ON CLOSURE OPERATORS ON MONOIDS

JAROMÍR FUCHS, Rožnov
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INTRODUCTION

The essential part of grammatical categories theory is based on the idea of Galois connection using the induced closure operator.

A groupoid is a set $G$ with a binary operation. If $x, y$ are elements of $G$, then we denote by $xy$ the element which is obtained by applying the operation to the ordered pair $(x, y)$; $xy$ is the product of $x, y$. An element $e \in G$ is called an identity if $ex = xe = x$ for each $x \in G$. Clearly each groupoid has at most one identity. A groupoid with an identity and with an associative operation is called a monoid.

If $x_1, x_2, \ldots, x_n$ are elements of a groupoid $G$ for $i = 1, 2, \ldots, n$, where $n \geq 0$ is an integer, then it is possible to form products of these elements in the given order in several ways, e.g. $((x_1 x_2) x_3 \ldots x_{n-1}) x_n$ or $x_1 (x_2 \ldots (x_{n-2} (x_{n-1} x_n)) \ldots)$. If the operation of $G$ is associative, then all these products are equal; we shall denote them by $x_1 x_2 \ldots x_n$.

Let $V$ be an arbitrary set. We denote by $V^*$ the set of all finite sequences of elements of $V$ including the empty sequence $\varepsilon$; these sequences are called strings. For any $x \in V$, we identify $x$ with the string $(x) \in V^*$. We define the operation of concatenation in $V^*$: If $x = (x_1, x_2, \ldots, x_m)$, $y = (y_1, y_2, \ldots, y_n)$ where $m, n \geq 0$ are integers and $x_i, y_i \in V$ for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, then we put $xy = (x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n)$. It is easy to see that $\varepsilon$ is an identity and that this operation is associative. Thus, $V^*$ is a monoid, if provided by the operation of concentration; this monoid is called the free monoid on $V$. We have $(x_1, x_2, \ldots, x_m) = (x_1)(x_2) \ldots (x_m) = x_1 x_2 \ldots x_m$ for each integer $m \geq 0$ and for arbitrary elements $x_i \in V$ ($i = 1, 2, \ldots, m$), which implies that each element $x \in V^*$ is of the form $x = x_1 x_2 \ldots x_m$ where $m \geq 0$ is an integer and $x_i \in V$ for $i = 1, 2, \ldots, m$. We put $|x| = m$ and $|x|$ is called the length of $x$. Let $V$ be a set, $L \subseteq V^*$ a subset of the free monoid $V^*$. Then the ordered pair $(V, L)$ is a called a language. Let $(V, L)$ be a language, $x \in V^*$, $(u, v) \in V^* \times V^*$. If $uxv \in L$, then we put $(x, (u, v)) \in q \subseteq V^* \times (V^* \times V^*)$. We say that $(u, v)$ is a context accepting $x$. The correspondence $q$ from $V^*$ to $V^* \times V^*$ induces a Galois connection between $2^V$ and $2^{V^* \times V^*}$. The last defines a closure operator on $2^V$. 
In [2], necessary and sufficient conditions have been found for obtaining a Galois connection between \(2^V^*\) and \(2^{*\times V^*}\) by means of some language \((V, L)\). This paper solves a similar problem for closure operators.

At first, we study some basic properties of the closure operators mentioned above. It has appeared that this study can be generalized and transferred from a free monoid to a general one. In solving the basic problem we start from general closure operators on monoids. We are looking for necessary and sufficient conditions for a closure operator to be derived from a Galois connection given by means of contexts. From the standpoint of linguistic interpretation of these results the following question formulated by prof. Novotný, is answered: Which are necessary and sufficient conditions for closure operator \(c\) on \(2^V^*\) having the property 
\[c(M) \subseteq c(MN)\]
for all \(M, N \subseteq V^*\), to be derived from a language \((V, L)\) by constructing the Galois connection by means of its contexts.

1. PRINCIPAL CLOSURE OPERATORS

1.1. Definition. Let \(G\) be a set, \((2^G, \subseteq)\) the set of all its subsets partially ordered by inclusion, \(\varphi\) a mapping of \(2^G\) into \(2^G\). Let the following three conditions be satisfied for arbitrary \(X, Y \subseteq G\):

(A) \(\varphi(X) \supseteq X\).
(B) \(\varphi(\varphi(X)) = \varphi(X)\).
(C) \(X \subseteq Y\) implies \(\varphi(X) \subseteq \varphi(Y)\).

Then \(\varphi\) is called a closure operator on \(2^G\). The set \(\varphi(X)\) is called the \(\varphi\)-closure of the set \(X\).

1.2. Definition. Let \(G\) be a set, \(\varphi\) be a closure operator on \(2^G\). A set \(X \subseteq G\) is called \(\varphi\)-closed if \(\varphi(X) = X\).

We denote by \(\Phi_G\) the set of all closure operators on \(2^G\).

1.3. Remark. If \(G\) is a set then we say a "closure operator on \(G\)" instead of a "closure operator on \(2^G\)" too.

In this paper we shall study the closures, which can belong to various closure operators on a given set. Therefore the distinction, introduced in 1.2, is necessary.

1.4. Theorem. (See [1], § 23). Let \(G\) be a set, \(\varphi\) a closure operator on \(G\). Then the following assertions hold:

(A) \(G\) is \(\varphi\)-closed.
(B) \(\varphi\) is defined, in a unique way, by the system of all \(\varphi\)-closed subsets of \(G\).
(C) The \(\varphi\)-closure of each subset \(X\) of \(G\) is the least \(\varphi\)-closed subset of \(G\) including \(X\).
1.5. Lemma. Let $G$ be a set. A subset $\Phi$ of $2^G$ is the system of all $\varphi$-closed subsets for a closure operator $\varphi$ iff $\Phi$ is closed with respect to intersections.

Proof. See [1], p. 75.

1.6. Definition. Let $G$ be a monoid, $P_1, P_2, \ldots, P_n$ subsets of $G$ where $n$ is a natural number. Then we put $P_1P_2 \ldots P_n = \{x_1x_2 \ldots x_n; x_i \in P_i, i = 1, 2, \ldots, n\}$.

1.7. Definition. Let $S$ and $T$ be a pair of partially ordered sets, $\sigma$ a mapping of $S$ into $T$ and $\tau$ a mapping of $T$ into $S$. We say that the ordered pair of mappings $(\sigma, \tau)$ establishes a Galois connection between the partially ordered sets $S$ and $T$, if the following conditions (1)–(4) are satisfied:

(A) $x_1 \leq x_2$ implies $\sigma(x_1) \geq \sigma(x_2)$ for arbitrary $x_1, x_2 \in S$.
(B) $y_1 \leq y_2$ implies $\tau(y_1) \geq \tau(y_2)$ for arbitrary $y_1, y_2 \in T$.
(C) $x \leq \tau\sigma(x)$ for every element $x$ of $S$.
(D) $y \leq \sigma\tau(y)$ for every element $y$ of $T$.

1.8. Theorem. If the ordered pair of mappings $(\sigma, \tau)$ establishes a Galois connection between the partially ordered sets $S$ and $T$, then $\tau\sigma$ is a closure operator on $S$, and $\sigma\tau$ is a closure operator on $T$.

Proof. See [1], Theorem 16.

1.9. Remark. Let $G$ be a monoid, $L \subseteq G$ its subset. For $X \subseteq G$ we put $\sigma_L(X) = \{(u, v); (u, v) \in G \times G, uvx \in L$ for each $x \in X\}$. For $Y \subseteq G \times G$ we put $\tau_L(Y) = \{x; x \in G, uvx \in L$ for each $(u, v) \in Y\}$. Then the ordered pair of mappings $(\sigma_L, \tau_L)$ is a Galois connection between $2^G$ and $2^{G \times G}$.

Indeed, if $X_1, X_2 \in 2^G$ are arbitrary sets such that $X_1 \subseteq X_2$, and $(u, v) \in \sigma_L(X_2)$, then $uvx \in L$ for each $x \in X_2$. However, $X_1 \subseteq X_2$ implies $uvx \in L$ for each $x \in X_1$. Thus, $(u, v) \in \sigma_L(X_1)$; we obtain $\sigma_L(X_1) \supseteq \sigma_L(X_2)$. Further, let $X \in 2^G$ be an arbitrary set, $x \in X$ its element. Then $uvx \in L$ for each $(u, v) \in \sigma_L(X)$, which implies $x \in \tau_L(\sigma_L(X))$. Therefore we have $\tau_L(\sigma_L(X)) \supseteq X$. Thus, we have verified the validity of (A) and (C) from 1.7. Similarly, we can prove that (B) and (D) holds true, too. Thus, $(\sigma_L, \tau_L)$ establishes a Galois connection between partially ordered sets $(2^G, \subseteq)$ and $(2^{G \times G}, \subseteq)$.

1.10. Corollary. Let $G$ be a monoid, $L \subseteq G$ its subset, $(\sigma_L, \tau_L)$ a Galois connection between $2^G$ and $2^{G \times G}$. We put $\tau_L(\sigma_L(X)) = \varphi_L(X)$ for arbitrary $X \subseteq G$. Then $\varphi_L$ is a closure operator on $G$.

1.11. Definition. Let $G$ be a monoid, $\varphi$ a closure operator on $G$. $\varphi$ is called principal, if there exists $L \subseteq G$ with the property $\varphi = \varphi_L$.

We denote by $\Phi_G$ the set of all principal closure operators on $G$.

1.12. Theorem. Let $G$ be a monoid, $L \subseteq G$ its subset, $\varphi_L$ a principal closure operator on $G$. Then $L$ is $\varphi_L$-closed.
Proof. By 1.1. (A) we obtain $L \subseteq \varphi_L(L)$.
Let us have $x \in \varphi_L(L)$. Then $uxv \in L$ for each $(u, v) \in \sigma_L(L)$. As $(e, e) \in \sigma_L(L)$, we have $x = exe \in L$ which implies $\varphi_L(L) \subseteq L$.

1.13. Theorem. Let $G$ be a monoid, $L \subseteq G$ its subset. Then the following conditions are equivalent:
(i) $\varphi_L(X) = G$ for each $X \subseteq G$.
(ii) $L = G$.

Proof. Let us have $L = G$. Then $\sigma_L(X) = G \times G$ for each $X \subseteq G$ and further $\tau_L(Y) = G$ for each $Y \subseteq G \times G$. Thus, $\varphi_L(X) = G$ for each $X \subseteq G$.

Let us have $\varphi_L(X) = G$ for each $X \subseteq G$. If $L \neq G$ then, according to 1.12, we have $\varphi_L(L) = L \neq G$, which is a contradiction. Thus $L = G$.

1.14. Theorem. Let $G$ be a monoid, $L \subseteq G$ its subset. Let $M, N \subseteq G$ be arbitrary sets. Then $\varphi_L(M) \varphi_L(N) \subseteq \varphi_L(MN)$.

Proof. Let $x \in \varphi_L(M), y \in \varphi_L(N), (u, v) \in \sigma_L(MN)$. If $m \in M$ and $n \in N$ are arbitrary elements, then $mn \in MN$. It yields $umnv \in L$. Thus $um(nv) \in L$ for each $m \in M$. Hence $(u, nv) \in \sigma_L(M)$; we have $uxnv \in L$ seeing that $x \in \tau_L(\sigma_L(M))$. It implies $(ux)nv \in L$ for each $n \in N$. We have proved that $(ux, v) \in \sigma_L(N)$. Since $y \in \tau_L(\sigma_L(N))$, we obtain $uxyv \in L$. It follows $xy \in \tau_L(\sigma_L(MN)) = \varphi_L(MN)$.

1.15. Example. Let $(V, L)$ be a language where $V = \{a\}$ and $L = \{a^2, a^3\}$. We put $M = \{a^3\}, N = \{A, a\}$.

Evidently, $M, N \subseteq V^*$. We have $\sigma_L(M) = \sigma_L(\{a^3\}) = \{(A, A)\}, \varphi_L(M) = \tau_L(\{(A, A)\}) = \{a^2, a^3\}$. Further, $\sigma_L(N) = \sigma_L(\{A, a\}) = \{(A, a^2), (a, a), (a^2, A)\}, \varphi_L(N) = \tau_L(\{(A, a^2), (a, a), (a^2, A)\}) = \{A, a\}$. Thus, $\varphi_L(M) \varphi_L(N) = \{a^2, a^3\} \times \times \{A, a\} = \{a^2, a^2, a^3, a^4\}$. Clearly, $MN = \{a^3, a^4\}$. It follows that $\sigma_L(MN) = \sigma_L(\{a^3, a^4\}) = \emptyset, \varphi_L(MN) = \tau_L(\emptyset) = V^*, \varphi_L(M) \varphi_L(N) = \{a^2, a^3, a^4\} \subseteq V^* = \varphi_L(MN)$.

2. ADMISSIBLE CLOSURE OPERATORS

2.1. Definition. Let $G$ be a monoid, $\varphi$ a closure operator on $G$. We say that $\varphi$ is admissible if $\varphi(M) \varphi(N) \subseteq \varphi(MN)$ for arbitrary $M, N \subseteq G$.

We denote by $\Phi_G$ the set of all admissible closure operators on $G$.

2.2. Remark. By 1.14, we see that every principal closure operator is admissible on a monoid.

2.3. Theorem. Let $G$ be a monoid. Let elements $a, x \in G$ exist such that $a \neq e$ and $ax \neq a$.

Then $\Phi_G \subseteq \Phi_G$.
Proof. We put $\mathfrak{N}_\varphi = \{X; X \subseteq G, e \notin X\}$. If $\emptyset \neq \mathfrak{N} \subseteq \mathfrak{N}_\varphi \cup G$ then $\bigcap_{A \in \mathfrak{N}} A = \varphi(M) \subseteq G$. Thus, by 1.4.(C), $\mathfrak{N}_\varphi \cup G$ is a system of all $\varphi$-closed subsets from $G$, where $\varphi$ is a suitable closure operator on $G$. According to 1.4.(B), the closure operator $\varphi$ is defined by this system.

By 1.4.(C) we have, for every $M \subseteq G$, that $\varphi(M) = M$ when $e \notin M$, and $\varphi(M) = G$ when $e \in M$.

Let $M = \{a\}$, $N = \{e\}$. Then $\varphi(M) = \{a\}$, $\varphi(N) = G$, $MN = \{a\}$, $\varphi(MN) = \{a\}$.

Thus, $\varphi(M) \varphi(N) = \{a\} G \oplus \{a\} = \varphi(MN)$.

2.5. Corollary. Let $V \neq \emptyset$ be a set.

Then $\Phi_{V^*} \subseteq \Phi_{V^*} \subseteq \Phi_{V^*}$.
Proof. 1. Let us have \( a \in V^* \), \( a \neq A \). Then \( ax \neq a \) for each \( x \in V^* \). Thus, according to 2.3, we have \( \Phi_{V^*} \subset \Phi_{V^*} \).

2. \( V \) is not empty. Thus, by proof of 2.4, \( \{0, \{A\}, \{A, a\}, \{a\}, V^*\} \) is the system of all \( \varphi \)-closed subsets from \( V^* \), where \( \varphi \) is an admissible closure operator not principal on \( V^* \). Therefore, by 2.2, the second part of our assertion holds true, too.

3. CHARACTERIZATION OF PRINCIPAL CLOSURE OPERATORS

3.1. Lemma. Let \( G \) be a monoid, \( L \subseteq G \) its subset. Let there exist \( \varphi_L \)-closed sets \( X, Y \subseteq G \) such that \( Y \notin X \). Then there exist \( \varphi_L \)-closed sets \( U, V \subseteq G \), such that \( UXV \subseteq L \) and \( UYV \notin L \).

Proof. There exist \((u_0, v_0) \in \sigma_L(X)\) and \( y_0 \in Y \), such that \( u_0y_0v_0 \notin L \). Namely, if \( uyv \in L \) for each \((u, v) \in \sigma_L(X)\) and each \( y \in Y \), then \( Y \subseteq \tau_L(\sigma_L(X)) = \varphi_L(X) = X \), which is a contradiction.

We put \( U = \varphi_L(\{u_0\}) \), \( V = \varphi_L(\{v_0\}) \). Then we have \( u_0y_0v_0 \in UYV \) and \( u_0y_0v_0 \notin L \). Thus, \( UYV \notin L \).

On the contrary, \( u_0xv_0 \in L \) holds for each \( x \in X \). We obtain \((e, xv_0) \in \sigma_L(\{u_0\})\) for each \( x \in X \). Then we have \( u xv_0 \in L \) for each \( x \in X \) and each \( u \in \tau_L(\sigma_L(\{u_0\})) = \varphi_L(\{u_0\}) = U \). It implies \((ux, e) \in \sigma_L(\{v_0\})\) for each \( u \in U \) and each \( x \in X \). Thus, \( u xv \in L \) for each \( u \in U \), \( x \in X \), \( v \in \tau_L(\sigma_L(\{v_0\})) = \varphi_L(\{v_0\}) = V \), which implies \( UXV \subseteq L \).

3.2. Definition. Let \( G \) be a monoid, \( L \subseteq G \) its subset, \( \varphi \) a closure operator on \( G \). We say that \( L \) is a disjunctive set for \( \varphi \) if, for arbitrary \( \varphi \)-closed sets \( X, Y \subseteq G \) with the property \( Y \notin X \), there exist \( \varphi \)-closed sets \( U, V \subseteq G \), such that \( UXV \subseteq L \) and \( UYV \notin L \).

3.3. Theorem. There exists a disjunctive closed set for any principal closure operator on a monoid.

Proof. It follows from 1. and 3.1.

3.4. Theorem. Let \( G \) be a monoid, \( \varphi \) an admissible closure operator on \( G \). If there exists a \( \varphi \)-closed set disjunctive for \( \varphi \), then \( \varphi \) is principal.

Proof. Let \( X \subseteq G \) be an arbitrary set.

(A) Let us suppose that \( y \in \varphi_L(X) - \varphi(X) \).

Clearly, \( \varphi(X) \) and \( \varphi(\{y\}) \) are \( \varphi \)-closed sets with the properties \( y \notin \varphi(\{y\}) \) and \( y \notin \varphi(X) \). Thus, \( \varphi(\{y\}) \notin \varphi(X) \). Since \( L \) is a disjunctive closed set for \( \varphi \), there exist \( \varphi \)-closed \( U, V \subseteq G \) such that \( U \varphi(X) \subseteq L \) and \( U \varphi(\{y\}) \notin L \). Evidently, \( U \neq \emptyset \neq V \).

Further, there exist \( u_0 \in U \), \( y_0 \in \varphi(\{y\}) \) and \( v_0 \in V \) such that \( u_0y_0v_0 \notin L \). But \( u_0xv_0 \in L \).
for each \( x \in X \), thus, \((u_0, v_0) \in \sigma_L(X)\). Moreover, \( y \in \varphi_L(X) = \tau_L(\sigma_L(X)) \) which implies \( u_0v_0 \in L \). It follows \( u_0y_0v_0 \in \varphi(\{u_0\}) \varphi(\{y\}) \varphi(\{v_0\}) \subseteq \varphi(\{u_0v_0\}) \subseteq L \) seeing that \( \varphi \) is an admissible closure operator and \( L \) is a \( \varphi \)-closed set. Thus we have a contradiction. Hence, we have \( \varphi_L(X) \subseteq \varphi(X) \).

(B) Let us suppose that \( y \in \varphi(X) \setminus \varphi_L(X) \).

Then there exists an ordered pair \((u_0, v_0) \in \sigma_L(X)\), such that \( u_0y_0v_0 \notin L \). Indeed, from the fact that \( uyv \in L \) for each \((u, v) \in \sigma_L(X)\) it follows that \( y \in \tau_L(\sigma_L(X)) = \varphi_L(X) \), which is a contradiction. It implies \( u_0v_0 \in \varphi(\{u_0\}) \varphi(\{v_0\}) \subseteq \varphi(\{u_0v_0\}) \), because \( \varphi \) is an admissible closure operator. The fact that \((u_0, v_0) \in \sigma_L(X)\) implies \( \{u_0\} X\{v_0\} \subseteq L \). It follows \( \varphi(\{u_0\} X\{v_0\}) \subseteq \varphi(L) = L \) seeing that \( L \) is \( \varphi \)-closed. Thus, we obtain \( u_0y_0v_0 \in L \), which is a contradiction. Therefore we have \( \varphi(X) \subseteq \varphi_L(X) \).

We have proved \( \varphi(X) = \varphi_L(X) \) for each \( X \subseteq G \).

3.5. Main Theorem. Let \( G \) be a monoid, \( \varphi \) a closure operator on \( G \). Then the following assertions are equivalent:

(A) \( \varphi \) is principal.

(B) \( \varphi \) is admissible and there exists a disjunctive \( \varphi \)-closed subset in \( G \).

Proof. It follows from 2.2, 3.3 and 3.4.

3.6. Example. Let \( V^* \) be a free monoid over \( V = \{a\} \). We put \( \mathcal{A}_\varphi = \{\emptyset, \{A\}, \{a\}, V^*\} \). It is easy to see that \( \mathcal{A}_\varphi \) is a system closed with respect to intersections, which defines a closure operator \( \mathcal{P} \) on \( V^* \).

1. We put \( L = \{a\} \).

Let \( X, Y \in \mathcal{A}_\varphi \) be sets with the property \( Y \notin X \).

(a) Let us have \( X = \emptyset \). Then \( Y = \{A\} \) or \( \{a\} \) or \( = V^* \). We put \( U = \{a\} = W \). Then we obtain \( UXW = \emptyset \subseteq L \) and \( UYW = \{a^2\} \) in the first case, \( = \{a^3\} \) in the second case, and \( = \{a^2\} V^* \) in the third. None of these sets is a subset of \( L \).

(b) Let us have \( X = \{A\} \). Then \( Y = \{a\} \) or \( = V^* \). If \( U = \{A\}, W = \{a\} \) then \( UXW = \{A\}\{a\} = \{a\} = L \). If \( Y = \{a\} \) then \( UYW = \{A\}\{a\}\{a\} = \{a^2\} \notin L = = \{a\} \). At last, if \( Y = V^* \) then \( UYW = \{A\} V^* = \{a\} \notin \{A\} \notin \{a\} \in L \).

(c) Let us have \( X = \{a\} \). Then \( Y = V^* \) or \( = \{A\} \). If \( U = \{A\} \) and \( W = \{A\} \), then \( UXW = \{A\}\{a\}\{A\} = \{a\} = L \). Further, \( UYW = \{A\} V^* = \{a\} \) or \( = \{A\}\{a\}\{A\} = \{A\} \). It follows that \( UYW \notin \{a\} = L \).

We have proved that to each \( \mathcal{P} \)-closed sets \( X, Y \subseteq V^* \) with the property \( Y \notin X \) there exist \( \mathcal{P} \)-closed sets \( U, W \subseteq V^* \) such that \( UXW \subseteq \) and \( UYW \notin L \). Thus \( L = \{a\} \) is a disjunctive set for \( \mathcal{P} \).

Let \( R \subseteq V^* \) be a \( \mathcal{P} \)-closed set, i.e. \( R \in \mathcal{A}_\varphi \).

(i) Let us have \( R = \emptyset \). Then \( \sigma_L(\emptyset) = V^* \times V^*, \tau_L(V^* \times V^*) = \emptyset, \varphi_L(\emptyset) = \tau_L(\sigma_L(\emptyset)) = = \emptyset \).

(ii) Let us have \( R = \{A\} \). Then \( \sigma_L(\{A\}) = \{(A, a), (a, A)\}, \varphi_L(\{A\}) = \tau_L(\{(A, a), (a, A)\}) = \{A\} \).
(iii) Let us have $R = \{a\}$. Then $\sigma_L(\{a\}) = \{(A, A)\}$, $\varphi_L(\{a\}) = \tau_L(\{(A, A)\}) = \{a\}$. 
(iv) Let us have $R = V^*$. Then $\sigma_L(V^*) = \emptyset$, $\varphi_L(V^*) = \tau_L(\emptyset) = V^*$.

We have proved that $\varphi_L(R) \in \mathcal{A}_\psi$. 

Let $Z \subseteq V^*$ be a set with the property $Z \notin \mathcal{A}_\psi$. By 1.4.(D) we have $\Psi(Z) = V^*$. Clearly it follows that $\sigma_L(Z) = \emptyset$ and $\varphi_L(Z) = \tau_L(\emptyset) = V^*$.

From this analysis it follows that $\Psi = \varphi_L$. Simultaneously, we have proved that $\Psi$ is obtained by constructing the Galois connection by means of contexts of the language $(V, L)$, where $L = \{a\}$ is a disjunctive set for $\Psi$.

2. We put $L = \{A\}$.

Let us denote $\mathfrak{D} = \{UXW; X = \{a\}, U, W \in \mathcal{A}_\psi\}$. It is easy to see that $\mathfrak{D} = \{\emptyset, \{a\}, \{a^2\}, \{a^3\}, \{V^* - \{a, A\}, \{V^* - \{A\}\}\}$, thus $UXW \notin L$ for any not empty $\Psi$-closed sets $U, W \subseteq V^*$. It follows that $UXW \subseteq V^*$ implies either $U = \emptyset$ or $W = \emptyset$. Thus $UYW = \emptyset \subseteq L$ for each $Y \subseteq V^*$. Therefore $L$ is not a disjunctive set for $\Psi$.

We have $\sigma_L(\{a\}) = \emptyset$ and $\varphi_L(\{a\}) = \tau_L(\emptyset) = V^* \uplus \{a\} = \Psi(\{a\})$. Thus, we obtain $\Psi \neq \varphi_L$.

We have proved that $L = \{A\}$ is not a disjunctive set for $\Psi$, and this closure operator on $V^*$ cannot be obtained by constructing the Galois connection by means of contexts of the corresponding language $(V, L)$.

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J. Fuchs
756 61 Rožnov p. R., Koryčanské Paseky 1568
Czechoslovakia

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