z_2, \ldots, z_m\}$. Let $f \in C(B)$; let us denote $\mu = \min_{Q \in W} ||Q - f||$.

Remark. We take $D = \langle a, b \rangle$, $\mathcal{N} = \{\{x\}|x \in \langle a, b \rangle - \{z_1, \ldots, z_m\} \cup \{z_1, \ldots, z_m\}, x = 0\}$. Let us denote $d_1 = 1$ and $q(x, y) = d_k/d_j$ for $k, j = 1, \ldots, m$. $W$ is an $n$-dimensional subspace of $Y(D, \mathcal{N}, x, q, R) = \{g \in R^{\langle a, b \rangle}|g(z_k) = d_k \cdot g(z_1)\}$ for $k = 2, \ldots, m$ satisfying the Haar decomposition condition with respect to $D, \mathcal{N}, x$ (as $Q(z_k) = 0$ implies $Q(z_1) = 0$). We have card $(\mathcal{M} - \{x\}) = \text{card } B \geq n + 1$. If $x, y \in B$ and $x \sim y$, then $x = y$, hence there is no condition for $f$. Let us put $I = J = \langle a, b \rangle$, $\xi(s) \equiv s$. If $x \in \mathcal{N}$ and $x \cap B \neq \emptyset$, then $x \neq \{z_1, \ldots, z_m\}$ and card $(x \cap I) = 1$. As $B$ is a representative subset, there exists a minimal set.

Theorem 21. All the three assertions hold also in this case, they are the same as in Theorem 17.

E. Generalized Even and Odd Functions

Assumption. Let $S = R$, $n \in N$, $0 < a \leq +\infty$, $d \in R$, $d \neq 0$. Let $W$ be an $n$-dimensional subspace of $C\langle -a, a \rangle$, let $Q(-x) = d \cdot Q(x)$ for all $x \in (0, a)$ and $Q(0) = 0$ for all $Q \in W$. Let every non-trivial polynomial $Q \in W$ have at most $n - 1$ zeros in $(0, a)$. Let $B \subset \langle -a, a \rangle$ be compact, let card $(\{|x|\}|x \in B, x \neq 0\}) \geq n + 1$. Let $f \in C(B)$ be such that $f(0) = 0$ in case $0 \in B$ and $f(-x) = d \cdot f(x)$ in case $x > 0$, $x \in B, -x \in B$. Let us denote $\mu = \min_{Q \in W} ||Q - f||$.

Remark. We take $D = \langle -a, a \rangle$, $\mathcal{N} = \{\{-x, x\}|x \in (0, a), x = 0\}, q(-x, x) = d$ for $0 < x \leq a$. We have card $(\mathcal{M} - \{x\}) = \text{card } (\{|x|\}|x \in B, x \neq 0\}) \geq n + 1$. $W$ is an $n$-dimensional subspace of $Y(D, \mathcal{N}, x, q, R) = \{g \in R^{\langle -a, a \rangle}|g(0) = 0, g(-x) = d \cdot g(x)\}$ for all $x \in (0, a)\}$ satisfying the Haar decomposition condition with respect to $D, \mathcal{N}, x$. The function $f$ satisfies the requirements. We can take either $I = J = (0, a)$ or $I = J = \langle -a, 0 \rangle$, $\xi(s) \equiv s$. Then $x \cap I = \emptyset$ and card $(x \cap I) = 1$ for all $x \in \mathcal{N} - \{x\}$. As $B$ is a representative subset, there exists a minimal set.

Theorem 22. (1) Let $P \in W$ have this property: there exist points $x_1 < \ldots < x_{n+1}$ in $I$, points $t_1, \ldots, t_{n+1} \in B$ and $h \in \{-1, +1\}$ such that for $k = 1, \ldots, n + 1$ we have either $t_k = s_k$ and $P(t_k) - f(t_k) = h \cdot (-1)^k \cdot d_k, 01 \ t_k = -x_k$ and $P(t_k) - f(t_k) = h \cdot (\text{sign } d) \cdot (-1)^k \cdot d_k$, where $d_k \geq 0$. Then $\mu \geq \mu(\{x_1, \ldots, x_{n+1}\}) \geq \min_{k=1,\ldots,n+1} d_k$. 129
(2) Let \( P \in W \). Then \( \| P - f \| = \mu \) iff there exist points \( x_1 < \ldots < x_{n+1} \) in \( I \), points \( t_1, \ldots, t_{n+1} \in B \) and \( h \in \{-1, +1\} \) such that for \( k = 1, \ldots, n + 1 \) we have either \( t_k = x_k \) and \( P(t_k) - f(t_k) = h \cdot (-1)^k \cdot \| P - f \| \), or \( t_k = -x_k \) and \( P(t_k) - f(t_k) = h \cdot (\text{sign } d) \cdot (-1)^k \cdot \| P - f \| \).

(3) There exists one and only one \( P \in W \) such that \( \| P - f \| = \mu \).

Proof. (1) follows from Theorems 12(2) and 9(1); (2) follows from Theorem 13; (3) follows from Theorem 14.

Remark. Let \( P \in W \), let \( x < -a, a > \) be such a point that \( x \in B \), \( -x \in B \) and \( |P(x) - f(x)| = \| P - f \| > 0 \). If \( |d| < 1 \), then \( x > 0 \); if \( |d| > 1 \), then \( x < 0 \).

Remark. (1) If \( d = -1 \), then the functions are odd.

(2) Let \( d = 1 \). We may change the assumptions in this way: we omit the assumptions \( Q(0) = 0 \) and \( f(0) = 0 \) and assume that every non-trivial polynomial \( Q \in W \) has at most \( n - 1 \) zeros in \( <0, a> \). Then we take \( x = 0, I = <0, a> \) or \( I = <-a, 0> \) etc. Then the functions are even and all the three assertions of Theorem 22 hold also in this case. We can substitute \( \text{sign } d = 1 \) and simplify the assertions (1) and (2).

F. The Approximation on a Generalized Arc

Assumption. Let \( S = R, n \in N, a, b \in R^*, a < b \). Let \( \xi(s) \) be a one-one mapping of \( <a, b> \) onto some set \( I \). Let \( W \) be an \( n \)-dimensional subspace of \( R^I \), let every non-trivial polynomial \( Q \in W \) have at most \( n - 1 \) zeros in \( I \) and for every \( Q \in W \) let the function \( Q[\xi(s)] \) be continuous in \( <a, b> \). Let \( B \subseteq I \) be such a subset that \( \xi^{-1}(B) \) is a compact subset of \( <a, b> \), let card \( B \geq n + 1 \). Let \( f \in R^B \) be such a function that \( f[\xi(s)] \) is continuous in \( \xi^{-1}(B) \). Let us denote \( \mu = \min_{Q \in W} \| Q - f \| \).

Remark. We take \( D = I, \mathcal{N} = \{\{x\}/x \in I\}, x = \emptyset \); \( q \) is defined implicitly. We have card \( (\mathcal{M} \setminus \{\omega\}) = \text{card } B \geq n + 1 \). \( W \) is an \( n \)-dimensional subspace of \( Y(D, \mathcal{N}, x, q, R) = R^I \) satisfying the Haar decomposition condition with respect to \( D, \mathcal{N}, x \). We take \( J = <a, b> \), we have card \( (\alpha \cap I) = 1 \) for all \( \alpha \in \mathcal{N} \).

We transfer the topology from \( <a, b> \) onto \( I \) by means of the mapping \( \xi \). Then each \( Q \in W \) is continuous in \( I, B \) is compact and \( f \) is continuous in \( B \). \( B \) is a representative subset and consequently there exists a minimal set.

Theorem 23. (1) Let \( P \in W \) have this property: there exist points \( x_1, \ldots, x_{n+1} \in B \) such that \( \xi^{-1}(x_1) < \ldots < \xi^{-1}(x_{n+1}) \) and the numbers \( P(x_k) - f(x_k) \) \((k = 1, \ldots, n + 1)\) alternate their signs. Then \( \mu \geq \mu(\{x_1, \ldots, x_{n+1}\}) \geq \min_{k=1,\ldots,n+1} |P(x_k) - f(x_k)| \).

(2) Let \( P \in W \). Then \( \| P - f \| = \mu \) iff there exist points \( x_1, \ldots, x_{n+1} \in B \) and \( h \in \{-1, +1\} \) such that \( \xi^{-1}(x_1) < \ldots < \xi^{-1}(x_{n+1}) \) and \( P(x_k) - f(x_k) = h \cdot (-1)^k \times x \| P - f \| \) for \( k = 1, \ldots, n + 1 \).
(3) There exists one and only one \( P \in W \) such that \( \| P - f \| = \mu \).

**Proof.** The same as that of Theorem 17.

**Remark.** Any theory formulated for an interval can be transferred in this way onto a generalized arc.

**G. Trigonometric Polynomials**

**Theorem 24.** (1) Let \( a_0, \ldots, a_m, b_1, \ldots, b_m \in R \) be not all zero. Then the trigonometric polynomial \( Q(x) = a_0 + \sum_{k=1}^{m} (a_k \cdot \cos kx + b_k \cdot \sin kx) \) has at most \( 2m \) zeros in \( \langle 0, 2\pi \rangle \).

(2) Let \( a_0, \ldots, a_m \in R \) be not all zero. Then the even trigonometric polynomial \( Q(x) = \sum_{k=0}^{m} a_k \cdot \cos kx \) has at most \( m \) zeros in \( \langle 0, \pi \rangle \).

(3) Let \( b_1, \ldots, b_m \in R \) be not all zero. Then the odd trigonometric polynomial \( Q(x) = \sum_{k=1}^{m} b_k \cdot \sin kx \) has at most \( m - 1 \) zeros in \( (0, \pi) \).

**Proof.** Theorem 24 is well-known and can be proved e.g. by expressing \( Q(x) \) by means of algebraic polynomials; we have \( Q(x) = e^{-imx} \cdot \sum_{k=0}^{2m} c_k \cdot (e^{ix})^k \) for (1), \( Q(x) = \sum_{k=0}^{m-1} c_k \cdot (\cos x)^k \) for (2), \( Q(x) = (\sin x) \cdot \sum_{k=0}^{m-1} c_k \cdot (\cos x)^k \) for (3).

**Definition 3.** Let the symbol \( C_{2\pi} \) denote the system of all the continuous functions in \( R \) which are periodic with the period \( 2\pi \).

**Remark.** Let \( W \) mean the system of all the trigonometric polynomials of at most the \( m \)-th degree, let \( f \in C_{2\pi} \). We shall approximate \( f \) by the polynomials \( Q \in W \) in \( R \). As \( \max_{x \in R} | Q(x) - f(x) | = \max_{x \in \langle 0, 2\pi \rangle} | Q(x) - f(x) | \) for all \( Q \in W \), we may investigate the problem only in \( \langle 0, 2\pi \rangle \). This problem can be solved according to 0§4.C, if we take \( a = 0, b = 2\pi, d = 1, B = \langle 0, 2\pi \rangle, n = 2m + 1 = \dim W \).

**Theorem 25.** (1) Let \( P \in W \) have this property: there exist points \( x_1 < \ldots < x_{2m+2} \) in \( \langle 0, 2\pi \rangle \) such that the numbers \( P(x_k) - f(x_k) \) \((k = 1, \ldots, 2m + 2)\) alternate their signs. Then \( \mu \geq \mu(x_1, \ldots, x_{2m+2}) \geq \min_{k=1, \ldots, 2m+2} | P(x_k) - f(x_k) | \).

(2) Let \( P \in W \). Then \( \| P - f \| = \mu \) iff there exist points \( x_1 < \ldots < x_{2m+2} \) in \( \langle 0, 2\pi \rangle \) and \( h \in \{-1, +1\} \) such that \( P(x_k) - f(x_k) = h \cdot (-1)^k \cdot \| P - f \| \) for \( k = 1, \ldots, 2m + 2 \).

(3) There exists one and only one \( P \in W \) such that \( \| P - f \| = \mu \).
Proof. See Theorem 19. To the assertion (2): Theorem 19 admits also the case 
\( x_1 > 0, x_{2m+2} = 2\pi \). Then we can put \( x_0 = 0; \) we have 
\[ P(x_0) − f(x_0) = P(x_{2m+2}) − f(x_{2m+2}) = h \cdot (-1)^{2m+2} \cdot \| P − f \| = h \cdot (-1)^0 \cdot \| P − f \|. \] 
We can take 
\( x_0, \ldots, x_{2m+1} \) and renumerate them.

Remark. Let now \( W \) represent the system of all the even trigonometric polynomials 
of at most the \( m \)-th degree, let \( f ∈ C_{2π} \) be even. We shall approximate \( f \) by the polynomials 
\( Q ∈ W \) in \( R \). As \( \max_{x ∈ R} |Q(x) − f(x)| = \max_{x ∈ ⟨0, π⟩} |Q(x) − f(x)| \) for all \( Q ∈ W \), 
we may investigate the problem only on \( ⟨0, π⟩ \). This problem can be solved according 
to § 4.A, if we take \( a = 0, b = π, B = ⟨0, π⟩, n = m + 1 = \dim W \). We shall not 
formulate the theorem since it would be the same as Theorem 17, if we substitute 
\( B = ⟨0, π⟩, n = m + 1 \).

Remark. Let now \( W \) mean the system of all the odd trigonometric polynomials 
of at most the \( m \)-th degree, let \( f ∈ C_{2π} \) be odd. We shall approximate \( f \) by the polynomials 
\( Q ∈ W \) in \( R \). Since \( \max_{x ∈ R} |Q(x) − f(x)| = \max_{x ∈ ⟨0, π⟩} |Q(x) − f(x)| \) for all \( Q ∈ W \), 
we can investigate the problem only on \( ⟨0, π⟩ \). This problem was mentioned in Remark 
(2) of § 4.B. We take \( a = 0, b = π, B = ⟨0, π⟩, n = m = \dim W \). We can formulate

Theorem 26. (1) Let \( P ∈ W \) have this property: there exist points 
\( x_1 < \ldots < x_{m+1} \) in \( ⟨0, π⟩ \) such that the numbers 
\( P(x_k) − f(x_k) (k = 1, \ldots, m + 1) \) alternate their signs. 
Then \( μ ≥ μ(\{x_1, \ldots, x_{m+1}\}) ≥ \min_{k=1,\ldots,m+1} |P(x_k) − f(x_k)|. \)

(2) Let \( P ∈ W \). Then \( \| P − f \| = μ \) iff there exist points 
\( x_1 < \ldots < x_{m+1} \) in \( ⟨0, π⟩ \) and \( h ∈ \{-1, +1\} \) such that 
\( P(x_k) − f(x_k) = h \cdot (-1)^k \cdot \| P − f \| \) for \( k = 1, \ldots, \ldots, m + 1. \)

(3) There exists one and only one \( P ∈ W \) such that \( \| P − f \| = μ. \)

H. Another Approach to the Trigonometric Polynomials

Remark. Let \( W \) be the system of all the trigonometric polynomials of at most the 
\( m \)-th degree, let \( f ∈ C_{2π} \). We shall approximate \( f \) by the polynomials 
\( Q ∈ W \) in \( R \), let 
\( μ = \min_{Q ∈ W} \| Q − f \|. \)

Let us denote \( n = 2m + 1, S = R, D = B = R \). Let us give a decomposition \( N \) 
of \( R \) by means of the equivalence on \( R: \ x ∼ y \iff \frac{x − y}{2π} \) is integer. Let \( x = 0, \)
\( q(x, y) = 1 \) for \( x ∼ y. \)

\( W \) is an \( n \)-dimensional subspace of \( Y(D,N, x, q, R) = \{g ∈ R^x | g(x) \) is \( 2π \)-periodic 
in \( R \} \) satisfying the Haar decomposition condition with respect to \( D,N, x. \) The function \( f \) 
satisfies the requirements of the Assumption for § 3.

Let \( I = J = ⟨0, 2π⟩, ζ(s) ≡ s. \) We have \( \text{card}(x ∩ I) = 1 \) for all \( x ∈ N. \) The set
$A = \langle 0, 2\pi \rangle$ is a representative subset (e.g. by Theorem 15), hence there exists a minimal set.

We can now derive Theorem 25 once again; (1) follows from Theorems 12(2) and 10(1); (2) follows from (1) and from Theorem 11(2c) (since we may assume $M \subset \subset \langle 0, 2\pi \rangle$ by Theorem 15 of [1]); (3) follows from Theorem 14.

**Remark.** In the same way we can investigate also the even trigonometric polynomials (we take $x \sim y$ iff either $\frac{x - y}{2\pi}$ or $\frac{x + y}{2\pi}$ is integer, $\pi = 0$, $q(x, y) = 1$ for $x \sim y, n = m + 1, I = \langle 0, \pi \rangle, A = \langle 0, \pi \rangle$) and the odd trigonometric polynomials (we take $x = \{k\pi/k \text{ integer}\}, x \sim y$ iff either $x, y \in x$ or $x, y \in R - x$ and one of the numbers $\frac{x - y}{2\pi}, \frac{x + y}{2\pi}$ is integer; if $x, y \in R - x$ and $\frac{x - y}{2\pi}$ is integer, we take $q(x, y) = 1$; if $x, y \in R - x$ and $\frac{x + y}{2\pi}$ is integer, we take $q(x, y) = -1$; $n = m, I = (0, \pi), A = \langle 0, \pi \rangle$).

**Remark.** We can investigate also the approximation on a subset, i.e. $B \subset R, f$ is defined only on $B$. We can solve the problem if $B$ has a representative subset $A$. The compactness of $A$ may be investigated with respect to the usual topology on $R$, but we may introduce also another topology on $R$ and investigate the compactness of $A$ with respect to it.

**Remark.** Let $U$ be the system of all the trigonometric polynomials of at most the $m$-th degree, $g \in C_{2x}$. Let $h(x)$ be a continuous positive real function in $R$. We can approximate the function $f = hg$ by the polynomials of $\{hQ/Q \in U\}$ if we are able to prove the existence of a representative subset (e.g. for $h(x) = e^{-x}$).

5. THE HAAR NODE CONDITION

**Remark.** In what follows we shall consider functions having common zeros (or values) at several points. We distinguish two types of the zeros according to the behaviour of the function in a neighbourhood of the zero point. We consider only real functions.

**Definition 4.** Let $g$ be a real function defined in some set $I \subset R^*$, let $z \in I$ be a point.

1. The point $z$ will be called a *cross zero* of the function $g$ iff there exists a number $u > 0$ such that $\langle z - u, z + u \rangle \subset I, g(z) = 0$ and either $g(x) < 0$ for $x \in \langle z - u, z \rangle$ and $g(x) > 0$ for $x \in (z, z + u)$, or $g(x) > 0$ for $x \in \langle z - u, z \rangle$ and $g(x) < 0$ for $x \in (z, z + u)$.

2. The point $z$ is called a *touch zero* of the function $g$ iff there exists a number $u > 0$ such that $\langle z - u, z + u \rangle \subset I, g(z) = 0$ and either $g(x) > 0$ for $0 < |x - z| \leq u$ or $g(x) < 0$ for $0 < |x - z| \leq u$. 

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Remark. If \( z \) is a cross zero or a touch zero of \( g \), then \( z \) is inside \( I \) and \( g \) has no other zeros in some neighbourhood of \( z \).

**Theorem 27.** (1) Let \( g \) be defined (at least) in \( (a, b) \), let \( a < z < b \). Let \( g(z) = 0 \) and \( g(x) \neq 0 \) for all \( x \in (a, z) \cup (z, b) \). Suppose that if either \( a \leq c \leq d < z \) or \( z < c \leq d \leq b \), then \( g(c) \cdot g(d) > 0 \). Then \( z \) is either a cross zero or a touch zero of \( g \), moreover, \( g(x) \) has a constant sign in \( (a, z) \) and a constant sign in \( (z, b) \).

(2) Let \( g \) be continuous in \( (a, b) \), let \( a < z < b \). Let \( g(z) = 0 \) and \( g(x) \neq 0 \) for all \( x \in (a, z) \cup (z, b) \). Then the assertions of (1) hold.

(3) Let \( g \) have derivatives up to the \( r \)-th order at a point \( z \) (\( r \in \mathbb{N} \)). Let \( g(z) = g'(z) = \ldots = g^{(r-1)}(z) = 0 \), \( g^{(r)}(z) \neq 0 \). If \( r \) is odd (even), then \( z \) is a cross (touch) zero of \( g \).

**Proof.** Assertions (1) and (2) are obvious, (3) follows immediately from a well-known theorem.

**Assumption** (for § 5.). Let \( n \in \mathbb{N} \), \( m \in \mathbb{N}_0 \), let \( I \subset \mathbb{R}^* \) be an interval. Suppose that there are given points \( z_1 < \ldots < z_m \) in \( I \) (called nodes) and numbers \( t_1, \ldots, t_m \in \{1, 2\} \). Let us denote \( I' = I - \{z_1, \ldots, z_m\} \).

**Remark.** Let us denote \( A(I) = \{g \in \mathbb{R}'/if (c, d) \subset I \text{ and } g(x) \neq 0 \text{ for all } x \in (c, d)\} \), then \( g(c) \cdot g(d) > 0 \}, X(I) = \{g \in A(I)/g(z_1) = \ldots = g(z_m) = 0\} \).

(1) We have \( C(I) \subset A(I) \).

(2) Let \( g \in A(I) \). Then \( g(x) \) keeps the sign in each subinterval of \( I \), in which \( g(x) \neq 0 \).

(3) Let \( g \in A(I) \) and let \( z \) be an isolated zero of \( g \) (inside \( I \)). Then \( z \) is either a cross zero or a touch zero of \( g \).

**Definition 5.** Let \( g \in X(I) \). A point \( x \in I \) will be called an additional zero of \( g \) iff either

(1) \( x \in I' \) and \( g(x) = 0 \); or

(2) \( x = z_k \), \( t_k = 1 \) and \( z_k \) is a touch zero of \( g \) (inside \( I \)); or

(3) \( x = z_k \), \( t_k = 2 \) and \( z_k \) is a cross zero of \( g \) (inside \( I \)).

**Remark.** If \( c, d \in I' \) and \( c \leq d \), then \( t(c, d) \) will denote the sum of all \( t_k \) for such \( k \) that \( c < z_k < d \).

**Remark.** If \( x_1 \leq \ldots \leq x_r \) are points in \( I' \) (\( r \geq 2 \)), then \( t(x_1, x_r) = t(x_1, x_2) + + \ldots + t(x_{r-1}, x_r) \).

**Theorem 28.** Let \( g \in X(I) \), let \( c \leq d \) be such points in \( I' \) that the function \( g \) has no additional zero in \( (c, d) \). Then \( \text{sign } g(d) = (-1)^{t(c, d)} \cdot \text{sign } g(c) \neq 0 \).

**Proof.** Suppose that there are exactly \( z_p < \ldots < z_q \) in \( (c, d) \). The function \( g \) keeps the sign in the intervals \( (c, z_p), (z_p, z_{p+1}), \ldots, (z_{q-1}, z_q), (z_q, d) \). Let \( k \in \mathbb{N} \).
If the number of such \( k \in \{p, \ldots, q\} \) for which \( t_k = 1 \) is odd (even), then \( g(c) \cdot g(d) < 0 \) (\( g(c) \cdot g(d) > 0 \)) and \( r(c, d) = t_p + \ldots + t_q \) is odd (even), hence the assertion holds.

**Remark.** The numbers \( t_k \) are of the following meaning. Suppose that \( g \in X(I) \) and \( z_k \) is an isolated zero of \( g \) (inside \( I \)). The number \( t_k \) determines the behaviour of \( g \) in some neighbourhood of \( z_k \) which is necessary for \( z_k \) to be an "allowed" zero of \( g \) (i.e. which is not additional). For \( t_k = 1 \) we allow a cross zero, for \( t_k = 2 \) we allow a touch zero; if \( z_k \) is a zero of the other type, then \( z_k \) is called an additional zero of \( g \).

If \( z_k \) is an end point of the interval \( I \), then \( t_k \) has no meaning.

**Definition 6.** Let \( W \) be an \( n \)-dimensional subspace of \( X(I) \). We shall say that \( W \) satisfies the Haar node condition (with respect to \( I, z_k, t_k \)) iff every non-trivial polynomial \( Q \in W \) has at most \( n - 1 \) additional zeros in \( I \).

**Remark.** If \( m = 0 \), then we have the classical Haar condition.

**Theorem 29.** Let \( W \) be an \( n \)-dimensional subspace of \( X(I) \) satisfying the Haar node condition. Let \( Q_1, \ldots, Q_n \) form a basis of \( W \).

1. If \( a_1, \ldots, a_n \in \mathbb{R} \) are not all zero, then \( \sum_{k=1}^{n} a_k Q_k \) has at most \( n - 1 \) additional zeros in \( I \).
2. If \( x_1, \ldots, x_n \in I' \) are distinct, then \( \det Q_k(x_j) \neq 0 \) and \( \dim_{\{x_1, \ldots, x_n\}} W = n \).
3. If \( x_1, \ldots, x_n \in I' \) are distinct and numbers \( y_1, \ldots, y_n \in \mathbb{R} \) are arbitrary, then there exists one and only one \( P \in W \) such that \( P(x_k) = y_k \) for \( k = 1, \ldots, n \).

**Proof.** All the assertions are obvious.

**Theorem 30.** Let \( W \) be an \( n \)-dimensional subspace of \( X(I) \) satisfying the Haar node condition, let \( Q_1, \ldots, Q_n \) form a basis of \( W \). Let \( x_1 < \ldots < x_{n+1} \) be points in \( I' \). For \( k = 1, \ldots, n+1 \) let us denote

\[
C_k = (-1)^{k-1} \cdot \begin{vmatrix}
Q_1(x_1) & \ldots & Q_1(x_{k-1}) & Q_1(x_{k+1}) & \ldots & Q_1(x_{n+1}) \\
\vdots & & & & & \vdots \\
Q_n(x_1) & \ldots & Q_n(x_{k-1}) & Q_n(x_{k+1}) & \ldots & Q_n(x_{n+1})
\end{vmatrix}
\]

The sign \( C_k = (-1)^{(x_1, x_k)+k-1} \cdot \text{sign } C_1 \neq 0 \) for \( k = 1, \ldots, n+1 \).

**Proof.** Let \( k \in \{1, \ldots, n\} \). For all \( x \in I \) let us put

\[
Q(x) = \begin{vmatrix}
Q_1(x_1) & \ldots & Q_1(x_{k-1}) & Q_1(x) & Q_1(x_{k+2}) & \ldots & Q_1(x_{n+1}) \\
\vdots & & & & & \vdots \\
Q_n(x_1) & \ldots & Q_n(x_{k-1}) & Q_n(x) & Q_n(x_{k+2}) & \ldots & Q_n(x_{n+1})
\end{vmatrix}
\]
We have \( Q \in W \) and \( Q \neq 0 \). Since \( Q \) has additional zeros \( x_1, \ldots, x_{k-1}, x_{k+2}, \ldots, x_{n+1} \), consequently \( Q \) has no other additional zero, namely \( Q \) has no additional zero in \( \langle x_k, x_{k+1} \rangle \). By Theorem 28, we have \( \text{sign} \ Q(x_{k+1}) = (-1)^{t(x_k, x_{k+1})} \cdot \text{sign} \ Q(x_k) \neq 0 \). As \( C_k = (-1)^{k-1} Q(x_{k+1}) \) and \( C_{k+1} = (-1)^k Q(x_k) \), we have \( \text{sign} \ C_{k+1} = (-1)^{t(x_k, x_{k+1})+1} \cdot \text{sign} \ C_k \). Hence \( \text{sign} \ C_k = (-1)^{t(x_{k-1}, x_k)+1} \cdot \cdots \cdot (-1)^{t(x_k, x_{k+2})+1} \times \text{sign} \ C_1 = (-1)^{t(x_1, x_k)+k-1} \cdot \text{sign} \ C_1 \) for \( k = 1, \ldots, n+1 \).

6. THE APPROXIMATION

Assumption (for § 6.). Let \( n \in \mathbb{N}, m \in \mathbb{N}_0 \), let \( I \subset \mathbb{R}^* \) be an interval. Suppose that there are given points \( z_1 < \cdots < z_m \) in \( I \) and numbers \( t_1, \ldots, t_m \in \{1, 2\} \). Let \( I' = I - \{z_1, \ldots, z_m\} \).

Let \( V \) be an \( n \)-dimensional subspace of \( X(I) \) satisfying the Haar node condition. Let \( Q_1, \ldots, Q_n \) form a basis of \( V \).

Let \( B \neq \emptyset \) be a subset of \( I \), let us denote \( B' = B - \{z_1, \ldots, z_m\} \). Let \( f \in \mathbb{R}^B \) be such a function that if \( z_k \in B \), then \( f(z_k) = 0 \).

Remark. Let us denote \( V = \{Q_b/\forall Q \in W\} \). Then \( V \) is a subspace of \( \mathbb{R}^B \), \( \dim V = \dim_B W \leq n \). We shall approximate \( f \) by the polynomials \( Q \in V \) on the set \( B \); let us denote \( \mu = \min \|Q - f\| \). If \( Q \in W \), we denote \( \|Q - f\| = \sup_{x \in B} |Q(x) - f(x)| = \|Q_b - f\| \); we have \( \mu = \min_{Q \in W} \|Q - f\| \).

Theorem 31. (1) If \( \text{card} \ B' \leq n \), then \( \mu = 0 \).
(2) If \( \text{card} \ B' > n \), then \( \dim V = n \) and the restrictions of \( Q_1, \ldots, Q_n \) to the set \( B \) form a basis of \( V \).

Proof. (1) follows from Theorem 29(3), (2) follows from Theorem 29(2).

Theorem 32. Let \( P \in W \) have this property: there exist points \( x_1 < \cdots < x_{n+1} \) in \( B' \) and a number \( h \in \{-1, +1\} \) such that for \( k = 1, \ldots, n + 1 \) we have

\[ P(x_k) - f(x_k) = h \cdot (-1)^{t(x_k, x_k)+k} \cdot d_k, \quad \text{where} \quad d_k \geq 0. \]

(1) Let us define \( C_1, \ldots, C_{n+1} \) as in Theorem 30. Then \( \mu \geq \mu(\{x_1, \ldots, x_{n+1}\}) = \sum \frac{|C_k| \cdot |P(x_k) - f(x_k)|}{\sum |C_k|} \geq \min_{k = 1, \ldots, n+1} |P(x_k) - f(x_k)| \).

(2) If \( |P(x_k) - f(x_k)| = \|P - f\| \) for \( k = 1, \ldots, n + 1 \), then \( \|P - f\| = \mu \).

Proof. (1) We have \( \dim_{\{x_1, \ldots, x_{n+1}\}} V = \dim_{\{x_1, \ldots, x_{n+1}\}} W = n \) by Theorem 29(2). For \( k = 1, \ldots, n + 1 \) we have \( (-h \cdot \text{sign} \ C_1) \cdot C_k \cdot |P(x_k) - f(x_k)| = -h \cdot \text{sign} \ C_1 \cdot C_k \cdot (-1)^{t(x_k, x_k)+k-1} \cdot \text{sign} \ C_1 \cdot h \cdot (-1)^{t(x_k, x_k)+k} \cdot d_k = |C_k| \cdot d_k \geq 0 \) by Theorem 30. Now the assertion follows from Theorem 28(6) of [1].

(2) follows from (1).
Remark. If \( B \) is compact and if all the polynomials \( Q \in W \) and the function \( f \) are continuous on \( B \), then \( B \) is a representative subset and there exists a minimal set \( M \subset B \). If \( M \neq \emptyset \), then \( \mu > 0 \) and necessarily \( \text{card } B' \geq n + 1 \) by Theorem 31(1).

**Theorem 33.** (1) Let \( M \neq \emptyset \) be a minimal set. Then \( M \subset B' \), card \( M = n + 1 \) and \( \dim_M V = \dim_M W = n \).

(2) Suppose that there exists a minimal set \( M \) and card \( B' \geq n \). Then there exists one and only one \( P \in W \) such that \( \| P - f \| = \mu \).

**Proof.** (1) Let us admit that \( z_k \in M \). Then \( \| Q - f \|_{M - \{ z_k \}} = \| Q - f \|_M \) for all \( Q \in V \), hence \( \mu(M - \{ z_k \}) = \mu(M) \), which is a contradiction; hence \( M \subset B' \). Let us admit \( \text{card } M \leq n \), then we have \( \mu = \mu(M) = 0 \) by Theorem 29(3), which is a contradiction; hence card \( M = n + 1 \). By Theorem 29(2), we have \( \dim_M V = \dim_M W = n \).

(2) By Theorem 29(3), two distinct polynomials of \( W \) cannot coincide on \( B' \). If \( M = \emptyset \), then \( \mu = 0 \), \( f \in V \) and the assertion is evident. If \( M \neq \emptyset \), then \( \dim_M V = \) by (1) and the assertion follows from Theorem 20(3) of [1].

**Theorem 34.** Let \( M = \{ x_1, \ldots, x_{n+1} \} \) be a minimal set, we can assume \( x_1 < \ldots < x_{n+1} \). Let \( P \in W \) be such a polynomial that \( \| P - f \| = \mu \). Then there exists a number \( h \in \{ -1, +1 \} \) such that \( P(x_k) - f(x_k) = h \cdot (-1)^{(x_k, x_{k+1})+k} \cdot \| P - f \| \) for \( k = 1, \ldots, n + 1 \).

**Proof.** By Theorem 31(2) of [1], there exists \( a \in \{ -1, +1 \} \) such that for \( k = 1, \ldots, n + 1 \) we have \( P(x_k) - f(x_k) = a \cdot \text{sign } C_k \cdot \| P - f \| = a \cdot (-1)^{(x_k, x_{k+1})+k-1} \times \text{sign } C_k \cdot \| P - f \| \); we take \( h = -a \cdot \text{sign } C_1 \).

**Theorem 35.** Let card \( B' \geq n + 1 \). Suppose that there exists a minimal set, let \( P \in W \). Then \( \| P - f \| = \mu \) iff there exist points \( x_1 < \ldots < x_{n+1} \) in \( B' \) and a number \( h \in \{ -1, +1 \} \) such that \( P(x_k) - f(x_k) = h \cdot (-1)^{(x_k, x_{k+1})+k} \cdot \| P - f \| \) for \( k = 1, \ldots, n + 1 \).

**Proof.** If the latter condition is fulfilled, then we have \( \| P - f \| = \mu \) by Theorem 32(2).

Let \( \| P - f \| = \mu \). If \( \mu = 0 \), then the assertion is trivial. If \( \mu > 0 \), then the assertion follows from Theorem 34.

**Remark.** The theory given in § 5 and § 6 corresponds to that of § 2 and § 3. The most important common fact is that we can find some relations between the signs of the numbers \( C_1, \ldots, C_{n+1} \). If we consider any other properties of the polynomials \( Q \in W \) which enable us to find some similar relations, we can derive all the theory analogous to these two theories. E.g., it is possible to construct a theory which is a common generalization of these two theories (such a theory is given in [4]).
7. THE CONNECTION WITH THE CLASSICAL
HAAR CONDITION

Assumption (for §7.). Let \( n \in \mathbb{N}, m \in \mathbb{N}_0 \), let \( I \subset \mathbb{R}^* \) be an interval. Let \( z_1 < \ldots < z_m \) be points in \( I \), let \( I' = I \setminus \{z_1, \ldots, z_m\} \).

Let \( Z \) be an \((n + m)\)-dimensional subspace of \( A(I) \) satisfying the classical Haar condition on \( I \). Let \( B \neq \emptyset \) be a subset of \( I \), let us denote \( B' = B \setminus \{z_1, \ldots, z_m\} \). Let \( w_1, \ldots, w_m \in \mathbb{R} \) be fixed numbers. Let \( f \in \mathbb{R}^B \) be such a function that if \( z_k \in B \), then \( f(z_k) = w_k \) (for \( k = 1, \ldots, m \)).

Remark. We take \( t_1 = \ldots = t_m = 1 \). A point \( x \in I \) is an additional zero of \( g \in X(I) \) iff either \( x \in I' \) and \( g(x) = 0 \) or \( x = z_k \) and \( z_k \) is a touch zero of \( g \).

Remark. Let us denote \( U = \{Q \in Z | Q(z_1) = \ldots = Q(z_m) = 0\} \), \( W = \{Q \in Z | Q(z_k) = w_k \ldots \} \).

Theorem 36. \( U \) is an \( n \)-dimensional subspace of \( X(I) \) satisfying the Haar node condition.

Proof. \( U \) is a subspace of \( X(I) \). Let us choose arbitrary distinct points \( x_1, \ldots, x_n \in I' \). By the Haar condition (see Lemma (4) in §2.4. of [1] where we take \( n + m \) instead of \( n \)), there exist \( Q_1, \ldots, Q_n \in Z \) such that for \( k = 1, \ldots, n \) we have \( Q_k(x_1) = 1, Q_k(x_j) = 0 \) for \( j = 1, \ldots, k - 1, k + 1, \ldots, n \) and \( Q_k(z_j) = 0 \) for \( j = 1, \ldots, m \). Then \( Q_1, \ldots, Q_n \) are independent polynomials of \( U \).

On the other hand, if \( Q \in U \), then the polynomials \( Q \) and \( \sum_{k=1}^n Q(x_k) \cdot \overline{Q}_k \) have the same values at \( m + n \) points \( x_1, \ldots, x_n, z_1, \ldots, z_m \), hence \( Q = \sum_{k=1}^n Q(x_k) \cdot \overline{Q}_k \) (see Lemma (4) in §2.4. of [1]). Therefore \( Q_1, \ldots, Q_n \) form a basis of \( U \), hence \( \dim U = n \).

Let \( P \in U, P \neq 0 \). Let \( P \) have \( n \) additional zeros in \( I \), let \( p \) of them (denoted by \( u_1, \ldots, u_p \)) be in \( \{z_1, \ldots, z_m\} \) and \( n - p \) of them (denoted by \( v_1, \ldots, v_{n-p} \)) be in \( I' \).

If \( p = 0 \), then \( P \) has \( n + m \) zeros \( v_1, \ldots, v_n, z_1, \ldots, z_m \), which is a contradiction. Hence \( p \geq 1 \). Let \( k \in \{1, \ldots, p\} \); then \( u_k \) is a touch zero of \( P \) (inside \( I \)). There exist points \( a_k, b_k \in I \) with these properties:

1. \( a_k < b_k \) for \( k = 1, \ldots, p \);
2. \( P \) has a constant sign in \( \langle a_k, b_k \rangle \setminus \{u_k\} \) for \( k = 1, \ldots, p \);
3. if \( j \neq k \), then \( b_j < a_j \) or \( b_j < a_k \).

There exists a polynomial \( F \in Z \) such that \( F(u_k) = \text{sign} \ P(a_k) \) for \( k = 1, \ldots, p \), \( F(v_k) = 0 \) for \( k = 1, \ldots, n - p \) and \( F(z_k) = 0 \) for \( z_k \notin \{u_1, \ldots, u_p\} \). We can choose such \( c > 0 \) that \( c \cdot |F(a_k)| < |P(a_k)| \) and \( c \cdot |F(b_k)| < |P(b_k)| \) for \( k = 1, \ldots, p \).

Let us put \( Q = P - cF \); we have \( Q \in Z, Q \neq 0 \). We have \( \text{sign} \ P(a_k) = \text{sign} \ Q(b_k) = -\text{sign} \ P(a_k) \) for \( k = 1, \ldots, p \); hence \( Q \) has a zero in
Moreover, \( Q(v_k) = 0 \) for \( k = 1, \ldots, n - p \) and \( Q(z_k) = 0 \) for \( z_k \notin \{ u_1, \ldots, u_p \} \); all these zeros are distinct. Hence \( Q \) has \( 2p + (n - p) + (m - p) = m + n \) zeros in \( I \), which is a contradiction; \( U \) satisfies the Haar node condition.

**Remark.** We shall approximate the function \( f \) by the polynomials \( Q \in W \) in the set \( B \). Let us denote \( \mu = \inf_{Q \in W} ||Q - f|| \).

**Theorem 37.** Let us choose arbitrary fixed \( T \in W \), let us denote \( g = f - T \). Then we have:

1. \( g \in \mathbb{R}^n \); if \( z_k \in B \), then \( g(z_k) = 0 \) \( (k = 1, \ldots, m) \).
2. \( W = \{ Q + T|Q \in U \} \).
3. Let \( P \in W \) and \( Q \in U \) be such that \( P = Q + T \). Then \( P(x) - f(x) = Q(x) - g(x) \) for all \( x \in B \), hence \( ||P - f|| = ||Q - g|| \).
4. \( \mu = \min_{Q \in U} ||Q - g|| \); hence there exists \( P \in W \) such that \( ||P - f|| = \mu \) and it may be written \( \mu = \min_{Q \in W} ||Q - f|| \).

**Corollary.** All the assertions of § 6. hold if we write \( U \) and \( g \) instead of \( W \) and \( f \). However, by Theorem 37(3), they hold also if we write \( W \) and \( f \) again (i.e. in the original formulation).

**Remark.** The meaning of the theory given in § 7. is the following: We approximate the function \( f \) in the set \( B \) only by the polynomials of \( Z \) which have the fixed given values \( w_1, \ldots, w_m \) at the points \( z_1, \ldots, z_m \). The numbers \( w_1, \ldots, w_m \) must be given so that \( f(z_k) = w_k \) in case \( z_k \in B \).

§ 7. gives this theory only for the case when \( Z \) satisfies the Haar condition. It is possible to give such a theory also for the case when \( Z \) satisfies the Haar decomposition condition (see [4]).

A special case of the theory of § 7. was solved e.g. in [5].

8. THE APPROXIMATION WITH GIVEN DERIVATIVES

**Assumption** (for § 8.). Let \( n \in \mathbb{N} \), \( m \in \mathbb{N}_0 \), let \( I \subset \mathbb{R}^* \) be an interval. Suppose that \( z_1 < \ldots < z_m \) are points in \( I \), let \( I' = I - \{ z_1, \ldots, z_m \} \).

Suppose that \( r_1, \ldots, r_m \in \mathbb{N}_0 \) are such numbers that \( r_k = 0 \) if \( z_k \) is at the end of \( I \).

Let us denote \( t_k = 1 \) if \( r_k \) is even and \( t_k = 2 \) if \( r_k \) is odd. Let us denote \( r = \sum_{k=1}^{m} (r_k + 1) \).

Let \( Z \) be an \((r + n)\)-dimensional subspace of \( A(I) \) with the following properties:

1. If \( z_k \) is inside \( I \), then every \( Q \in Z \) has derivatives up to the order \( r_k + 1 \) at \( z_k \).
2. If we give
(a) \( q \in N_0 \) and points \( u_1, \ldots, u_q \in I' \);
(b) numbers \( s_1, \ldots, s_m \in N_0 \) such that
\( (b1) \) if \( z_k \) is inside \( I \), then \( r_k \leq s_k \leq r_k + 1 \);
\( (b2) \) if \( z_k \) is at the end of \( I \), then \( s_k = 0 \);
\( (b3) \) \( \sum_{k=1}^{m} (s_k + 1) + q = r + n \);
(c) numbers \( w_1, \ldots, w_q, v_1^{(0)}, \ldots, v_1^{(s_1)}, \ldots, v_m^{(0)}, \ldots, v_m^{(s_m)} \in R \),
then there exists one and only one \( P \in Z \) such that \( P(u_k) = w_k \) for \( k = 1, \ldots, q \) and \( P^{(i)}(z_k) = v_k^{(i)} \) for \( k = 1, \ldots, m \) and \( i = 0, \ldots, s_k \).
Let us denote \( U = \{ Q \in Z | Q^{(i)}(z_k) = 0 \text{ for } k = 1, \ldots, m \text{ and } i = 0, \ldots, r_k \} \).
Let \( y_k^{(i)}(k = 1, \ldots, m \text{ and } i = 0, \ldots, r_k) \) be fixed real numbers; let us denote \( W = \{ Q \in Z | Q^{(i)}(z_k) = y_k^{(i)} \text{ for } k = 1, \ldots, m \text{ and } i = 0, \ldots, r_k \} \).
Let \( B \neq \emptyset \) be a subset of \( I \), let us denote \( B' = B - \{ z_1, \ldots, z_m \} \). Let \( f \in R^B \) be such a function that if \( z_k \in B \), then \( f(z_k) = y_k^{(i)} \).

**Theorem 38.** \( U \) is an \( n \)-dimensional subspace of \( X(I) \) satisfying the Haar node condition.

**Proof.** \( U \) is a subspace of \( X(I) \). Let us choose arbitrary distinct points \( x_1, \ldots, x_n \in I' \). Let us take \( q = n, u_k = x_k \text{ for } k = 1, \ldots, n \text{ and } s_k = r_k \text{ for } k = 1, \ldots, m \); by (2), there exist \( Q_1, \ldots, Q_n \in U \) such that for \( k = 1, \ldots, n \) we have \( Q_k(x_k) = 1 \), \( Q_k(x_j) = 0 \text{ for } j = 1, \ldots, k-1, k+1, \ldots, n \) and \( Q^{(i)}(z_k) = 0 \text{ for } j = 1, \ldots, m \text{ and } i = 0, \ldots, r_j \). Then \( Q_1, \ldots, Q_n \) are independent polynomials of \( U \).

On the other hand, if \( Q \in U \), then the polynomials \( Q \) and \( \sum_{k=1}^{n} Q(x_k) \cdot Q_k \) have the same values at the points \( x_1, \ldots, x_n \) and zero derivatives at each \( z_j \) up to the order \( r_j \) \( (j = 1, \ldots, m) \). By (2), we have \( Q = \sum_{k=1}^{n} Q(x_k) \cdot Q_k \). Hence \( Q_1, \ldots, Q_n \) form a basis of \( U \) and \( \dim U = n \).

Let \( P \in U, P \neq 0 \). Let \( P \) have \( n \) additional zeros in \( I \) and let \( p \) of them be in \( \{ z_1, \ldots, z_m \} \). Let us consider one of these \( z_k \); it is inside \( I \). Let us admit that \( P^{(n+1)}(z_k) \neq 0 \). Then for \( r_k \) odd (even) \( z_k \) is a touch (cross) zero of \( P \) (see Theorem 27(3)) and \( z_k \) is not an additional zero of \( P \). Hence \( P^{(n+1)}(z_k) = 0 \).

We shall apply (2). If \( z_k \) is an additional zero of \( P \), we put \( s_k = r_k + 1 \), otherwise \( s_k = r_k \). Let \( u_1, \ldots, u_{n-p} \) be the additional zeros of \( P \) in \( I' \); we put \( q = n - p \). We have \( \sum_{k=1}^{m} (s_k + 1) + q = (r + p) + (n - p) = r + n \). We have \( P(u_k) = 0 \) for \( k = 1, \ldots, q \) and \( P^{(i)}(z_k) = 0 \) for \( k = 1, \ldots, m \text{ and } i = 1, \ldots, s_k \). By (2), there exists one and only one polynomial of \( Z \) with these properties. Hence \( P \equiv 0 \), which is a contradiction. \( U \) satisfies the Haar node condition.

**Remark.** We shall approximate the function \( f \) by the polynomials \( Q \in W \) in the set \( B \). Let us denote \( \mu = \inf_{Q \in W} ||Q - f|| \).
Theorem 39. Since $W \neq \emptyset$ by (2), let us choose arbitrary fixed $T \in W$ and let us denote $g = f - T_B$. Then we have:

1. $g \in R^B$; if $z_k \in B$, then $g(z_k) = 0$ ($k = 1, \ldots, m$).

2. $W = \{Q + T \mid Q \in U\}$.

3. Let $P \in W$ and $Q \in U$ be such that $P = Q + T$. Then $P(x) - f(x) = Q(x) - f(x)$ for all $x \in B$, hence $\| P - f \| = \| Q - g \|$.

4. $\mu = \min_{Q \in U} \| Q - g \|$; hence there exists $P \in W$ such that $\| P - f \| = \mu$ and it may be written $\mu = \min_{Q \in W} \| Q - f \|$.

Corollary. All the assertions of § 6. hold if we write $U$ and $g$ instead of $W$ and $f$. However, by Theorem 39(3), they hold also if we write $W$ and $f$ again (i.e. in the original formulation).

Theorem 40. Let $I = R$. Let $Z$ be the system of all the algebraic polynomials of at most the order $r + n - 1$. Then $Z$ satisfies the Assumption for § 8.

Proof. (1) is evident, (2) follows from the well-known theorem of the interpolation theory.

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